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Notes On The Schwarzschild Line-Element

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Introduction.

The Schwarzschild line-element [(1.1) below] is familiar to all relativists. It represents the gravitational field outside a spherically symmetric distribution of matter at rest. The formula is simple and exact, and seems to offer no possibility of confusion provided we refrain (as we do here) from attempting to elucidate the so-called "singularity" at $r = 2m$. There are, however, two possible sources of confusion, and it is the purpose of the present work to clarify them by explicit and detailed discussion.

First, there is the question of the transformation of coordinates. The only transformation of interest is that of the radial coordinate r. This gives us an infinity of line-elements, including the well-known "isotropic line-element", in which the spatial part is conformally flat. Each of these line-elements represents the same gravitational field as does the original Schwarzschild line-element, and is, in fact, the Schwarzschild line-element "in disguise". We give in Sections 1 and 2 rules for detecting whether a given spherically symmetric line-element is of this type.

Secondly, and this is a much more dangerous source of confusion than the above, we have approximations based on the weakness of the field, or, equivalently, on the smallness of the mass m of the central (spherical) material system which causes the field. Roughly speaking, terms of order m give Newtonian gravitational effects (elliptic orbits), and terms of order m^2 refinements on this (advance of perihelion), while the effects of terms of order m^3 lie far beyond the limits of astronomical observation. However, such a rough statement must be accepted with great caution, for it has led one of us into an error [cf. J. L. Synge, Relativity: The General Theory (North-Holland, Amsterdam, 1960), footnote to p. 296]. The fact is that, if in the usual Schwarzschild line-element [(1.1) below] we neglect terms in m^2, obtaining the linearized form (4.31) below, we still get the correct formula for the advance of perihelion. Thus it is unwise to say that advance of perihelion is an m^2-effect. The matter is discussed fully in Section 4.

The maximum confusion arises when we combine the above two confusions, applying an infinitesimal transformation to the radial coordinate and at the same time approximating for small m. Consideration of this has been forced upon us in developing a method of successive approximations to calculate stationary weak gravitational fields in a paper by A. Das, P. S. Florides and J. L. Synge entitled "Stationary weak gravitational fields to any approximation" (this paper, not yet published*, will be referred to as DFS). To test the method, we applied it to the case of spherical symmetry, expecting to derive the Schwarzschild line-element without much difficulty. However, the work proved formidable, and we had to be satisfied with verifying that the method yields a disguised Schwarzschild line-element up to terms of order m^2 inclusive. Details are given in Section 3 below. The paper ends with an appendix in which certain integrals appearing in Section 3 are evaluated.

1. Transformations in polar coordinates.

The most familiar form of the Schwarzschild exterior metric is

$$\Phi = (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\sigma^2 - (1 - \frac{2m}{r}) dt^2 , \qquad (1.1)$$

where

$$d\sigma^2 = d\theta^2 + \sin^2\theta \, d\phi^2 , \qquad (1.2)$$

and m is a constant (the mass of the central body). The four coordinates (r, θ, ϕ, t) have geometrico-physical interpretations. The 2-space with equations r = constant, t = constant, is a sphere of constant Gaussian

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curvature $1/r^2$, and (θ, ϕ) are polar angles (co-latitude and longitude on the sphere); t is the measure of proper time by a clock at rest at infinity.

If we apply a transformation

$$r = f(\rho) , \tag{1.3}$$

we get

$$\Phi = A\, d\rho^2 + B\, \rho^2\, d\sigma^2 - C\, dt^2 , \tag{1.4}$$

where A , B , C are functions of ρ such that

$$A\, d\rho^2 = (1 - \frac{2m}{r})^{-1}\, dr^2 , \quad B\rho^2 = r^2 , \quad C = 1 - \frac{2m}{r} . \tag{1.5}$$

The form (1.4) may be described as a Schwarzschild form disguised by the transformation (1.3).

Suppose now that we are presented with a spherically symmetric statical line-element, how are we to test whether it is, or is not, a disguised Schwarzschild line-element? In other words, given the three functions $A(\rho)$, $B(\rho)$, $C(\rho)$, under what conditions on these functions does there exist a transformation $r = f(\rho)$ which will turn the form (1.4) into the form (1.1)? And how is the mass m expressed in terms of A, B, C, if the transformation is possible?

We proceed as follows. If (1.4) is a disguised Schwarzschild line-element, then (1.5) are true for some transformation $r = f(\rho)$. We have, from the second of (1.5),

$$r = \rho\, B^{\frac{1}{2}} , \tag{1.6}$$

and, from the last of (1.5),

$$\rho\, B^{\frac{1}{2}}\, (1 - C) = 2m ; \tag{1.7}$$

hence

$$\frac{d}{d\rho}\,[\rho\, B^{\frac{1}{2}}(1 - C)] = 0 \,. \tag{1.8}$$

By (1.6) we have, denoting $d/d\rho$ by a prime,

$$r\, dr = (B + \tfrac{1}{2}\rho B')\,\rho\, d\rho \,, \tag{1.9}$$

and so the first of (1.5) becomes, on multiplication by r^2,

$$A\,\rho^2\, B\, d\rho^2 = C^{-1}\,(B + \tfrac{1}{2}\rho B')^2\,\rho^2\, d\rho^2 \,. \tag{1.10}$$

Hence

$$ABC = (B + \tfrac{1}{2}\rho B')^2 \,. \tag{1.11}$$

Since the argument can be put into reverse, we have the following theorem:

Theorem I: <u>In order that the form</u>

$$\Phi = A(\rho)\, d\rho^2 + B(\rho)\,\rho^2\, d\sigma^2 - C(\rho)\, dt^2 \tag{1.12}$$

<u>may be a Schwarzschild form in disguise, it is necessary and sufficient that A, B, C be positive functions satisfying the two conditions</u>

$$\frac{d}{d\rho}\,[\rho B^{\frac{1}{2}}\,(1 - C)] = 0 \,, \tag{1.13}$$

$$ABC = (B + \tfrac{1}{2}\rho B')^2 \,. \tag{1.14}$$

<u>When these conditions are satisfied, the central mass m is given by</u>

$$m = \tfrac{1}{2}\rho B^{\frac{1}{2}}\,(1 - C) \,. \tag{1.15}$$

In all cases of physical interest, m/r is so small outside the central body that the discussion of physical phenomena can be carried out with sufficient accuracy if we retain terms in m^2, but reject m^3

and higher powers. Accordingly we write the Schwarzschild line-element (1.1) in the form

$$\Phi = \left(1 + \frac{2m}{r} + \frac{4m^2}{r^2}\right) dr^2 + r^2\, d\sigma^2 - \left(1 - \frac{2m}{r}\right) dt^2 + O_3 , \quad (1.16)$$

the symbol O_3 indicating terms involving m^3.

Consider any spherically symmetric statical space-time which is nearly flat, the deviation from flatness depending on a small parameter in such a way that the line-element may be written in the form

$$\left.\begin{aligned} &\Phi = A\, d\rho^2 + B\, \rho^2\, d\sigma^2 - C\, dt^2 , \\ &A = 1 + A_1 + A_2 + O_3 , \\ &B = 1 + B_1 + B_2 + O_3 , \\ &C = 1 + C_1 + C_2 + O_3 , \end{aligned}\right\} \quad (1.17)$$

where A_1, B_1, C_1 are small of the first order, A_2, B_2, C_2 are small of the second order, and O_3 indicates terms of the third order. What are the conditions that (1.17) should be the line-element of a weak Schwarzschild field in disguise? The answer is given by expanding (1.13) and (1.14). The conditions read

$$\frac{d}{d\rho}\left[\rho\left(C_1 + \tfrac{1}{2} B_1 C_1 + C_2\right)\right] = O_3 , \quad (1.18)$$

$$A_1 - B_1 + C_1 - \rho B_1' + B_1 C_1 + C_1 A_1 + A_1 B_1 - \left(B_1 + \tfrac{1}{2}\rho B_1'\right)^2 + A_2 - B_2 + C_2 - \rho B_2' = O_3 . \quad (1.19)$$

When these conditions are satisfied, with O_3 unspecified except as to magnitude, we may say that the metric (1.17) is a disguised Schwarzschild metric <u>correct to the second order</u>. The central mass is

$$m = -\tfrac{1}{2}\rho(C_1 + \tfrac{1}{2}B_1 C_1 + C_2) + O_3 . \qquad (1.20)$$

2. <u>Transformations in Cartesian coordinates.</u>

Greek suffixes take the values 1, 2, 3, and Latin suffixes the values 1, 2, 3, 4, with summation for a repeated suffix in each case.

If we introduce Cartesian coordinates x_α by

$$x_1 = r \sin\theta \cos\phi , \quad x_2 = r \sin\theta \sin\phi , \quad x_3 = r \cos\theta , \qquad (2.1)$$

we have

$$r^2 = x_\alpha x_\alpha , \quad r\,dr = x_\alpha dx_\alpha , \qquad (2.2)$$

and, putting $x_4 = it$, we may change the Schwarzschild line-element (1.1) into

$$\Phi = g_{ij}\, dx_i\, dx_j , \qquad (2.3)$$

where

$$g_{\alpha\beta} = \delta_{\alpha\beta} + [(1 - \frac{2m}{r})^{-1} - 1]\frac{x_\alpha x_\beta}{r^2} ,$$

$$g_{\alpha 4} = 0 , \qquad (2.4)$$

$$g_{44} = (1 - \frac{2m}{r}) .$$

We have to consider the disguises of the Schwarzschild line-element (2.3). Any spherically symmetric statical line-element may be written in the form

$$\Phi = \gamma_{ij}\, d\xi_i\, d\xi_j ,$$

where

$$\gamma_{\alpha\beta} = P(\rho)\,\delta_{\alpha\beta} + Q(\rho)\frac{\xi_\alpha \xi_\beta}{\rho^2} , \qquad (2.5)$$

$$\gamma_{\alpha 4} = 0 ,$$

$$\gamma_{44} = R(\rho) , \qquad \rho^2 = \xi_\alpha \xi_\alpha ,$$

and the coordinate ξ_4 is a pure imaginary. We seek the conditions on P, Q, R under which the metric (2.5) is a disguised Schwarzschild metric.

In view of the results already established in Section 1, the best plan is to go back to polar coordinates, instead of comparing (2.3) with (2.5). Accordingly we write

$$\xi_1 = \rho \sin\theta \cos\phi, \quad \xi_2 = \rho \sin\theta \sin\phi, \quad \xi_3 = \rho \cos\theta, \quad \xi_4 = it. \tag{2.6}$$

Then (2.5) becomes

$$\begin{aligned}\Phi &= P(d\rho^2 + \rho^2 d\sigma^2) + Q\, d\rho^2 - R\, dt^2 \\ &= (P+Q)\, d\rho^2 + P \rho^2\, d\sigma^2 - R\, dt^2.\end{aligned} \tag{2.7}$$

This is the same as (1.12) if we put

$$A = P + Q, \qquad B = P, \qquad C = R. \tag{2.8}$$

Hence Theorem I gives the following:

Theorem II: <u>In order that the form</u> $\Phi = \gamma_{ij}\, d\xi_i\, d\xi_j$, <u>with</u> ξ_4 <u>a pure imaginary and</u>

$$\begin{aligned}\gamma_{\alpha\beta} &= P(\rho)\, \delta_{\alpha\beta} + Q(\rho) \frac{\xi_\alpha \xi_\beta}{\rho^2}, \\ \gamma_{\alpha 4} &= 0, \qquad \gamma_{44} = R(\rho),\end{aligned} \tag{2.9}$$

<u>may be a Schwarzschild form in disguise, it is necessary and sufficient that</u> P, Q, R <u>satisfy</u>

$$P > 0, \qquad P + Q > 0, \qquad R > 0, \tag{2.10}$$

and

$$\frac{d}{d\rho}\left[\rho P^{\frac{1}{2}} (1 - R)\right] = 0, \tag{2.11}$$

$$PR\,(P+Q) = (P + \tfrac{1}{2} \rho P')^2. \tag{2.12}$$

<u>When these conditions are satisfied, the central mass m is given by</u>

$$m = \tfrac{1}{2}\,\rho\, P^{\frac{1}{2}}\,(1 - R)\,. \qquad (2.13)$$

In the case of a space-time which is spherically symmetric, statical, and nearly flat, we may use in (2.9) the expansion

$$\begin{aligned} P &= 1 + P_1 + P_2 + O_3\,, \\ Q &= Q_1 + Q_2 + O_3\,, \qquad (2.14) \\ R &= 1 + R_1 + R_2 + O_3\,. \end{aligned}$$

Then (2.11) and (2.12), on expansion, give the conditions that the line-element (2.9) shall be a Schwarzschild line-element in disguise: these conditions read

$$\frac{d}{d\rho}\left[\rho(R_1 + \tfrac{1}{2}R_1P_1 + R_2)\right] = O_3\,, \qquad (2.15)$$

$$Q_1 + R_1 - \rho P_1' + P_1Q_1 + Q_1R_1 + 2R_1P_1 - \rho P_1P_1' - \frac{1}{4}\rho^2P_1' + Q_2 + R_2 - \rho P_2' = O_3\,. \qquad (2.16)$$

When these conditions are satisfied, the central mass m is given by

$$m = -\tfrac{1}{2}\rho\,(R_1 + \tfrac{1}{2}R_1P_1 + R_2) + O_3\,. \qquad (2.17)$$

3. <u>The Schwarzschild line-element obtained by successive approximations.</u>

In this section we apply the method of successive approximations developed in DFS to find the Schwarzschild field up to the second approximation.

We take the body to be at rest as in Schwarzschild case, but we shall not assume for the moment that the body is spherical; it can be of any shape whatever. We shall denote the interior and exterior regions of the body by $\bar{I}$

and $\bar{\bar{E}}$ respectively, and its surface by $\bar{\bar{B}}$.

To start the process indicated in (4.1) of DFS we choose

$$\underset{1}{T}_{\alpha\beta} = 0 \ , \quad \underset{1}{T}_{\alpha 4} = 0 \ , \quad \underset{1}{T}_{44} = -\rho \ , \ \text{in } \bar{\bar{I}} \ , \qquad (3.1)$$
$$\underset{1}{T}_{ij} = 0 \quad \text{in } \bar{\bar{E}} \ ,$$

where ρ is a time-independent variable*assigned in $\bar{\bar{I}}$. Since we are dealing with a stationary field the unit normal vector n_i to the history of $\bar{\bar{B}}$ is of the form

$$n_i = (n_\alpha, 0) \ . \qquad (3.2)$$

It follows then that $\underset{1}{T}_{ij}$ satisfies conditions (4.3) of DFS, namely,

$$\underset{1}{T}_{ij,4} = 0 \ \text{ in } \bar{\bar{I}} \text{ and } \bar{\bar{E}} \ ,$$
$$\underset{1}{T}_{ij,j} = 0 \ \text{ in } \bar{\bar{I}} \text{ and } \bar{\bar{E}} \ , \qquad (3.3)$$
$$\underset{1}{T}_{ij} \, n_j = 0 \ \text{ on } \bar{\bar{B}} \ .$$

We note from (3.1) that the star conjugate of $\underset{1}{T}_{ij}$, defined by

$$\underset{1}{T}_{ij}{}^* = \underset{1}{T}_{ij} - \tfrac{1}{2}\,\delta_{ij}\,\underset{1}{T}_{kk} \ , \qquad (3.4)$$

is given by

$$\underset{1}{T}_{\alpha\beta}{}^* = \tfrac{1}{2}\,\rho\,\delta_{\alpha\beta} \ , \quad \underset{1}{T}_{4\alpha}{}^* = 0 \ , \quad \underset{1}{T}_{44}{}^* = -\tfrac{1}{2}\,\rho \ , \ \text{in } \bar{\bar{I}} \ , \qquad (3.5)$$
$$\underset{1}{T}_{ij}{}^* = 0 \ \text{ in } \bar{\bar{E}} \ .$$

Indicating 3-vectors by underlined symbols, we now define $\underset{1}{g}_{ij}$ by [cf. (4.4) of DFS]

$$\underset{1}{g}_{ij}(\underline{y}) = 4 \int \underset{1}{T}_{ij}{}^*(\underline{x}) \, \frac{d_3 x}{|\underline{x}-\underline{y}|} \ , \qquad (3.6)$$

and so obtain

$$\underset{1}{g}_{\alpha\beta} = 2V\,\delta_{\alpha\beta} \ , \quad \underset{1}{g}_{\alpha 4} = 0 \ , \quad \underset{1}{g}_{44} = -2V \ , \qquad (3.7)$$

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* Not to be confused with the radial coordinate ρ of Section 2 .

where V is the usual Newtonian potential, given by

$$V(\underline{y}) = \int \frac{\rho(x)\, d_3x}{|\underline{x} - \underline{y}|} \qquad (3.8)$$

with integration throughout $\bar{\bar{I}}$.

Next we evaluate the quantity $\underset{2}{M}_{ij}(\underset{1}{g})$ given by [cf. (3.7) of DFS]

$$\underset{2}{M}_{ij} = -\tfrac{1}{2}\, \underset{1}{g}_{rs}\, (\underset{1}{g}_{rs,ij} + \underset{1}{g}_{ij,rs} - \underset{1}{g}_{rj,is} - \underset{1}{g}_{is,rj}) - [rj,a]_1\, [ir,a]_1 +$$

$$+ \tfrac{1}{2}\, \delta_{ij}\, [rk,a]_1\, [rk,a]_1 + \delta_{ij}\, \underset{1}{g}_{ab}\, \underset{1}{L}^*_{ab} - \tfrac{1}{2}\, \underset{1}{g}_{ij}\, \underset{1}{L}^*_{aa}\,, \qquad (3.9)$$

where

$$[ij,k]_1 = \tfrac{1}{2}\, (\underset{1}{g}_{ik,j} + \underset{1}{g}_{jk,i} - \underset{1}{g}_{ij,k})\,, \qquad (3.10)$$

and

$$\underset{1}{L}^*_{ij} = \tfrac{1}{2}\, (\underset{1}{g}_{aa,ij} + \underset{1}{g}_{ij,aa} - \underset{1}{g}_{ia,aj} - \underset{1}{g}_{ja,ai})\,. \qquad (3.11)$$

Using (3.7) and (3.9) we get

$$\underset{2}{M}_{\lambda\mu} = 4V\, (V_{,\alpha\alpha}\, \delta_{\lambda\mu} - V_{,\lambda\mu}) + 3\, V_{,\alpha}\, V_{,\alpha}\, \delta_{\lambda\mu} - 2V_{,\lambda}\, V_{,\mu}\,,$$

$$\underset{2}{M}_{\lambda 4} = 0\,, \qquad (3.12)$$

$$\underset{2}{M}_{44} = 12\, V\, V_{,\alpha\alpha} + 3\, V_{,\alpha}\, V_{,\alpha}\,,$$

or equivalently

$$\underset{2}{M}^*_{\lambda\mu} = -6\, V\, V_{,\alpha\alpha}\, \delta_{\lambda\mu} - 2\, V_{,\alpha}\, V_{,\alpha}\, \delta_{\lambda\mu} - 4\, V\, V_{,\lambda\mu} - 2\, V_{,\lambda}\, V_{,\mu}\,,$$

$$\underset{2}{M}^*_{\lambda 4} = 0\,, \qquad (3.13)$$

$$\underset{2}{M}^*_{44} = 2\, V\, V_{,\alpha\alpha} - 2\, V_{,\alpha}\, V_{,\alpha}\,.$$

Up to this point the formulae are completely general; they hold for a body (at rest) of any shape and for any time-independent $\rho(\underline{x})$. From now on we shall assume that the body is spherical (of radius a) and that ρ is a function of r only, r being the "distance" from the centre of the body, given at the point y_α by

$$r = (y_\alpha y_\alpha)^{\frac{1}{2}} . \tag{3.14}$$

By (3.8)

$$V = \frac{\bar{m}}{r} \text{ in } \bar{E} , \quad \bar{m} = \int \rho \, d_3x , \tag{3.15}$$

$\bar{m}$ being the "mass" of the body in a rough sense; by (3.7) we have, in $\bar{E}$,

$$\underset{1}{g}_{\alpha\beta} = \frac{2\bar{m}}{r} \delta_{\alpha\beta} , \quad \underset{1}{g}_{\alpha 4} = 0 , \quad \underset{1}{g}_{44} = - \frac{2\bar{m}}{r} . \tag{3.16}$$

It is easily seen that $\underset{1}{g}_{ij}$ satisfies the relations (2.15), (2.16) when $\bar{m}^2$ is neglected, and so (3.16) is a disguised Schwarzschild field to the first order; by (2.17) the central mass is

$$m = \bar{m} \tag{3.17}$$

to the first order.

We proceed to find the field in the second approximation. For this we need $\underset{2}{M}_{ij}$ as given by (3.9), with substitution of $\underset{1}{g}_{ij}$ as in (3.16). Direct calculation yields

$$\underset{2}{M}_{\lambda\mu} = - \frac{14 \bar{m}^2}{r^6} y_\lambda y_\mu + \frac{7 \bar{m}^2}{r^4} \delta_{\lambda\mu} ,$$
$$\underset{2}{M}_{\lambda 4} = 0 , \quad \underset{2}{M}_{44} = \frac{3 \bar{m}^2}{r^4} , \tag{3.18}$$

or equivalently*

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* Throughout this work we use x_α or y_α indifferently as current coordinates, with $r^2 = x_\alpha x_\alpha$ in the former case, and $r^2 = y_\alpha y_\alpha$ in the latter.

$$\underset{2}{M}^*_{\lambda\mu} = -\frac{14\,\bar{m}^2}{r^6} y_\lambda y_\mu + \frac{2\,\bar{m}^2}{r^4}\delta_{\lambda\mu} \ , \tag{3.19}$$

$$\underset{2}{M}^*_{\lambda 4} = 0 \ , \qquad \underset{2}{M}^*_{44} = -\frac{2\,\bar{m}^2}{r^4} \ .$$

Having found $\underset{2}{M}_{ij}$ in $\bar{\bar{E}}$, our next task is to find a time-independent tensor $\underset{2}{P}_{ij}$ satisfying the conditions (4.13), (4.14). (4.15) of DFS, viz.

$$\underset{2}{P}_{ij} = \underset{2}{M}_{ij} \ \text{ in } \bar{\bar{E}} \ , \tag{3.20}$$

$$\underset{2}{P}_{\alpha\beta,\beta} = 0 \ \text{ in } \bar{I} \ , \qquad \underset{2}{P}_{\alpha\beta}\, n_\beta = \underset{2}{M}_{\alpha\beta}\, n_\beta \ \text{ on } \bar{\bar{B}} \ , \tag{3.21}$$

$$\underset{2}{P}_{4\beta,\beta} = 0 \ \text{ in } \bar{I} \ , \qquad \underset{2}{P}_{4\beta}\, n_\beta = \underset{2}{M}_{4\beta}\, n_\beta \ \text{ on } \bar{\bar{B}} \ , \tag{3.22}$$

$$\underset{2}{P}_{44} = 0 \ \text{ in } \bar{I} \ . \tag{3.23}$$

Now

$$n_\alpha = \frac{y_\alpha}{a} \ , \tag{3.24}$$

and so

$$M_{\alpha\beta}\, n_\beta = -\frac{7\,\bar{m}^2}{a^5} y_\alpha \ ; \tag{3.25}$$

thus (3.21) reduce to

$$\underset{2}{P}_{\alpha\beta,\beta} = 0 \ \text{ in } \bar{I} \ , \qquad \underset{2}{P}_{\alpha\beta}\, n_\beta = -\frac{7\,\bar{m}^2}{a^5} y_\alpha \ \text{ on } \bar{\bar{B}} \ . \tag{3.26}$$

We note that the only solution of (3.22) consistent with spherical symmetry is

$$\underset{2}{P}_{4\alpha} = 0 \ \text{ in } \bar{I} \ . \tag{3.27}$$

It is an essential feature of the DFS-method that, in the second and higher orders, an indeterminacy enters, and can be resolved only by assigning a structure (e.g. elastic or fluid) to the body producing the field. Now, although the usual complete Schwarzschild field is that of a fluid sphere of constant density, the exterior form (1.1) in no way depends on this particular hypothesis, and so, in applying the DFS-method we should avoid any structural hypothesis. However, to simplify the work we shall not be so general, but take, as solution of (3.26),

$$\underset{2}{P}_{\alpha\beta} = -\frac{7\,\bar{m}^2}{a^4}\,\delta_{\alpha\beta} \quad \text{in } \bar{I}, \tag{3.28}$$

which, since $\underset{2}{P}_{\alpha\beta}$ is to be regarded as a stress, corresponds to the (unique) hydrostatic pressure satisfying (3.26). Hence we have

$$\begin{aligned} &\underset{2}{P}_{ij} = \underset{2}{M}_{ij} \quad \text{in } \bar{E}, \\ &\underset{2}{P}_{\alpha\beta} = -\frac{7\,\bar{m}^2}{a^4}\,\delta_{\alpha\beta}, \qquad \underset{2}{P}_{\alpha 4} = 0, \qquad \underset{2}{P}_{44} = 0, \text{ in } \bar{I}. \end{aligned} \tag{3.29}$$

Equivalently

$$\begin{aligned} &\underset{2}{P}^*_{\alpha\beta} = \underset{2}{M}^*_{\alpha\beta} = \frac{2\,\bar{m}^2}{r^4}\,\delta_{\alpha\beta} - \frac{14\,\bar{m}^2}{r^6}\,y_\alpha\,y_\beta, \\ &\underset{2}{P}^*_{\alpha 4} = \underset{2}{M}^*_{\alpha 4} = 0, \qquad \underset{2}{P}^*_{44} = \underset{2}{M}^*_{44} = -\frac{2\,\bar{m}^2}{r^4} \end{aligned} \tag{3.30}$$

in $\bar{E}$, and

$$\underset{2}{P}^*_{\alpha\beta} = \frac{7}{2}\,\frac{\bar{m}^2}{a^4}\,\delta_{\alpha\beta}, \qquad \underset{2}{P}^*_{\alpha 4} = 0, \qquad \underset{2}{P}^*_{44} = \frac{21}{2}\,\frac{\bar{m}^2}{a^4} \tag{3.31}$$

in $\bar{I}$.

In accordance with (4.28) of DFS, we now define $\underset{2}{g}_{ij}$ by

$$\underset{2}{g}_{ij}(\underline{x}) = 4\,\kappa^{-1} \int \underset{2}{P}^*_{ij}(\underline{y})\,\frac{d_3 y}{|\underline{x}-\underline{y}|}, \quad \kappa = 8\pi. \tag{3.32}$$

Using (3.30), (3.31), and the integrals (A.28) in the Appendix, we get in $\bar{E}$ $(r > a)$

$$\underset{2}{g}_{\alpha\beta} = \bar{m}^2 \left[\delta_{\alpha\beta}\left(-\frac{3}{ar} + \frac{5}{r^2} - \frac{28}{15}\frac{a}{r^3} \right) + \left(-\frac{7}{r^2} + \frac{28\,a}{5\,r^3} \right) \frac{x_\alpha x_\beta}{r^2} \right],$$

$$\underset{2}{g}_{\alpha 4} = 0\,, \qquad (3.33)$$

$$\underset{2}{g}_{44} = \frac{\bar{m}^2}{r}\left(\frac{3}{a} + \frac{2}{r} \right).$$

This may be checked by verifying that $\underset{2}{g}^*_{\alpha\beta,\beta} = 0$.

Writing γ_{ij} for the complete metric tensor up to the second approximation (inclusive), we have

$$\gamma_{ij} = \delta_{ij} + \underset{1}{g}_{ij} + \underset{2}{g}_{ij}\,, \qquad (3.34)$$

where $\underset{1}{g}_{ij}$, $\underset{2}{g}_{ij}$ are given by (3.16) and (3.33) respectively: thus in $\bar{E}$

$$\gamma_{\alpha\beta} = \left[1 + \frac{2\bar{m}}{r} + \bar{m}^2\left(-\frac{3}{ar} + \frac{5}{r^2} - \frac{28}{15}\frac{a}{r^3} \right) \right] \delta_{\alpha\beta} + \bar{m}^2\left(-\frac{7}{r^2} + \frac{28\,a}{5\,r^3} \right) \frac{x_\alpha x_\beta}{r^2},$$

$$\gamma_{\lambda 4} = 0\,, \qquad (3.35)$$

$$\gamma_{44} = 1 - \frac{2\bar{m}}{r} + \frac{\bar{m}^2}{r}\left(\frac{3}{a} + \frac{2}{r} \right).$$

<u>To sum up: Application of the DFS-method to the case of spherical symmetry, combined with the special choice (3.28) of $\underset{2}{P}_{\alpha\beta}$, gives, in the exterior region $\bar{E}$, the metric form $\gamma_{ij}\,dx_i dx_j$ with γ_{ij} as in (3.35). Here a is the radius of the sphere ($a^2 = x_\sigma x_\sigma$) and $\bar{m}$ is the integral (3.15).</u>

At first sight we do not recognize the Schwarzschild metric in (3.35). But it is in fact a disguised Schwarzschild metric (to the second order), as we shall now show by applying Theorem II. The notation must be changed, however, reading r, x_α for ρ, ξ_α in (2.9). From (3.35) we have

$$P(r) = 1 + \frac{2\bar{m}}{r} + \bar{m}^2 \left(- \frac{3}{ar} + \frac{5}{r^2} - \frac{28}{15}\frac{a}{r^3} \right) ,$$

$$Q(r) = \bar{m}^2 \left(- \frac{7}{r^2} + \frac{28\, a}{5\, r^3} \right) , \qquad (3.36)$$

$$R(r) = 1 - \frac{2\bar{m}}{r} + \frac{\bar{m}^2}{r} \left(\frac{3}{a} + \frac{2}{r} \right) ,$$

or, by (2.14),

$$P_1 = \frac{2\bar{m}}{r} , \qquad P_2 = \bar{m}^2 \left(- \frac{3}{ar} + \frac{5}{r^2} - \frac{28}{15}\frac{a}{r^3} \right) ,$$

$$Q_1 = 0 , \qquad Q_2 = \left(- \frac{7}{r^2} + \frac{28\, a}{5\, r^3} \right) \bar{m}^2 , \qquad (3.37)$$

$$R_1 = - \frac{2m}{r} , \qquad R_2 = \frac{\bar{m}^2}{r} \left(\frac{3}{a} + \frac{2}{r} \right) .$$

It is easily proved that these values satisfy equations (2.15) and (2.16), and this shows that (3.35) is, in the second order of approximation, a disguised Schwarzschild field. By (2.17) the central mass is

$$m = \bar{m} - \frac{3}{2}\frac{\bar{m}^2}{a} . \qquad (3.38)$$

4. Advance of perihelion.

We shall start with the general spherically symmetric statical metric form

$$\Phi = A\, dr^2 + B\, r^2\, d\sigma^2 - C\, dt^2 , \qquad (4.1)$$

where A, B, C are any positive functions of r ; this may, but need not, be the Schwarzschild metric, in its usual form or disguised.

To study the geodesics, without loss of generality we consider those in the hyperplane $\theta = \frac{1}{2}\pi$. Then the usual geodesic equations give

$$B\, r^2 \dot{\phi} = \alpha^{-1} , \qquad (4.2)$$

$$C\, \dot{t} = \sqrt{(1 - \gamma^2)} , \qquad (4.3)$$

$$A\, \dot{r}^2 + B\, r^2\, \dot{\phi}^2 - C\, \dot{t}^2 = -1 , \qquad (4.4)$$

where α, γ are constants of integration, and dots denote differentiation with respect to proper time s.

Eliminating t from (4.2), (4.3), (4.4), and writing $r = u^{-1}$, we get

$$\left(\frac{du}{d\phi} \right)^2 = \frac{B}{AC} f(u) , \qquad (4.5)$$

where

$$f(u) = -C\, u^2 - B\, \alpha^2 \gamma^2 + B\, \alpha^2 (1 - C) . \qquad (4.6)$$

It will be remembered, of course, that A, B, C are themselves functions of u. From (4.5) we may obtain the orbit by a quadrature. The apsides of the orbit are given by $du/d\phi = 0$, and the reciprocals of the apsidal "distances" are zeros of $f(u)$; we recall that, by hypothesis, $B > 0$, and so no zero can come from that factor. If an orbit has two apsides, it oscillates between two concentric circles; if u', u'' ($u'' > u'$) are the reciprocal apsidal distances, the apsidal angle is

$$\chi = \int_{u'}^{u''} \sqrt{\frac{AC}{B}} \frac{du}{\sqrt{f(u)}} . \qquad (4.7)$$

Passing to the case of a weak field, we assume for A, B, C expansions

$$A = 1 + A_1 + A_2 + \dots ,$$
$$B = 1 + B_1 + B_2 + \dots , \qquad (4.8)$$
$$C = 1 + C_1 + C_2 + \dots ,$$

the subscripts indicating orders of magnitude in terms of some small parameter (e.g. the mass of the central body). Further, we consider only orbits with finite apsidal distances. By (4.6), the equation for the reciprocal of the apsidal distances is

$$-f(u) \equiv u^2 (1 + C_1 + C_2 + \dots) + \alpha^2\gamma^2 (1 + B_1 + B_2 + \dots) +$$
$$\alpha^2(1 + B_1 + B_2 + \dots) (C_1 + C_2 + \dots) = 0 . \qquad (4.9)$$

If α and γ were both finite, this would give no real apse. To get two finite real apsidal distances, we must choose α large, so that $\alpha^2 C_1$ is finite, at the same time choosing γ small, so that $\alpha\gamma$ is finite. In fact, if we define

$$\beta = \alpha\gamma , \qquad (4.10)$$

we must take

$$\beta \text{ finite}, \qquad \alpha = 0_{-\frac{1}{2}} . \qquad (4.11)$$

Rearranging (4.9) in orders of magnitude, we have

$$-f(u) = (u^2 + \beta^2 + \alpha^2 C_1) + (u^2 C_1 + \beta^2 B_1 + \alpha^2 C_2 + \alpha^2 B_1 C_1) + 0_2 , \qquad (4.12)$$

the first part being finite, and the second of the first order (0_1).

In (4.7), the terms on the right are functions of u and of a small parameter. We can proceed no further without specifying the forms of these functions; we shall take

$$A_1 = a_1 u \ , \qquad A_2 = a_2 u \ , \ \cdots \ ,$$

$$B_1 = b_1 u \ , \qquad B_2 = b_2 u \ , \ \cdots \ , \qquad (4.13)$$

$$C_1 = - c_1 u \ , \qquad C_2 = - c_2 u \ , \ \cdots \ ,$$

where the coefficients are constants, small of the orders indicated by the subscripts. Note the minus signs in the last line, a notational convenience, since, as we shall see, c_1 is positive. Substitution from (4.13) in (4.12) gives

$$f(u) = c_1 u^3 - (1 - b_1 c_1 \alpha^2 - c_2 \alpha^2)\, u^2 + (c_1 \alpha^2 - b_1 \beta^2)\, u - \beta^2 + 0_2 \ . \qquad (4.14)$$

Let us drop the 0_2 term. Then $f(u)$ is a cubic. It has two finite zeros u' , u'' $(u'' > u')$ which are approximately the roots of the quadratic equation

$$u^2 - c_1 \alpha^2 u + \beta^2 = 0 \ , \qquad (4.15)$$

the left hand side of which is the finite part of $f(u)$ as given by (4.14). It follows that

$$u' = \tfrac{1}{2}\, [c_1 \alpha^2 - (c_1{}^2 \alpha^4 - 4\beta^2)^{\frac{1}{2}}] + 0_1 \ ,$$

$$u'' = \tfrac{1}{2}\, [c_1 \alpha^2 + (c_1{}^2 \alpha^4 - 4\beta^2)^{\frac{1}{2}}] + 0_1 \ , \qquad (4.16)$$

$$u' + u'' = c_1 \alpha^2 + 0_1 \ , \qquad u' u'' = \beta^2 + 0_1 \ .$$

The cubic $f(u)$ has a third real zero, u''' , and we can write

$$f(u) = c_1\, (u - u')\, (u - u'')\, (u - u''') \ . \qquad (4.17)$$

Comparing this with (4.14), we have

$$c_1\, (u' + u'' + u''') = 1 - c_1 b_1 \alpha^2 - c_2 \alpha^2 + 0_2 \ ,$$

$$(4.18)$$

$$c_1 u' u'' u''' = \beta^2 + 0_2 \ .$$

It is clear that u''' is large of order O_{-1} . By (4.16) we have

$$c_1 u''' = 1 + O_1 , \tag{4.19}$$

or, more accurately, using (4.18) and (4.16) we get

$$\begin{aligned} c_1 u''' &= 1 - b_1 c_1 \alpha^2 - c_2 \alpha^2 - c_1(u' + u'') + O_2 \\ &= 1 - \alpha^2 (b_1 c_1 + c_1 + c_2) + O_2 . \end{aligned} \tag{4.20}$$

Hence

$$(c_1 u''')^{-\frac{1}{2}} = 1 + \tfrac{1}{2} \alpha^2 (b_1 c_1 + c_1 + c_2) + O_2 . \tag{4.21}$$

By (4.7) the apsidal angle is

$$\chi = \int_{u'}^{u''} [1 + \tfrac{1}{2} u(a_1 - b_1 - c_1)] \frac{du}{\sqrt{c_1 u'''(u''-u)(u-u')(1-\frac{u}{u'''})}} . \tag{4.22}$$

Expanding $(1 - u/u''')^{-\frac{1}{2}}$, we get, remembering (4.18),

$$\chi = (c_1 u''')^{-\frac{1}{2}} I + \tfrac{1}{2}(a_1 - b_1) J , \tag{4.23}$$

where

$$I = \int_{u'}^{u''} \frac{du}{\sqrt{(u'' - u)(u-u')}} = \pi ,$$

$$J = \int_{u'}^{u''} \frac{u\, du}{\sqrt{(u'' - u)(u-u')}} = \tfrac{1}{2} \pi (u' + u'') . \tag{4.24}$$

Substituting from (4.16) and (4.21), we have

$$\begin{aligned} \chi &= \pi + \tfrac{1}{2} \pi \alpha^2 (b_1 c_1 + c_1{}^2 + c_2) + \frac{1}{4} \pi \alpha^2 c_1 (a_1 - b_1) + O_2 \\ &= \pi + \frac{1}{4} \pi \alpha^2 (a_1 c_1 + b_1 c_1 + 2c_1{}^2 + 2c_2) + O_2 . \end{aligned} \tag{4.25}$$

Hence the advance of perihelion per revolution is

$$\boxed{\delta\phi = 2\chi - 2\pi = \tfrac{1}{2}\pi\alpha^2(a_1c_1 + b_1c_1 + 2c_1{}^2 + 2c_2) + O_2 \ .} \qquad (4.26)$$

This expresses the advance in terms of the angular momentum constant α and the constants a_1 , b_1 , c_1 , c_2 of the metric. We note that, in this approximation, a_2 and b_2 do not appear. By (4.16) we have the alternative form in terms of apsidal distances:

$$\boxed{\delta\phi = \tfrac{1}{2}\pi(u' + u'')(a_1 + b_1 + 2c_1 + 2c_2/c_1) + O_2 \ .} \qquad (4.27)$$

For the Schwarzschild metric (1.1), we have

$$A = (1 - 2mu)^{-1} = 1 + 2mu + 4m^2u^2 + \dots ,$$

$$B = 1 , \qquad (4.28)$$

$$C = 1 - 2mu ,$$

and so

$$a_1 = 2m , \qquad a_2 = 4\,m^2 ,$$

$$b_1 = 0 , \qquad b_2 = 0 , \qquad (4.29)$$

$$c_1 = 2m , \qquad c_2 = 0 .$$

Hence, by (4.26) and (4.27), to the first order,

$$\delta\phi = 6\pi\, m^2\alpha^2 = 3m\pi(u' + u'') . \qquad (4.30)$$

In more usual notation, $\alpha = 1/h$; we have here the well-known formula for advance of perihelion. It is important to note that the m^2-term in the expansion of A (viz. a_2u^2) plays no part; we would have got the same advance (4.30) had we used the Schwarzschild form in its first approximation, viz.

$$\Phi = \left(1 + \frac{2m}{r} \right) dr^2 + r^2 d\sigma^2 - \left(1 - \frac{2m}{r} \right) dt^2. \qquad (4.31)$$

The isotropic form of the Schwarzschild metric reads

$$\Phi = \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\sigma^2) - \left[\frac{1 - \frac{1}{2}m/r}{1 + \frac{1}{2}m/r}\right]^2 dt^2 , \qquad (4.32)$$

so that

$$A = B = (1 + \tfrac{1}{2}mu)^4 = 1 + 2mu + \ldots ,$$

$$C = (1 - mu + \frac{1}{4} m^2u^2) (1 - mu + \frac{3}{4} m^2u^2 + \ldots) = 1 - 2mu + 2m^2u^2 + \ldots , \qquad (4.33)$$

Hence

$$a_1 = b_1 = 2m , \qquad c_1 = 2m , \qquad c_2 = - 2m^2 . \qquad (4.33a)$$

Substituting in (4.26) and (4.27), we obtain for the advance of perihelion the formulae (4.30), as of course we must. As remarked by Eddington [The Mathematical Theory of Relativity (Cambridge University Press, 1924), p. 101.] the term c_2 is significant, and we would not get the correct advance if we used the "linearized" isotropic form

$$\Phi = \left(1 + \frac{2m}{r}\right) (dx^2 + dy^2 + dz^2) - \left(1 - \frac{2m}{r}\right) dt^2 . \qquad (4.34)$$

We shall now apply formula (4.26) to find the advance of perihelion for the disguised Schwarzschild metric (3.35) obtained by the DFS method. To do this we first express $\bar{m}$ in terms of the central mass m by using (3.38). To the second approximation we get

$$\bar{m} = m + \frac{3}{2a} m^2 . \qquad (4.35)$$

Hence the field (3.35) becomes

$$\gamma_{\alpha\beta} = P(r)\, \delta_{\alpha\beta} + Q(r) \frac{y_\alpha y_\beta}{r^2} ,$$

$$\gamma_{\alpha 4} = 0 , \qquad \gamma_{44} = R(r) , \qquad (4.36)$$

where (neglecting terms in m^3)

$$P(r) = 1 + \frac{2m}{r} + m^2 \left(\frac{5}{r^2} - \frac{28}{15} \frac{a}{r^3} \right) ,$$

$$Q(r) = m^2 \left(- \frac{7}{r^2} + \frac{28\, a}{5\, r^3} \right) , \tag{4.37}$$

$$R(r) = 1 - \frac{2m}{r} + \frac{2m^2}{r^2} .$$

Introducing spherical polar coordinates as in Section 2 , the line-element can be put in the form

$$\Phi = A\, dr^2 + B\, r^2\, d\sigma^2 - C\, dt^2 , \tag{4.38}$$

where

$$A = P + Q = 1 + \frac{2m}{r} + m^2 \left(- \frac{2}{r^2} + \frac{56}{15} \frac{a}{r^3} \right) ,$$

$$B = P = 1 + \frac{2m}{r} + m^2 \left(\frac{5}{r^2} - \frac{28}{15} \frac{a}{r^3} \right) , \tag{4.39}$$

$$C = R = 1 - \frac{2m}{r} + \frac{2m^2}{r^2} .$$

Therefore in this case we have

$$a_1 = 2m , \quad b_1 = 2m , \quad c_1 = 2m , \quad c_2 = - 2m^2 . \tag{4.40}$$

Hence equation (4.26) gives $\delta\phi = 6\pi m^2 \alpha^2$ as in (4.30).

Remembering that α^2 is large of order O_{-1} , and that u' , u'' are finite, we see from (4.26) and (4.27) that advance of perihelion is a "first-order effect". Nevertheless, m^2 is involved in the products of the type $a_1 c_1$, and so we are inclined to say that it is, if exhibited in the form (4.26), an "m^2-effect". The key to this riddle is that α^2 is itself a large quantity of order m^{-1} , since otherwise we would not get an orbit with finite apsidal distances. One must beware, in this confusing situation, of making statements about orders of magnitude without careful consideration.

We thank Mr. A. Das for discussions.

APPENDIX

Evaluation of certain integrals (Section 3).

Let $\underline{x}$ and $\underline{y}$ be any two points in Euclidean 3-space. We write $|\underline{x}| = x$, $|\underline{y}| = y$, and denote by $|\underline{x} - \underline{y}|$ the distance between the two points. Let $\mu = \cos\theta$, where θ is the angle between the vectors $\underline{x}$, $\underline{y}$. Then

$$\frac{1}{|\underline{x} - \underline{y}|} = \frac{1}{(x^2 + y^2 - 2xy\mu)^{\frac{1}{2}}}$$

$$= \frac{1}{x}\left(1 - 2\mu\,\frac{y}{x} + \frac{y^2}{x^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{y}\left(1 - 2\mu\,\frac{x}{y} + \frac{x^2}{y^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\alpha}\left(P_0(\mu) + \beta P_1(\mu) + \beta^2 P_2(\mu) + \dots\right), \qquad \text{(A-1)}$$

where the P's are Legendre polynomials and

$$\left.\begin{array}{ll} \alpha = x, & \beta = y/x \ \text{ if } \ y < x, \\ \alpha = y, & \beta = x/y \ \text{ if } \ y > x. \end{array}\right\} \qquad \text{(A-2)}$$

We have [cf. E. W. Hobson, Spherical and Ellipsoidal Harmonics (Cambridge University Press, 1931), p. 37.]

$$\int_{-1}^{1} P_m P_n \, d\mu = 0 \quad \text{for } m \neq n, \qquad \text{(A-3)}$$

$$\int_{-1}^{1} P_n{}^2 \, d\mu = \frac{2}{2n+1}.$$

Also

$$P_0 = 1 , \quad P_1 = \mu , \quad P_2 = \tfrac{1}{2}(3\mu^2 - 1) ,$$
$$P_3 = \tfrac{1}{2}(5\mu^3 - 3\mu) , \quad P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3) , \ldots , \tag{A-4}$$

and so

$$\mu = P_1 , \quad \mu^2 = \frac{1}{3}(1 + 2P_2) , \quad \mu^3 = \frac{1}{5}(3P_1 + 2P_3) ,$$
$$\mu^4 = \frac{1}{35}(7 + 20\,P_2 + 8\,P_4) , \ldots \tag{A-5}$$

Let $f(y)$ be any function of $y\ (=|\underline{y}|)$. Consider the integral

$$I(x) = \int f(y) \frac{d_3 y}{|\underline{x} - \underline{y}|} , \tag{A-6}$$

taken throughout the whole of space. Let dS_y be an element of the sphere $y =$ constant. Then we have $d_3 y = dS_y\, dy$, and

$$I(x) = \int_{y=0}^{x} f(y)\, dy \int_{S_y} \frac{dS_y}{|\underline{x} - \underline{y}|} + \int_{y=x}^{\infty} f(y)\, dy \int_{S_y} \frac{dS_y}{|\underline{x} - \underline{y}|} . \tag{A-7}$$

Now, by (A-1)

$$\int_{S_y} \frac{dS_y}{|\underline{x} - \underline{y}|} = 2\pi y^2 \int_{-1}^{1} \frac{d\mu}{|\underline{x} - \underline{y}|} = \frac{4\pi y^2}{\alpha} , \tag{A-8}$$

and so

$$I(x) = \frac{4\pi}{x} \int_{y=0}^{x} y^2 f(y)\, dy + 4\pi \int_{y=x}^{\infty} y\, f(y)\, dy , \tag{A-9}$$

if $f(y)$ is such that these integrals converge.

In particular, if

$$f(y) = 0 \text{ for } y < a , \quad f(y) = \frac{1}{y^4} \text{ for } y > a ,$$
$$x > a , \tag{A-10}$$

we have

$$I(x) = \int_{y>a} \frac{1}{y^4} \frac{d_3y}{|\underline{x} - \underline{y}|} = \frac{4\pi}{x} \int_{y=a}^{x} \frac{1}{y^2}\, dy + 4\pi \int_{y=x}^{\infty} \frac{1}{y^3}\, dy$$

$$= \frac{4\pi}{x} \left(\frac{1}{a} - \frac{1}{x}\right) + \frac{2\pi}{x^2} = \frac{2\pi}{x^2} \left(-1 + \frac{2x}{a}\right) . \qquad \text{(A-11)}$$

Consider now the integral

$$I_{\rho\sigma}(\underline{x}) = \int f(y) \frac{y_\rho y_\sigma d_3 y}{|\underline{x} - \underline{y}|} . \qquad \text{(A-12)}$$

From tensor form it is evident that

$$I_{\rho\sigma} = \phi(x)\, \delta_{\rho\sigma} + \psi(x) \frac{x_\rho x_\sigma}{x^2} , \qquad \text{(A-13)}$$

where ϕ and ψ are functions of x ($= |\underline{x}|$) . Then

$$\left.\begin{aligned} I_{\rho\rho} &= 3\phi + \psi , \\ I_{\rho\sigma} x_\rho x_\sigma &= x^2 \,(\phi + \psi) , \end{aligned}\right. \qquad \text{(A-14)}$$

and so

$$\left.\begin{aligned} \phi(x) &= \tfrac{1}{2} \left(I_{\rho\rho} - I_{\rho\sigma} \frac{x_\rho x_\sigma}{x^2} \right) , \\ \psi(x) &= \tfrac{1}{2} \left(3\, I_{\rho\sigma} \frac{x_\rho x_\sigma}{x^2} - I_{\rho\rho}\right) . \end{aligned}\right\} \qquad \text{(A-15)}$$

Now

$$\left.\begin{aligned} I_{\rho\rho} &= \int y^2 f(y) \frac{d_3 y}{|\underline{x} - \underline{y}|} , \\ I_{\rho\sigma} \frac{x_\rho x_\sigma}{x^2} &= \int y^2 f(y) \frac{\mu^2\, d_3 y}{|\underline{x} - \underline{y}|} , \end{aligned}\right\} \qquad \text{(A-16)}$$

and so

$$I_{\rho\rho} = \int_{y=0}^{\infty} y^2 \, f(y) \, dy \int_{S_y} \frac{dS_y}{|\underline{x} - \underline{y}|} ,$$

(A-17)

$$I_{\rho\sigma} \frac{x_\rho x_\sigma}{x^2} = \int_{y=0}^{\infty} y^2 \, f(y) \, dy \int_{S_y} \frac{\mu^2 \, dS_y}{|\underline{x} - \underline{y}|} .$$

We have, as in (A-8),

$$\int_{S_y} \frac{dS_y}{|\underline{x} - \underline{y}|} = \frac{4\pi y^2}{\alpha} , \qquad \text{(A-18)}$$

and, by (A-1),

$$\int_{S_y} \frac{\mu^2 \, dS_y}{|\underline{x} - \underline{y}|} = \frac{2\pi y^2}{3\alpha} \int_{\mu=-1}^{1} (1 + 2P_2)(1 + \beta P_1 + \beta^2 P_2 + \dots) \, d\mu$$

$$= \frac{4\pi y^2}{3\alpha} \left(1 + \frac{2}{5} \beta^2\right) . \qquad \text{(A-19)}$$

Thus

$$I_{\rho\rho} = \frac{4\pi}{x} \int_{y=0}^{x} y^4 \, f(y) \, dy + 4\pi \int_{y=x}^{\infty} y^3 \, f(y) \, dy , \qquad \text{(A-20)}$$

and

$$I_{\rho\sigma} \frac{x_\rho x_\sigma}{x^2} = \frac{4\pi}{3x} \int_{y=0}^{x} y^4 \, f(y) \left(1 + \frac{2}{5} \frac{y^2}{x^2}\right) dy + \frac{4\pi}{3} \int_{y=x}^{\infty} y^3 \, f(y) \left(1 + \frac{2}{5} \frac{x^2}{y^2}\right) dy$$

$$= \frac{8\pi}{15} \frac{1}{x^3} \int_{y=0}^{x} y^6 \, f(y) \, dy + \frac{4\pi}{3x} \int_{y=0}^{x} y^4 \, f(y) \, dy +$$

$$\frac{4\pi}{3} \int_{y=x}^{\infty} y^3 \, f(y) \, dy + \frac{8\pi x^2}{15} \int_{y=x}^{\infty} y \, f(y) \, dy . \qquad \text{(A-21)}$$

Thus, by (A-15),

$$\phi(x) = -\frac{4\pi}{15}\frac{1}{x^3}\int_{y=0}^{x} y^6\ f(y)\ dy + \frac{4\pi}{3x}\int_{y=0}^{x} y^4\ f(y)\ dy + \frac{4\pi}{3}\int_{y=x}^{\infty} y^3\ f(y)\ dy$$

$$- \frac{4\pi x^2}{15}\int_{y=x}^{\infty} y\ f(y)\ dy\ , \qquad \text{(A-22)}$$

$$\psi(x) = \frac{4\pi}{5}\frac{1}{x^3}\int_{y=0}^{x} y^6\ f(y)\ dy + \frac{4\pi}{5}x^2\int_{y=x}^{} y\ f(y)\ dy\ .$$

Provided these integrals converge, we have

$$I_{\rho\sigma} = \int f(y)\ \frac{y_\rho y_\sigma}{|\underline{x} - \underline{y}|}\ d_3y = \phi(x)\ \delta_{\rho\sigma} + \psi(x)\ \frac{x_\rho x_\sigma}{x^2}\ . \qquad \text{(A.23)}$$

As a particular case, we take

$$f(y) = 0 \ \text{ for } \ y < a\ , \quad f(y) = 1/y^6 \ \text{ for } \ y > a\ ,$$

$$x > a\ . \qquad \text{(A-24)}$$

Then

$$\int_{y=0}^{x} y^6\ f(y)\ dy = \int_{y=a}^{x} dy = x - a\ ,$$

$$\int_{y=0}^{x} y^4\ f(y)\ dy = \int_{y=a}^{x} \frac{1}{y^2}\ dy = \frac{1}{a} - \frac{1}{x}\ , \qquad \text{(A-25)}$$

$$\int_{y=x}^{\infty} y^3\ f(y)\ dy = \int_{y=x}^{\infty} \frac{1}{y^3}\ dy = \frac{1}{2x^2}\ ,$$

$$\int_{y=x}^{\infty} y\ f(y)\ dy = \int_{y=x}^{\infty} \frac{1}{y^5}\ dy = \frac{1}{4x^4}\ .$$

Thus, by (A-22),

$$\phi(x) = \frac{4\pi}{15}\left[-\frac{1}{x^3}(x-a) + \frac{5}{x}\left(\frac{1}{a}-\frac{1}{x}\right) + \frac{5}{2}\cdot\frac{1}{x^2} - x^2\frac{1}{4x^4}\right]$$

$$= \frac{4\pi}{15}\left[\frac{a}{x^3} - \frac{15}{4}\frac{1}{x^2} + \frac{5}{a}\frac{1}{x}\right], \qquad \text{(A-26)}$$

$$\psi(x) = \frac{4\pi}{5}\left[\frac{1}{x^3}(x-a) + x^2\frac{1}{4x^4}\right] = \frac{4\pi}{5}\left[-\frac{a}{x^3} + \frac{5}{4}\frac{1}{x^2}\right],$$

or equivalently,

$$\phi(x) = \frac{\pi}{x^2}\left[-1 + \frac{4}{15}\frac{a}{x} + \frac{4}{3}\frac{x}{a}\right],$$

$$\text{(A-27)}$$

$$\psi(x) = \frac{\pi}{x^2}\left[1 - \frac{4}{5}\frac{a}{x}\right].$$

For purpose of reference, let us repeat some results from (A-11) and (A-23) : for $x > a$,

$$\int_{y>a} \frac{1}{y^4}\,\frac{d_3y}{|\underline{x}-\underline{y}|} = \frac{2\pi}{x^2}\left(-1 + \frac{2x}{a}\right),$$

$$\text{(A-28)}$$

$$\int_{y>a} \frac{1}{y^6}\,\frac{y_\rho y_\sigma}{|\underline{x}-\underline{y}|}\,d_3y = \frac{\pi}{x^2}\left(-1 + \frac{4}{15}\frac{a}{x} + \frac{4}{3}\frac{x}{a}\right)\delta_{\rho\sigma} + \frac{\pi x_\rho x_\sigma}{x^4}\left(1 - \frac{4}{5}\frac{a}{x}\right).$$

As a check on these formulae, put $\rho = \sigma$ in the second. Then it should agree with the first. From the second we get

$$\frac{\pi}{x^2}\left[3\left(-1 + \frac{4}{15}\frac{a}{x} + \frac{4}{3}\frac{x}{a}\right) + 1 - \frac{4}{5}\frac{a}{x}\right] = \frac{\pi}{x^2}\left[-2 + 4\frac{x}{a}\right],$$

which agrees.

E R R A T A

p. 9, equation (2.16): for $-\frac{1}{4}\rho^2 P_1'$ read $-\frac{1}{4}\rho^2 P_1'^2$.

p. 12, equations (3.18), first line: for ${}_2M_{\lambda 4}$ read ${}_2M_{\lambda\mu}$.

p. 19, equations (4.13): for a_2u read a_2u^2 ,

b_2u read b_2u^2 ,

c_2u read c_2u^2 .