# Super (a,d)-Edge-antimagic Total Labeling of Shackle of Fan Graph

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**Abstract**

A graph $G$ of order $p$ and size $q$ is called an $(a,d)$-*edge-antimagic total* if there exist a bijection $f : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$ such that the edge-weights, $w(uv) = f(u) + f(v) + f(uv), uv \in E(G)$, form an arithmetic sequence with first term $a$ and common difference $d$. Such a graph $G$ is called *super* if the smallest possible labels appear on the vertices. In this paper we study super $(a,d)$-edge-antimagic total properties of connected of shackle of Fan Graph. The result shows that shackle of Fan Graph admit a super edge antimagic total labeling for $d \in 0, 1, 2$ for $n \geq 1$. It can be concluded that the result of this research has convered all the feasible $n$, $d$.

**Key Words** :*$(a,d)$-edge-antimagic total labeling, super $(a,d)$-edge-antimagic total labeling, Fan Graph.*

## Introduction

Defnitions of (a,d)-EAT labeling and super (a,d)-EAT labeling were introduced by Simanjuntak at al [7]. These labelings are natural extensions of the notion of edge- magic labeling, dened by Kotzig and Rosa [6], where edge-magic labeling is called magic valuation, and the notion of super edge-magic labeling, is natural extension of the notion of edge-magic labeling dened by Kotzig and Rosa [6]. The super $(a,d)$-edge-antimagic total labeling [8] is natural extension of the notion of super edge-magic labeling. For more information about graph can be found in [1],[3],[4],[2],[5]. In this paper we will now concentrate on the connected shackle of Fan graph denoted by $\mathbb{F}_n$. The example of figure 1.

## Super $(a,d)$-edge Antimagic Total Labeling

An $(a,d)$-edge-antimagic total labeling on a graph G is a bijective function $f$:$V(G) \cup E(G) \to \{1, 2, 3, ...p+q\}$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v)$; $uv \, \epsilon \, E(G)$, form an arithmetic progression $\{a, a+b, a+2b, \dots, a+(q-1)d\}$, where $a > 1$ and $d \geq 0$ are two fixed integers. If such a labeling exists then $G$ is said to be an $(a,d)$-edge-antimagic total graph. Such a graph $G$ is called super if the smallest possible labels appear on the vertices.

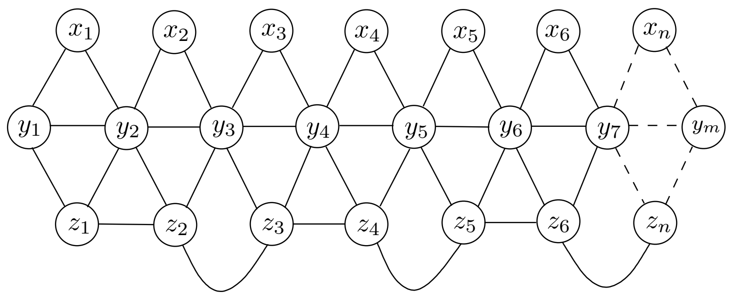


Figure 1:

Thus, a super $(a, d)$-edge-antimagic total graph is a graph that admits a super $(a, d)$-edgeantimagic total labeling.

Shackle of fan graph denoted by $\mathbb{F}_n$ with $n \geq 1$ is a connected graph with vertex set. $V(\mathbb{F}\ _n)$=$\{x_i, y_j, z_i; 1 \leq i \leq n; 1 \leq j \leq m; m, n\epsilon N\}$ and $E(\mathbb{F}\ _n)$= $\{x_iy_i, x_iy_{i+1}, y_iz_i, z_iz_{i+1}, z_iy_{i+1}; 1 \leq j \leq n \cup y_jy_{j+1}; 1 \leq j \leq m\}$. Thus $|V(\mathbb{F}\ _n)|$=p=$3n+$ 1 and $|E(\mathbb{F}\ _n)|$=q=$6n - 1$.

We continue this section by a necessary condition for a graph to be super $(a, d)$-edge antimagic total, providing a least upper bound for feasible values of $d$.

**Lemma 1** *If a $(p, q)$-graph is super $(a, d)$-edge-antimagic total then* $d \leq \frac{2p+q-5}{q-1}$

**Proof** Assume that a $(p, q)$-graph has a super $(a, d)$-edge-antimagic total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$. The minimum possible edge-weight in the labeling $f$ is at least $1 + 2 + p + 1 = p + 4$. Thus, $a \geq p + 4$. On the other hand, the maximum possible edge-weight is at most $(p - 1) + p + (p + q) = 3p + q - 1$. So we obtain $a + (q - 1)d \leq 3p + q - 1$ which gives the desired upper bound for the difference $d$. Or we can write:

$$\begin{aligned}
&\Leftrightarrow \quad a+(q-1)d \leq 3p+q-1 \\
&\Leftrightarrow \quad (p+4)+(q-1)d \leq 3p+q-1 \\
&\Leftrightarrow \quad d \leq \frac{3p+q-1-(p+4)}{q-1} \\
&\Leftrightarrow \quad d \leq \frac{2p+q-5}{q-1} \\
&\Leftrightarrow \quad d \leq \frac{2(3n+1)+(6n-1)-5}{(6n-1)-1} \\
&\Leftrightarrow \quad d \leq \frac{6n+2+6n-6}{6n-2} \\
&\Leftrightarrow \quad d \leq \frac{12n-4}{6n-2} \\
&\Leftrightarrow \quad d \leq 2 \\
&\Leftrightarrow \quad d \in \{0,1,2\} \qquad (1)
\end{aligned}$$

□

**Lemma 2** *A $(p,q)$-graph $G$ is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1,2,\ldots,p\}$ such that the set $S = \{f(u)+f(v) : uv \in E(G)\}$ consists of $q$ consecutive integers. In such a case, $f$ extends to a super edge-magic labeling of $G$ with magic constant $a = p+q+m$, where $m = min(M)$ and $S = \{a-(p+1), a-(p+2), \ldots, a-(p+q)\}$.*

The two above lemma will be used for develop theorem 1.

## Result

If shackle of Fan graph has a super $(a,d)$-edge-antimagic total labeling then for $p = 3n+1$ and $q = 6n-1$ it follows from Lemma 1 that the upper bound of $d$ is $d\leq 2$ or $d \in \{0,1,2\}$. The following Lemma describes an $a,1$-edge-antimagic vertex labeling for shackle of Fan graph.

**Lemma 3** *If $n \geq 1$, then the Shackle of Fan graph $\mathbb{F}_n$ has an $(a,1)$-edge-antimagic vertex labeling.*

**Proof.** Define the vertex labeling $f_1 : V(\mathbb{F}_n) \rightarrow \{1,2,\ldots,3n+1\}$ in the following way:

$$\begin{aligned}
f_1(x_i) &= 3i, \text{for } 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
f_1(x_i) &= 3i-1, \text{for } 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
f_1(y_i) &= 3j-2, \text{for } 1 \leq j \leq m \\
f_1(z_i) &= 3i-1, \text{for } 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
f_1(z_i) &= 3i, \text{for } 1 \leq i \leq n \text{ and i } \epsilon \text{ even number}
\end{aligned}$$

The vertex labeling $f_1$ is a bijective function. The edge-weights of $\mathbb{F}_n$, under the labeling $f_1$, constitute the following sets

$$\begin{aligned}
w_{f_1}(x_i y_i) &= 5i-1, \text{for} 1 \leq j \leq n, \\
w_{f_1}(x_i y_{i+1}) &= 6i+1, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
w_{f_1}(x_i y_{i+1}) &= 6i, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
w_{f_1}(y_i z_i) &= 6i-3, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
w_{f_1}(y_i z_i) &= 6i-2, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
w_{f_1}(z_i z_{i+1}) &= 6i+2, \text{for} 1 \leq i \leq n, \\
w_{f_1}(z_i y_{i+1}) &= 6i, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ odd number} \\
w_{f_1}(z_i y_{i+1}) &= 6i+1, \text{for} 1 \leq i \leq n \text{ and i } \epsilon \text{ even number} \\
w_{f_1}(y_j y_{j+1}) &= 6j-1, \text{for} 1 \leq j \leq m,
\end{aligned}$$

It is not difficult to see that the set $w_{f_1} = \{3, 4, 5, \ldots, 6n-1\}$ consists of consecutive integers. Thus $f_1$ is a $(3,1)$-edge antimagic vertex labeling.

Bača, Y. Lin, M. Miller and R. Simanjuntak [5], Theorem 5) have proved that if $(p,q)$-graph $G$ has an $(a,d)$-edge antimagic vertex labeling then $G$ has a super$(a+p+q, d-1)$-edge antimagic total labeling and a super$(a+p+1, d+1)$-edge antimagic total labeling. With the theorem Lemma 3 in hand, we obtain the following result.

◇ **Teorema 1** *If* $n \geq 1$ *then the graph* $\mathbb{F}_n$ *has a super* $(9n+3, 0)$*-edge-antimagic total labeling and a super* $(3n+5, 2)$*-edge-antimagic total labeling.*

**Proof.**
*Case 1.* $d = 0$
We have proved that the vertex labeling $f_1$ is a $(3,1)$-edge antimagic vertex labeling. With respect to Lemma 2, by completing the edge labels $p+1, p+2, \ldots, p+q$, we are able to extend labeling $f_1$ to a super $(a,0)$-edge-antimagic total labeling, where, for $p = 3n+1$ and $q = 6n-1$, the value $a = 9n+3$.

*Case 2.* $d = 2$
Label the vertices of $\mathbb{F}_n$ with $f_3$ that the edge labeling for $d = 2$, so we can that

label the edges with the following way.

$$
\begin{aligned}
f_3(x_iy_i) &= 3n+6i-3, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_3(x_iy_i) &= 3n+6i-4, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_3(x_iy_{i+1}) &= 3n+6i, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_3(x_iy_{i+1}) &= 3n+6i-1, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_3(y_iz_i) &= 3n+6i-4, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_3(y_iz_i) &= 3n+6i-3, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_3(z_iz_{i+1}) &= 3n+6i+1, \text{for} 1 \le i \le n,\\
f_3(z_iy_{i+1}) &= 3n+6i-1, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_3(z_iy_{i+1}) &= 3n+6i, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_3(y_jy_{j+1}) &= 3n+6j-2, \text{for} 1 \le j \le m,
\end{aligned}
$$

The total labeling $f_3$ is a bijective function from $V(\mathbb{F}_n) \cup E(\mathbb{F}_n)$ onto the set $\{1, 2, 3, \ldots, 3n+1\}$. The edge-weights of $\mathbb{F}_n$, under the labeling $f_3$, constitute the sets

$$
\begin{aligned}
W_{f_3} &= \{w_{f_3} + f_3(x_iy_i); \text{ for } 1 \le i \le n\} = 3n+11i-4 \text{ and i } \epsilon \text{ odd number}\\
W_{f_3} &= \{w_{f_3} + f_3(x_iy_i); \text{ for } 1 \le i \le n\} = 3n+11i-5 \text{ and i } \epsilon \text{ even number}\\
W_{f_3} &= \{w_{f_3} + f_3(x_iy_{i+1}); \text{ for } 1 \le i \le n\} = 3n+12+1 \text{ and i } \epsilon \text{ odd number}\\
W_{f_3} &= \{w_{f_3} + f_3(x_iy_{i+1}); \text{ for } 1 \le i \le n\} = 3n+12i-1 \text{ and i } \epsilon \text{ even number}\\
W_{f_3} &= \{w_{f_3} + f_3(y_iz_i); \text{ for } 1 \le i \le n\} = 3n+12-7 \text{ and i } \epsilon \text{ odd number}\\
W_{f_3} &= \{w_{f_3} + f_3(y_iz_i); \text{ for } 1 \le i \le n\} = 3n+12i-5 \text{ and i } \epsilon \text{ even number}\\
W_{f_3} &= \{w_{f3} + f_3(z_iz_{i+11}); \text{ for } 1 \le i \le n\} = 3n+12i+3\\
W_{f_3} &= \{w_{f_3} + f_3(z_iy_{i+1}); \text{ for } 1 \le i \le n\} = 3n+12i-1 \text{ and i } \epsilon \text{ odd number}\\
W_{f_3} &= \{w_{f_3} + f_3(z_iy_{i+1}); \text{ for } 1 \le i \le n\} = 3n+12i+1 \text{ and i } \epsilon \text{ even number}\\
W_{f_3} &= \{w_{f3} + f_3(y_jy_{j+1}); \text{ and } 1 \le j \le m\} = 3n+12j-3
\end{aligned}
$$

It is not difficult to see that the set $W_{f_3} = \{3n+5, 3n+7, 3n+9, , \ldots, 15n+1\}$ contains an arithmetic sequence with $a = 3n+5$ and $d = 2$. Thus $f_3$ is a super $(3n+5, 2)$-edge-antimagic total labeling. This concludes the proof. □

**Theorem 2** *If $n \ge 1$, then the graph $\mathbb{F}_n$ has a super $(6n+4, 1)$-edge-antimagic total labeling.*

**Proof.** Label the vertices of $\mathbb{F}_n$ with $f_4(x_iy_i) = f_1(x_iy_i)$,$f_4(x_iy_{i+1}) = f_1(x_iy_{i+1})$, $f_4(y_iz_i) = f_1(y_iz_i)$,$f_4(z_iz_{i+1}) = f_1(z_iz_{i+1})$, $f_4(y_jy_{j+1}) = f_1(y_jy_{j+1})$, $f_4(z_iy_{i+1}) = f_1(z_iy_{i+1})$ untuk $1 \le i \le n, 1 \le j \le m$ and label the edges with the following way.

$$
\begin{aligned}
f_4(x_iy_i) &= 9n-3i+3, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_4(x_iy_i) &= 6n-3i+4, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_4(x_iy_{i+1}) &= 6n-3i+5, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_4(x_iy_{i+1}) &= 9n-3i+2, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_4(y_iz_i) &= 6n-3i+4, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_4(y_iz_i) &= 9n-3i+3, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_4(z_iz_{i+1}) &= 9n-3i+1, \text{for} 1 \le i \le n,\\
f_4(z_iy_{i+1}) &= 9n-3i+2, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ odd number}\\
f_4(z_iy_{i+1}) &= 6n-3i+2, \text{for} 1 \le i \le n \text{ and i } \epsilon \text{ even number}\\
f_4(y_jy_{j+1}) &= 6n-3j+3, \text{for} 1 \le j \le m,
\end{aligned}
$$

The total labeling $f_4$ is a bijective function from $V(\mathbb{F}_n) \cup E(\mathbb{F}_n)$ onto the set $\{1, 2, 3, \ldots, 3n+1\}$. The edge-weights of $\mathbb{F}_n$, under the labeling $f_4$, constitute the sets

$$
\begin{aligned}
W_{f_4} &= \{w_{f_4} + f_4(x_iy_i); \text{ for } 1 \le i \le n\} = 9n+2i+2 \text{ and i } \epsilon \text{ odd number}\\
W_{f_4} &= \{w_{f_4} + f_4(x_iy_i); \text{ for } 1 \le i \le n\} = 6n+2i+3 \text{ and i } \epsilon \text{ even number}\\
W_{f_4} &= \{w_{f_4} + f_4(x_iy_{i+1}); \text{ for } 1 \le i \le n\} = 6n+3i+6 \text{ and i } \epsilon \text{ odd number}\\
W_{f_4} &= \{w_{f_4} + f_4(x_iy_{i+1}); \text{ for } 1 \le i \le n\} = 9n+3i+2 \text{ and i } \epsilon \text{ even number}\\
W_{f_4} &= \{w_{f_4} + f_4(y_iz_i); \text{ for } 1 \le i \le n\} = 6n+3i+1 \text{ and i } \epsilon \text{ odd number}\\
W_{f_4} &= \{w_{f_4} + f_4(y_iz_i); \text{ for } 1 \le i \le n\} = 9n+3i+1 \text{ and i } \epsilon \text{ even number}\\
W_{f_4} &= \{w_{f_4} + f_4(z_iz_{i+1}); \text{ for } 1 \le i \le n\} = 9n+3i+3\\
W_{f_4} &= \{w_{f_4} + f_4(z_iy_{i+1}); \text{ for } 1 \le i \le n\} = 9n+3i+2 \text{ and i } \epsilon \text{ odd number}\\
W_{f_4} &= \{w_{f_4} + f_4(z_iy_{i+1}); \text{ for } 1 \le i \le n\} = 6n+3i+3 \text{ and i } \epsilon \text{ even number}\\
W_{f_4} &= \{w_{f_4} + f_4(y_jy_{j+1}); \text{ jika } 1 \le j \le m\} = 6n+3j+2
\end{aligned}
$$

It is not difficult to see that the set $W_{f_4} = \{6n+4,\ 6n+5,\ \ldots, 12n+2\}$ contains an arithmetic sequence with the first term $8n+6$ and common difference 1. Thus $\alpha_3$ is a super $(6n+4, 1)$-edge-antimagic total labeling. This concludes the proof. □

## Conclusion

We can conclude that the graph $\mathbb{F}_n$ admit a super $(a, d)$-edge-antimagic total labeling for all feasible d and $n \ge 1$ .

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