Optimal switching problems with an infinite set of modes: an approach by randomization and constrained backward SDEs

Marco FUHRMAN * Marie-Amélie MORLAIS †

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Abstract

We address a general optimal switching problem over finite horizon for a stochastic system described by a differential equation driven by Brownian motion. The main novelty is the fact that we allow for infinitely many modes (or regimes, i.e. the possible values of the piecewise-constant control process). We allow all the given coefficients in the model to be path-dependent, that is, their value at any time depends on the past trajectory of the controlled system. The main aim is to introduce a suitable (scalar) backward stochastic differential equation (BSDE), with a constraint on the martingale part, that allows to give a probabilistic representation of the value function of the given problem. This is achieved by randomization of control, i.e. by introducing an auxiliary optimization problem which has the same value as the starting optimal switching problem and for which the desired BSDE representation is obtained. In comparison with the existing literature we do not rely on a system of reflected BSDE nor can we use the associated Hamilton-Jacobi-Bellman equation in our non-Markovian framework.

Keywords: stochastic optimal switching, backward SDEs, randomization of controls.

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*Università degli Studi di Milano, Dipartimento di Matematica, via Saldini 50, 20133 Milano, Italy; e-mail: marco.fuhrman@unimi.it

†Le Mans Université, Laboratoire Manceau de Mathématiques, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France; e-mail: Marie_Amelie.Morlais@univ-lemans.fr

1 Introduction

Stochastic switching control problems arise when a controller acts on a random system by choosing a piecewise constant control process of the form

$$\alpha(t) = \xi_0 \, 1_{[0,\tau_1)}(t) + \sum_{n\geq 1} \xi_n \, 1_{[\tau_n,\tau_{n+1})}(t), \qquad t \geq 0.$$

Here the switching times τ_n are an increasing sequence of stopping times with respect to some given filtration $(\mathcal{F}_t)$ and the chosen actions ξ_n are $\mathcal{F}_{\tau_n}$-measurable random variables with values in some set A, called the set of modes (or regimes). Thus, when the initial mode $\xi_0 = a \in A$ is fixed, choosing a switching strategy amounts to choosing the double sequence $\alpha = (\tau_n, \xi_n)_{n\geq 1}$. This special form of the strategies is justified when the controller incurs in some cost whenever the control action is changed, so that only piecewise constant control processes may have finite cost. Since optimal switching problems are commonly used as models for management issues, they have attracted interest since a long time in the economic literature: the interested reader is referred for instance to [9], [12] or [13].

In the classical framework the set of control actions A is finite, say $A = \{1, \ldots, m\}$. Our main concern is to deal with the case when the set A is arbitrary which is quite natural for many applications. For instance, each mode $a \in A$ may correspond to a working regime of a plant, or a production level of a firm; one may then conceive a situation when the regime or the production level can be chosen freely within an interval of possible values, still retaining the feature that switching from a value to another one entails some cost.

Let us now describe our framework. In this paper we will only consider stochastic differential equations in $\mathbb{R}^n$ driven by the Brownian motion. Suppose initially that the controlled system is described by an equation on the time interval $[0, T]$ of the form

$$dX_t^\alpha = b(X_t^\alpha, \alpha(t))\, dt + \sigma(X_t^\alpha, \alpha(t))\, dW_t, \qquad t \in [s, T] \subset [0, T], \tag{1.1}$$

with a given initial condition $X_s^\alpha = x \in \mathbb{R}^n$, where W is an $\mathbb{R}^d$-valued Brownian motion and the coefficients b, σ satisfy standard Lipschitz and growth conditions. The controller maximizes the reward functional

$$J(s, x, a, \alpha) = \mathbb{E}\Big[\int_s^T f(X_t^\alpha, \alpha(t))\, dt + g(X_T^\alpha, \alpha(T)) - \sum_{n\geq 1} 1_{\tau_n < T}\, c_{\tau_n}(X_{\tau_n}^\alpha, \xi_{n-1}, \xi_n) \Big],$$

where f and g represent the running and terminal rewards and $c_t(x, a, a')$ is the cost incurred when switching at time t from the mode a to the mode a' when the present state is x. The corresponding (so-called) primal value function of the optimal switching problem, with set of modes equal to A, is given at time s by

$$v_s(x, a) = \sup_\alpha J(s, x, a, \alpha). \tag{1.2}$$

Different approaches have been proposed to tackle this problem, that we briefly mention below, while we refer the reader to [17] for a much more detailed discussion.

The classical dynamic programming approach to this problem consists in studying the associated Hamilton-Jacobi-Bellman equation, which in this case is a system of partial differential equations coupled by an obstacle condition: for $a = 1, \dots, m$

$$\begin{cases} \min \left\{ -\partial_s v_s(x,a) - \mathcal{L}^a v_s(x,a) - f(x,a), v_s(x,a) - \max_{a' \neq a} [v_s(x,a') - c_s(x,a,a')] \right\} = 0, \\ v_T(x,a) = g(x,a), \qquad x \in \mathbb{R}^n, \ s \in [0,T], \end{cases} \tag{1.3}$$

where

$$\mathcal{L}^a v_s(x,a) = \frac{1}{2} \operatorname{Trace}\left[\sigma(x,a)\sigma(x,a)^T D_x^2 v_s(x,a)\right] + D_x v_s(x,a) b(x,a)$$

is the Kolmogorov operator corresponding to the controlled coefficients $b(x,a)$, $\sigma(x,a)$.

However, such an approach restricts to the Markovian framework. Among first studies relating the optimal switching problem (with finite number of modes) with a system of quasi-variational inequalities of the form (1.3) one can cite [29], [37] or [42] and, for general theory concerning stochastic control problems, the interested reader is referred to [5]. More recently, [38] and [39] have further investigated these systems in the context of filtrations allowing jumps (in that case, the Kolmogorov operator involves an extra non local term). Recent results on numerical approximation can be found in [23] for optimal multiple switching problems or in [6] for impulse control problems.

Another approach is based on the introduction of a system of Backward Stochastic Differential Equations (BSDEs). Letting the initial time $s = 0$ for simplicity, one solves a system of reflected BSDEs with interconnected obstacles looking for unknown adapted processes $(\bar{Y}_t^{x,a}, \bar{Z}_t^{x,a}, \bar{K}_t^{x,a})_{t \in [0,T]}$, parameterized by $x \in \mathbb{R}^n$ ans $a \in A$ and satisfying suitable conditions, such that

$$\begin{cases} \bar{Y}_t^{x,a} + \int_t^T \bar{Z}_s^{x,a} \, dW_s = g(\bar{X}_T^{x,a}) + \int_t^T f(\bar{X}_s^{x,a}, a) \, ds + \bar{K}_T^{x,a} - \bar{K}_t^{x,a}, \\ \bar{Y}_t^{x,a} \geq \max_{a' \neq a} [\bar{Y}_t^{x,a'} - c_t(\bar{X}_t^{x,a}, a, a')], \\ \int_0^T \left[\bar{Y}_t^{x,a} - \max_{a' \neq a} [\bar{Y}_t^{x,a'} - c_t(\bar{X}_t^{x,a}, a, a')] \right] d\bar{K}_t^{x,a} = 0, \end{cases} \tag{1.4}$$

where, in particular, $\bar{K}^{x,a}$ are non decreasing processes, $\bar{K}_0^{x,a} = 0$, and $\bar{X}^{x,a}$ are defined by the equations

$$d\bar{X}_t^{x,a} = b(\bar{X}_t^{x,a}, a) \, dt + \sigma(\bar{X}_t^{x,a}, a) \, dW_t, \quad t \in [0,T], \qquad \bar{X}_0^{x,a} = x.$$

Under suitable conditions this system is well-posed and one has a probabilistic representation for the value function: $v_0(x,a) = \bar{Y}_0^{x,a}$.

In the two last decades, such a BSDE approach has been extensively used to characterize the primal value function $v_0(x,a)$ (corresponding to (1.2) taken at time $s = 0$). Among the first papers relating the standard optimal switching problem (with m modes) to system of reflected BSDEs of the type (1.4) one may refer to [25], [26] or [28]. Some extensions can be found in [10], [24], [27], [20], this list being non exhaustive. In particular, the authors in [24] and [27] combine the BSDE and PDE approach in the Markovian setting where, under appropriate conditions, one can show that the solutions of (1.4) and (1.3) are related through a standard relation of Feynman-Kac type.

Another approach has been devised, also based on the introduction of a suitable BSDE, but of different type. Suppose that we are given a Poisson random measure (with finite intensity) on $(0,\infty) \times A$, independent of W, and let I^a denote the corresponding piecewise constant A-valued process starting from $a \in A$. Let further $X^{x,a}$ be the solution to

$$dX_t^{x,a} = b(X_t^{x,a}, I_t^a) \, dt + \sigma(X_t^{x,a}, I_t^a) \, dW_t, \quad t \in [0,T], \qquad X_0^{x,a} = x. \tag{1.5}$$

This will be called the randomized equation, since the switching control process has been replaced by a random (Poisson) process. Let us then consider the BSDE

$$\begin{cases} Y_t^{x,a} + \int_t^T Z_s^{x,a}\, dW_s + \int_{(t,T]} \int_A U_s^{x,a}(a')\, \mu(ds\, da') = \ g(X_T^{x,a}, I_T^a) \\ \qquad\qquad + \int_t^T f(X_s^{x,a}, I_s^a)\, ds + K_T^{x,a} - K_t^{x,a}, \\ U_t^{x,a}(a) \leq c_{t-}(X_t^{x,a}, I_{t-}^a, a). \end{cases} \tag{1.6}$$

Here the solution is $(Y_t^{x,a}, Z_t^{x,a}, K_t^{x,a}, U_t^{x,a}(a'))$ ($t \in [0,T]$, $a' \in A$) where the additional martingale term U is a predictable random field needed to solve the equation with respect to the filtration generated by the Brownian motion W and the Poisson random measure μ. This equation is called constrained BSDE, with reference to the inequality required to hold in (1.6). Under suitable assumptions there exists a unique minimal solution (in a sense to be defined) and it is proved that the value function is also represented by the formula $v_0(x,a) = Y_0^{x,a}$. This control randomization method was introduced in [8]. There the author also formulates a corresponding randomized optimal control problem (i.e. an auxiliary or dual problem) and a stochastic target problem related to optimal switching. In the framework of switching problems and associated BSDEs the method was further developed and extended in [16], [17], [18] and later applied to different contexts by many authors, see for instance [34], [35], [2], [1] [21], [11], [19], [20], [22], [4], [3].

We note that the two approaches based on BSDEs have immediate generalization, which already appear in many of the references cited above, to the case of path-dependent coefficients (also called the non-Markovian case), that is when the value at time t of the drift and the diffusion also depend on the past history $(X_s^\alpha)_{s\in[0,t]}$ of the controlled process. Moreover the approach based on the contrained BSDE is more promising from a computational point of view since one has to deal with a single equation instead of a system: as such, numerical methods have been devised to treat this equation, see [32], [33].

Finally, we cite another special approach to optimal switching developed in [15], which works both in Markovian and non-Markovian situations, where BSDEs are replaced by an implicit optimal stopping problem.

As mentioned above, our main concern in this paper is to address the switching problem when the set A is infinite (not necessarily countable). For greater generality we will consider path-dependent coefficients and try to generalize the approaches based on BSDEs. While addressing an infinite system of reflected BSDEs of the form (1.4), or using the approach of [15], seems difficult, it turns out that a generalization of the approach based on the constrained BSDE is possible, and this is in fact the main content of the present paper. Another motivation is the fact that we will still be concerned with a single BSDE even if the number of modes in infinite, so the feasibility of numerical approximation will be preserved, although we will not deal with this issue in this paper.

Following [8] and [17], we introduce an auxiliary optimization problem, called randomized control problem (see section 3 for a precise formulation), having the same value as the original switching problem and we show that this common value can be represented by means of the solution to the constrained BSDE (1.6), even when the set of modes A is infinite. To this aim we have to find entirely new proofs. Indeed, in [8] the result was proved by showing that the switching problem and the randomized problem correspond to the same Hamilton-Jacobi-Bellman equation, since in that paper only the Markovian case was addressed. On the contrary in [17] the non-Markovian situation was studied, but the link between the randomized problem and the switching problem was proved by means of the system of reflected BSDE (1.4), which does not seem easy to solve in the case when A is infinite. In fact, we establish a direct link between the switching problem and the randomized one, and between the latter and the constrained BSDE (1.6). As a consequence our treatment is almost entirely self-contained, except for some technical results related to the randomization technique.

A drawback of the randomization method is that it does not immediately provide a description nor even the existence of an optimal control, but it rather aims at a convenient representation of the value function. However, it also works in situation where an optimal control may not exist, for instance without compactness assumptions on the set of modes A.

The model that we formulate for the switching problem is fairly general: all coefficients, including the switching costs, are path-dependent and may be unbounded. On the diffusion coefficient (the volatility), that can also be controlled, we do not impose any nondegeneracy condition which implies that the case of deterministic optimal switching falls under the scope of our results.

To complete our discussion on the possible approaches to optimal switching with infinitely many modes we finally mention that results based on Hamilton-Jacobi-Bellman equations have been obtained, but limited to the Markovian case when there is no path-dependence in the coefficients. In fact in this case optimal switching can be considered as a special case of optimal impulse problem, where the state of the system is the pair (X_t, I_t). The randomization method has been successfully used in this context as well in [34]. However from a technical point of view these results are not always satisfactory since they impose stringent assumptions, being designed to hold true in a more general (or simply different) context. We believe that building on our approach more refined results can be obtained in the case of Markovian optimal switching with infinitely many modes, and this will be the object of future research. On the contrary, there are not many results on optimal impulse control in the non-Markovian context that apply to general models; one example is [14], which however seems difficult to generalize to the case of an infinite set A.

The plan of the paper is as follows: in section 2 we formulate our assumptions and introduce the optimal switching problem. In section 3 we formulate the auxiliary randomized optimization problem and prove that its value coincides with the value of the optimal switching problem. The proof is rather technical and is presented in section 4. Finally in section 5 we show that a constrained BSDE of the form (1.6) can be associated to the randomized problem thus giving the desired representation of the value for the starting optimal switching problem as well.

2 General assumptions and formulation of the optimal switching problem

2.1 General notations and assumptions

We start this section by an informal description of our optimization problem.

In the following we will consider controlled stochastic equations in $\mathbb{R}^n$ of the form

$$dX_t^\alpha = b_t(X^\alpha, \alpha(t))\, dt + \sigma_t(X^\alpha, \alpha(t))\, dW_t, \tag{2.1}$$

for $t \in [0, T]$, where $T > 0$ is a fixed deterministic and finite terminal time, and an initial condition $X_0^\alpha = x_0$, a given deterministic point in $\mathbb{R}^n$. W is a standard Brownian motion with values in $\mathbb{R}^d$. The control process $\alpha(\cdot)$ is a switching process: it takes values in a set A, called set of modes (or regimes), and it is piecewise constant: it starts at a deterministic mode $\xi_0 \in A$ and at random jumps times τ_n it jumps from ξ_n to ξ_{n+1} ($n \geq 1$). τ_n are stopping times for the filtration $(\mathcal{F}_t^W)$ generated by W and modes ξ_n are also random A-valued variables, each assumed to be $\mathcal{F}_{\tau_n}^W$-measurable.

In our framework we include path-dependent (or hereditary) systems, i.e. exhibiting memory effects with respect to the state. Indeed, the value of the coefficients b, σ at time t depend on the values X_s^α for $s \in [0,t]$: this non-anticipative dependence will be expressed below in a standard way by requiring that the coefficients should be progressive with respect to the canonical filtration on the space of continuous paths.

The reward functional, to be maximized over an appropriate class of switching processes α, has the form $J(\alpha) = J_1(\alpha) - J_2(\alpha)$, where

$$J_1(\alpha) = \mathbb{E}\Big[\int_0^T f_t(X^\alpha, \alpha(t))\,dt + g(X^\alpha, \alpha(T))\Big], \qquad J_2(\alpha) = \mathbb{E}\Big[\sum_{n\geq 1} 1_{\tau_n < T}\, c_{\tau_n}(X^\alpha, \xi_{n-1}, \xi_n)\Big].$$

The functional J_1 has a classical form, and also contains real-valued path dependent coefficients f, g; the functional J_2 takes into account the cost of switching: the (path-dependent) nonnegative function $c_t(x, a, a')$ is interpreted as the cost incurred when switching at time t from the mode a to the mode a' when the trajectory is $x(\cdot)$.

Now let us introduce notations and precise assumptions on the data $A, b, \sigma, f, g, c, x_0, \xi_0$. In the next paragraph we will formulate the optimization problem by describing in particular the class of admissible switching strategies.

Let us denote by $\mathbf{C}_n$ the space of continuous paths from $[0,T]$ to $\mathbb{R}^n$, equipped with the usual supremum norm $\|x\|_\infty = x_T^*$, where we set $x_t^* := \sup_{s\in[0,t]} |x(s)|$, for $t \in [0,T]$ and $x \in \mathbf{C}_n$. We introduce the filtration $(\mathcal{C}_t^n)_{t\in[0,T]}$, where by $\mathcal{C}_t^n$ we denote the σ-algebra generated by the canonical coordinate maps $\mathbf{C}_n \to \mathbb{R}^n$, $x(\cdot) \mapsto x(s)$ up to time t, namely

$$\mathcal{C}_t^n \;:=\; \sigma\{x(\cdot) \mapsto x(s) \;:\; s \in [0,t]\},$$

and we denote $Prog(\mathbf{C}_n)$ and $\mathcal{P}(\mathbf{C}_n)$ the progressive and predictable σ-algebra on $[0,T] \times \mathbf{C}_n$ with respect to $(\mathcal{C}_t^n)$, respectively. [Indeed, one can prove that these σ-algebras essentially coincide: see Remark (8.4) in Chapter V of [40], but we will not need this for the sequel.]

We require the space of control actions A to be a Borel space. We recall that a Borel space is a topological space homeomorphic to a Borel subset of a Polish space The terminology Lusin space, instead of Borel space, is sometimes used. The space A will be endowed with its Borel σ-algebra $\mathcal{B}(A)$.

Throughout the paper, the following assumptions will be in force.

(A1)

(i) A is a Borel space.

(ii) The functions b, σ, f are defined on $[0,T] \times \mathbf{C}_n \times A$ with values in $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$ and $\mathbb{R}$ respectively, they are assumed to be $Prog(\mathbf{C}_n) \otimes \mathcal{B}(A)$-measurable (see also Remark 2.1 below).
The function c is defined on $[0,T] \times \mathbf{C}_n \times A \times A$, it takes nonnegative real values and it is assumed to be $\mathcal{P}(\mathbf{C}_n) \otimes \mathcal{B}(A) \otimes \mathcal{B}(A)$-measurable.
The function g is defined on $\mathbf{C}_n \times A$ and takes real values.

(iii) For every $t \in [0,T]$, the functions $g(x,a)$, $b_t(x,a)$, $\sigma_t(x,a)$ and $f_t(x,a)$ are continuous functions of $(x,a) \in \mathbf{C}_n \times A$ ($\mathbf{C}_n$ being equipped with the supremum norm).
The function $c_t(x,a,a')$ is a continuous functions of $(t,x,a,a') \in [0,T] \times \mathbf{C}_n \times A \times A$.

(iv) There exist nonnegative constants L and r such that

$$|b_t(x,a) - b_t(x',a)| + |\sigma_t(x,a) - \sigma_t(x',a)| \leq L(x-x')^*_t, \tag{2.2}$$

$$|b_t(0,a)| + |\sigma_t(0,a)| \leq L, \tag{2.3}$$

$$|f_t(x,a)| + |g(x,a)| + |c_t(x,a,a')| \leq L\big(1 + (x^*_t)^r\big), \tag{2.4}$$

for all $(t,x,x',a,a') \in [0,T] \times \mathbf{C}_n \times \mathbf{C}_n \times A \times A$.

(v) $x_0 \in \mathbb{R}^n$ and $\xi_0 \in A$ are given: they represent the initial state and mode, respectively.

Remark 2.1 The measurability conditions in **(A1)**-(ii) entail the following property, which is easily verified:

(ii)' Whenever $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a filtration $\mathbb{F}$, X is an $\mathbb{F}$-progressive process with values in $\mathbb{R}^n$, and $a, a' \in A$, then the processes $b_t(X,a)$, $\sigma_t(X,a)$, $f_t(X,a)$, $c_t(X,a,a')$, defined for $t \in [0,T]$, are also $\mathbb{F}$-progressive.

All the results in this paper still hold, without any change in the proofs, if property (ii)' is assumed to hold instead of (ii). In some cases (ii)' is easier to be checked directly.

We finally note that the function g, being continuous, is also Borel measurable (equivalently, it is $\mathcal{C}^n_T \otimes \mathcal{B}(A)$-measurable). □

Remark 2.2 We mention that no non-degeneracy assumption on the diffusion coefficient σ is imposed. In particular the case of deterministic switching, where $\sigma = 0$, is included, and in this special case there is of course no need to introduce a Wiener process nor a probability space. □

2.2 Formulation of the optimal switching problem

We assume that $A, b, \sigma, f, g, c, x_0, \xi_0$ are given and satisfy the assumptions **(A1)**. A setting (Ω, $\mathcal{F}$, $\mathbb{P}$, W) for the optimization problem consists of a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathbb{R}^d$-valued process W which is a standard Wiener process with respect to $\mathbb{P}$.

Let us denote $\mathbb{F}^W = (\mathcal{F}^W_t)_{t\geq 0}$ the right-continuous and $\mathbb{P}$-complete filtration generated by W. We define the set $\mathcal{A}$ of admissible control strategies: its elements are the double sequences of the form

$$\alpha = (\tau_n, \xi_n)_{n\geq 1},$$

where:

(i) each τ_n is an $\mathbb{F}^W$-stopping time;

(ii) each τ_n takes values in $(0,\infty]$ and the sequence $(\tau_n)_{n\geq 1}$ is nondecreasing, a.s.;

(iii) if $\tau_n < \infty$ then $\tau_n < \tau_{n+1}$, for every $n \geq 1$, a.s.;

(iv) each ξ_n is a random variable with values in A, which is $\mathcal{F}^W_{\tau_n}$-measurable;

(v) $\tau_n \to \infty$ a.s. and $\tau_n \neq T$ a.s. for every $n \geq 1$.

Remark 2.3 Conditions $(i) - (iv)$ can be restated by saying that α is a marked (or multivariate) point process in A. It is convenient in the following to use this definition although the control horizon T is finite. The condition $\tau_n \to \infty$ can be expressed by saying that the explosion time $\lim_n \tau_n$ is infinite a.s. We comment further on this condition and on the condition $\tau_n \neq T$ in Remark 2.4. □

Given $\alpha \in \mathcal{A}$, we introduce the associated piecewise constant process, denoted by $\alpha(\cdot)$ (with a slight abuse of notation) and defined as

$$\alpha(t) = \xi_0 \, 1_{[0,\tau_1)}(t) + \sum_{n\geq 1} \xi_n \, 1_{[\tau_n,\tau_{n+1})}(t), \qquad t \in [0,T],$$

where ξ_0 is the given starting mode. Notice that the formal sum makes obvious sense even if there is no addition operation defined in A.

The corresponding trajectory X^α is defined as the solution to the controlled equation

$$dX_t^\alpha \;=\; b_t(X^\alpha, \alpha(t))\, dt + \sigma_t(X^\alpha, \alpha(t))\, dW_t \tag{2.5}$$

on the interval $[0,T]$ with initial condition $X_0^\alpha = x_0$. Since we assume that **(A1)** holds, by standard results (see e.g. [40] Thm V. 11.2, or [31] Theorem 14.23), there exists an almost surely unique $\mathbb{F}$-adapted strong solution $X^\alpha = (X_t^\alpha)_{t\in[0,T]}$ to (2.5) with continuous trajectories a.s. and such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^\alpha|^p\Big] \leq C_p < \infty, \tag{2.6}$$

for every $p \in [1,\infty)$, where the constant C_p, depends only on p, T, n, d and the constants L, r appearing in Assumption **(A1)**. The stochastic optimal control problem under partial observation consists in maximizing, over all $\alpha \in \mathcal{A}$, the reward functional

$$J(\alpha) = J_1(\alpha) - J_2(\alpha), \tag{2.7}$$

where

$$J_1(\alpha) \;=\; \mathbb{E}\Big[\int_0^T f_t(X^\alpha, \alpha(t))\, dt + g(X^\alpha, \alpha(T))\Big], \tag{2.8}$$

$$J_2(\alpha) \;=\; \mathbb{E}\Big[\sum_{n\geq 1} 1_{\tau_n<T}\, c_{\tau_n}(X^\alpha, \xi_{n-1}, \xi_n)\Big]. \tag{2.9}$$

We define the value of the optimal switching problem as

$$\upsilon_0 = \sup_{\alpha\in\mathcal{A}} J(\alpha). \tag{2.10}$$

Since we do not impose growth conditions on the cost function c, it is possible that $J_2(\alpha) = \infty$ for some admissible $\alpha \in \mathcal{A}$. However, we have the following simple result.

Lemma 2.1 *There exists a finite constant C, depending only on T, n, d and the constants L, r appearing in assumptions* **(A1)***, such that $|\upsilon_0| \leq C$.*

Proof. By standard estimates on the state equation (the same ones leading to (2.6)) and the growth conditions imposed in **(A1)**-(iv) it is easily shown that $|J_1(\alpha)| \leq C$ for every $\alpha \in \mathcal{A}$. Since c is nonnegative we have $J(\alpha) \leq J_1(\alpha) \leq C$ for $\alpha \in \mathcal{A}$ and it follows that $\mathcal{V}_0 \leq C$.

Now let us consider the strategy $\bar{\alpha}$ without switchings (i.e. such that $\tau_n = \infty$ for $n \geq 1$). Then we have $J_2(\bar{\alpha}) = 0$ and so

$$\mathcal{V}_0 \geq J(\bar{\alpha}) = J_1(\bar{\alpha}) \geq -C,$$

and we conclude that $|\mathcal{V}_0| \leq C$. $\square$

We end this section with several comments on the previous formulation of the optimization problem and its possible variants.

Remark 2.4 1. According to large part of the literature on optimal switching, we do not allow for a switching at initial time $t = 0$. This is not a real loss of generality, since a switching at time 0 does not affect the controlled trajectory X^α and it is easy to reduce the problem to the formulation that we adopt.

2. In our definition of admissible strategy we have imposed the condition of being non-explosive. This implies that

$$N_T := \sum_{n \geq 1} 1_{\tau_n \leq T}$$

is finite a.s., meaning that infinitely many switchings in the time interval within the control horizon T are not allowed. Alternatively, one may impose that there exists $\delta > 0$ such that $c_t(x, a, a') \geq \delta$ for every $t \in [0, T]$, $x \in \mathbf{C}_n$, $a, a' \in A$, which is a common requirement in the literature on switching problems. Under this additional assumption, any strategy α with $N_T = \infty$ has $J(\alpha) = -\infty$ and cannot be optimal. We will not need that $c_t(x, a, a') \geq \delta$ and will only assume the weaker conditions that $c_t(x, a, a') \geq 0$ and $\tau_n \to \infty$ for every admissible strategy.

3. Often, the following assumption is imposed on the cost function: for every distinct $a_1, a_2, a_3 \in A$ and for every $t \in [0, T]$, $x \in \mathbf{C}_n$,

$$c_t(x, a_1, a_3) < c_t(x, a_1, a_2) + c_t(x, a_2, a_3). \tag{2.11}$$

This says that switching from mode a_1 to mode a_3 directly is more convenient than a double switching from mode a_1 to a_2 followed immediately by a switching from a_2 to a_3. This condition entails that any strategy for which a switching time τ_n equals τ_{n+1} cannot be optimal. We will not need the condition (2.11), but we have imposed that $\tau_n < \tau_{n+1}$ (whenever τ_n is finite).

4. A variant of the optimal switching problem is obtained by allowing for a switching at the terminal time, that is by removing the requirement that $\tau_n \neq T$ and modifying the functional J_2, introduced in (2.9), in the following way:

$$J_2(\alpha) = \mathbb{E}\Big[\sum_{n \geq 1} 1_{\tau_n \leq T}\, c_{\tau_n}(X^\alpha, \xi_{n-1}, \xi_n)\Big], \tag{2.12}$$

in order to take into account the cost of a switching at the final time. In some papers, the following condition is imposed on the data: for every $a \in A$ and $x \in \mathbf{C}_n$,

$$g(x, a) > \sup_{a' \in A, a' \neq a} (g(x, a') - c_T(x, a, a')). \tag{2.13}$$

This says that at the final time it is more convenient to remain in the current mode a rather than switching to any another mode a', which would give a reward $g(x,a')$ but would incur in a cost $c_T(x,a,a')$. If (2.13) is required, the optimization problem has the same value (and the same optimal control, if it exists) whether J_2 is defined by (2.9) or by (2.12).

In this paper we will not impose condition (2.13) but we require that $\tau_n \neq T$ a.s.

□

3 The randomized stochastic optimal control problem

We still assume that $A, b, \sigma, f, g, c, x_0, \xi_0$ are given and satisfy the assumptions **(A1)**. We introduce an auxiliary optimization problem, that we call randomized optimal control problem, and we will eventually prove that it has the same value as the optimal switching problem formulated in section 2.2. However, the randomized problem has the advantage that it can be directly related to a suitable BSDE, as we will see in the following sections.

To this end we need one additional datum, that will play the role of an intensity measure for a Poisson process:

(A2) Let λ be a finite positive measure on $(A, \mathcal{B}(A))$ with full topological support.

Since A is separable (as a Borel space), such a measure always exists: for instance, one could choose a convex combination of Dirac measures at points $a_i \in A$, where (a_i) is a dense sequence in A. In general there are many possible choices for the measure λ and in any case **(A2)** is not a restriction imposed on the original optimization problem. It will be assumed to hold from now on.

3.1 Formulation of the randomized control problem

We say that $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{W}, \hat{\mu})$ is a setting for the randomized control problem if $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ is an arbitrary complete probability space, the process $\hat{W}$ is a standard Wiener process in $\mathbb{R}^d$ under $\hat{\mathbb{P}}$, $\hat{\mu}$ is a Poisson random measure on A with intensity $\lambda(da)$ under $\hat{\mathbb{P}}$, independent of W. Thus, $\hat{\mu}$ is a sum of random Dirac measures and it has the form $\hat{\mu} = \sum_{n\geq 1} \delta_{(\hat{\sigma}_n, \hat{\eta}_n)}$, where $(\hat{\eta}_n)_{n\geq 1}$ is a sequence of A-valued random variables and $(\hat{\sigma}_n)_{n\geq 1}$ is a strictly increasing sequence of random variables with values in $(0,\infty)$, and for any $C \in \mathcal{B}(A)$ the process $\hat{\mu}((0,t]\times C) - t\lambda(C)$, $t \geq 0$, is a $\hat{\mathbb{P}}$-martingale. We also define the piecewise-constant A-valued process associated to μ and starting at the initial mode ξ_0:

$$\hat{I}_t = \xi_0 \, 1_{[0,\hat{\sigma}_1)}(t) + \sum_{n\geq 1} \hat{\eta}_n \, 1_{[\hat{\sigma}_n, \hat{\sigma}_{n+1})}(t), \qquad t \geq 0. \tag{3.1}$$

The formal sum in (3.1) makes sense even if there is no addition operation defined in A, but when A is a subset of a linear space formula (3.1) can be written as

$$\hat{I}_t = \xi_0 + \int_0^t \int_A (a - \hat{I}_{s-}) \, \hat{\mu}(ds\, da), \qquad t \geq 0.$$

Let $\hat{X}$ be the solution to the equation

$$d\hat{X}_t = b_t(\hat{X}, \hat{I}_t)\, dt + \sigma_t(\hat{X}, \hat{I}_t)\, dW_t, \tag{3.2}$$

for $t \in [0,T]$, starting from $\hat{X}_0 = x_0$, the initial state fixed at the beginning.

We introduce the filtration $\mathbb{F}^{\hat{W},\hat{\mu}} = (\mathcal{F}_t^{\hat{W},\hat{\mu}})_{t\geq 0}$ generated by $\hat{W}, \hat{\mu}$ and defined by the formula:

$$\mathcal{F}_t^{\hat{W},\hat{\mu}} \quad = \quad \sigma(\hat{W}_s, \hat{\mu}((0,s] \times C) \,:\, s \in [0,t],\, C \in \mathcal{B}(A)) \vee \mathcal{N}, \tag{3.3}$$

where $\mathcal{N}$ denotes the family of $\hat{\mathbb{P}}$-null sets of $\hat{\mathcal{F}}$. We denote $\mathcal{P}(\mathbb{F}^{\hat{W},\hat{\mu}})$ the corresponding predictable σ-algebra.

Under **(A1)** it is well-known (see e.g. Theorem 14.23 in [31]) that there exists an almost surely unique $\mathbb{F}^{\hat{W},\hat{\mu}}$-adapted strong solution $\hat{X} = (\hat{X}_t)_{t\in[0,T]}$ to (3.2), satisfying $\hat{X}_0 = x_0$, with continuous trajectories a.s. and such that for every $p \in [1,\infty)$,

$$\hat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |\hat{X}_t|^p\Big] \quad \leq \quad C_p, \tag{3.4}$$

where C_p is a finite constant whose value depends only on p, T, n, d and the constants L, r occurring in **(A1)**-(iv).

We can now formulate the randomized optimal control problem as follows. We introduce the set $\hat{\mathcal{V}}$ of admissible controls as the set of all $\hat{\nu} = \hat{\nu}_t(\hat{\omega}, a) : \hat{\Omega} \times \mathbb{R}_+ \times A \to (0,\infty)$, which are $\mathcal{P}(\mathbb{F}^{\hat{W},\hat{\mu}}) \otimes \mathcal{B}(A)$-measurable and bounded. For any $\hat{\nu}$ in $\hat{\mathcal{V}}$, we associate its Doléans exponential process $\kappa_t^{\hat{\nu}}$ defined as follows

$$\begin{aligned} \kappa_t^{\hat{\nu}} \quad &= \quad \mathcal{E}_t\bigg(\int_0^\cdot \int_A (\hat{\nu}_s(a) - 1)\,(\hat{\mu}(ds\,da) - \lambda(da)\,ds)\bigg) \\ &= \quad \exp\bigg(\int_0^t \int_A (1 - \hat{\nu}_s(a))\lambda(da)\,ds\bigg) \prod_{0<\hat{\sigma}_n\leq t} \nu_{\hat{\sigma}_n}(\hat{\eta}_n), \qquad t \geq 0. \end{aligned} \tag{3.5}$$

It is known that $\kappa^{\hat{\nu}}$ is a martingale with respect to $\hat{\mathbb{P}}$ and $\mathbb{F}^{\hat{W},\hat{\mu}}$ and thus we define a new probability measure by setting $\hat{\mathbb{P}}^{\hat{\nu}}(d\hat{\omega}) = \kappa_T^{\hat{\nu}}(\hat{\omega})\,\hat{\mathbb{P}}(d\hat{\omega})$. From the Girsanov theorem for multivariate point processes ([30]) it follows that under $\hat{\mathbb{P}}^{\hat{\nu}}$ the $\mathbb{F}^{\hat{W},\hat{\mu}}$-compensator of $\hat{\mu}$ on the set $[0,T] \times A$ is the random measure $\hat{\nu}_t(a)\lambda(da)dt$. Moreover, $\hat{W}$ remains a standard Wiener process under $\hat{\mathbb{P}}^{\hat{\nu}}$, so that using both Assumptions (2.2)-(2.3) and standard results we obtain the following generalization of the estimate (3.4):

$$\sup_{\hat{\nu}\in\mathcal{V}} \hat{\mathbb{E}}^{\hat{\nu}}\Big[\sup_{t\in[0,T]} |\hat{X}_t|^p\Big] \quad \leq \quad C_p, \tag{3.6}$$

where $\hat{\mathbb{E}}^{\hat{\nu}}$ denotes the expectation with respect to $\hat{\mathbb{P}}^{\hat{\nu}}$ and C_p is the same as in (3.4). We finally introduce the reward functional of the randomized control problem

$$J^{\mathcal{R}}(\hat{\nu}) \quad = \quad J_1^{\mathcal{R}}(\hat{\nu}) - J_2^{\mathcal{R}}(\hat{\nu}), \tag{3.7}$$

where

$$J_1^{\mathcal{R}}(\hat{\nu}) \quad = \quad \hat{\mathbb{E}}^{\hat{\nu}}\Big[\int_0^T f_t(\hat{X}, \hat{I}_t)\,dt + g(\hat{X}, \hat{I}_T)\Big], \tag{3.8}$$

$$J_2^{\mathcal{R}}(\hat{\nu}) \quad = \quad \hat{\mathbb{E}}^{\hat{\nu}}\Big[\sum_{n\geq 1} 1_{\hat{\sigma}_n<T}\, c_{\hat{\sigma}_n}(\hat{X}, \hat{\eta}_{n-1}, \hat{\eta}_n)\Big], \tag{3.9}$$

where we use the convention $\eta_0 = \xi_0$. We note that

$$J^{\mathcal{R}}(\hat{\nu}) = \hat{\mathbb{E}}\left[\kappa_T^{\hat{\nu}}\left(\int_0^T f_t(\hat{X}, \hat{I}_t)\, dt + g(\hat{X}, \hat{I}_T) - \sum_{n\geq 1} 1_{\hat{\sigma}_n < T}\, c_{\hat{\sigma}_n}(\hat{X}, \hat{\eta}_{n-1}, \hat{\eta}_n)\right)\right]$$

is always finite: indeed, letting $\hat{N}_T = \sum_{n\geq 1} 1_{\hat{\sigma}_n \leq T}$ and recalling the growth conditions in (2.4) we see that

$$0 \leq \kappa_T^{\hat{\nu}} \leq \exp(T\lambda(A)\,(1 + \sup \nu)) \cdot (\sup \nu)^{\hat{N}_T}, \tag{3.10}$$

$$\left|\int_0^T f_t(\hat{X}, \hat{I}_t)\, dt + g(\hat{X}, \hat{I}_T)\right| + \sum_{n\geq 1} 1_{\hat{\sigma}_n < T}\, c_{\hat{\sigma}_n}(\hat{X}, \hat{\eta}_{n-1}, \hat{\eta}_n) \leq C(1 + \sup_{t\in[0,T]} |\hat{X}_t|)^r\,(1 + \hat{N}_T), \tag{3.11}$$

for a suitable constant C; noting that $\hat{N}_T$ has Poisson law with parameter $\lambda(A)\,T$ under $\hat{\mathbb{P}}$ and recalling (3.4), we see that the right-hand sides in the above expressions lie in $L^p(\hat{\mathbb{P}})$ for every $p \in [1, \infty)$ and the finiteness of $J^{\mathcal{R}}(\hat{\nu})$ follows.

The randomized stochastic optimal control problem consists in maximizing $J^{\mathcal{R}}(\hat{\nu})$ over all $\hat{\nu} \in \hat{\mathcal{V}}$. Its value is defined as

$$\mathcal{V}_0^{\mathcal{R}} \;=\; \sup_{\hat{\nu}\in\hat{\mathcal{V}}} J^{\mathcal{R}}(\hat{\nu}). \tag{3.12}$$

Remark 3.1 A comparison between the starting optimal switching problem and the randomized problem may be useful. In the switching problem, the switching process $\alpha(\cdot)$ is chosen to control the system. In the randomized problem $\alpha(\cdot)$ is first replaced by the Poisson point process $\hat{I}(\cdot)$ (associated with random measure $\hat{\mu}$) in the coefficients of the equation solved by $\hat{X}$. In this new problem, the effect of a control strategy $\hat{\nu}$ is to modify the intensity of $\hat{I}$ (more precisely, to change its compensator from $\lambda(da)dt$ to $\hat{\nu}_t(a)\lambda(da)dt$) and thus also to affect the law of the process $\hat{X}$. This is done by introducing the probabilities $\hat{\mathbb{P}}^{\hat{\nu}}$ via the Girsanov theorem, and optimizing the reward functional $\mathcal{J}^{\mathcal{R}}(\hat{\nu})$ among this family of equivalent probability measures parameterized by the set of all bounded predictable random fields $\hat{\nu}$. □

Remark 3.2 Let us define $\hat{\mathcal{V}}_{\inf > 0} = \{\hat{\nu} \in \hat{\mathcal{V}} \,:\, \inf_{\hat{\Omega}\times[0,T]\times A} \hat{\nu} > 0\}$. Then we claim that

$$\mathcal{V}_0^{\mathcal{R}} \;=\; \sup_{\hat{\nu}\in\hat{\mathcal{V}}_{\inf > 0}} J^{\mathcal{R}}(\hat{\nu}). \tag{3.13}$$

Indeed, given $\hat{\nu} \in \hat{\mathcal{V}}$ and $\epsilon > 0$, define $\hat{\nu}^\epsilon = \hat{\nu} \vee \epsilon \in \hat{\mathcal{V}}_{\inf > 0}$ and write the gain (3.7) in the form

$$J^{\mathcal{R}}(\hat{\nu}^\epsilon) = \hat{\mathbb{E}}\left[\kappa_T^{\hat{\nu}^\epsilon}\left(\int_0^T f_t(\hat{X}, \hat{I}_t)\, dt + g(\hat{X}, \hat{I}_T) - \sum_{n\geq 1} 1_{\hat{\sigma}_n < T}\, c_{\hat{\sigma}_n}(\hat{X}, \hat{\eta}_{n-1}, \hat{\eta}_n)\right)\right].$$

As noted earlier, the expression in curve brackets lies in $L^p(\hat{\mathbb{P}})$ for every $p \in [1, \infty)$. Moreover we have $\kappa_T^{\hat{\nu}^\epsilon} \to \kappa_T^{\hat{\nu}}$ a.s. as $\epsilon \to 0$, and using the estimate (3.10) with ν^ϵ instead of ν we conclude that $\kappa_T^{\hat{\nu}^\epsilon} \to \kappa_T^{\hat{\nu}}$ in $L^p(\hat{\mathbb{P}})$ for every $p \in [1, \infty)$ as well. It follows that $J^{\mathcal{R}}(\hat{\nu}^\epsilon) \to J^{\mathcal{R}}(\hat{\nu})$, which implies

$$\mathcal{V}_0^{\mathcal{R}} = \sup_{\hat{\nu}\in\hat{\mathcal{V}}} J^{\mathcal{R}}(\hat{\nu}) \leq \sup_{\hat{\nu}\in\hat{\mathcal{V}}_{\inf > 0}} J^{\mathcal{R}}(\hat{\nu}).$$

The other inequality being obvious, we obtain (3.13). □

Remark 3.3 We stress the fact that the value $\upsilon_0^{\mathcal{R}}$ of the randomized control problem defined in (3.12) does not depend on the specific setting $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{W}, \hat{\mu})$ that is chosen in its formulation.
More precisely, this means that if $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mu})$ is another setting with the properties described at the beginning of this section, and if the corresponding value $\tilde{\upsilon}_0^{\mathcal{R}}$ is defined in analogy with what was done before then we have the equality $\upsilon_0^{\mathcal{R}} = \tilde{\upsilon}_0^{\mathcal{R}}$.
We do not write down the proof of this statement, since it is entirely analogous to Proposition 3.1 of [2], where a classical optimization problem with continuous control was addressed instead of a switching problem, but the arguments remain the same.
As a consequence, we obtain the rather intuitive conclusion that the value $\upsilon_0^{\mathcal{R}}$ is just a functional of the (deterministic) elements $A, b, \sigma, f, g, c, x_0, \xi_0, \lambda$ appearing in the assumptions **(A1)** and **(A2)**.
Later on, in Theorem 3.1, we will prove that in fact $\upsilon_0^{\mathcal{R}}$ does not depend on the choice of λ either.
□

Remark 3.4 Starting from a setting $(\Omega, \mathcal{F}, \mathbb{P}, W)$ for the optimal switching problem one can always obtain a setting for a randomized optimal control problem by the following direct construction.
Take an arbitrary probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ where a Poisson random measure μ with intensity λ is defined. Thus in particular, for every $\omega' \in \Omega'$, $\mu(\omega', dt\, da)$ is a measure on $(0, \infty) \times A$. Let us define $\hat{\Omega} = \Omega \times \Omega'$, let us denote by $\hat{\mathcal{F}}$ the completion of the product σ-algebra $\mathcal{F} \otimes \mathcal{F}'$ with respect to $\mathbb{P} \otimes \mathbb{P}'$ and by $\hat{\mathbb{P}}$ the extension of $\mathbb{P} \otimes \mathbb{P}'$ to $\hat{\mathcal{F}}$. One can introduce canonical extensions $\hat{W}$ and $\hat{\mu}$ of W and μ to $\hat{\Omega}$ by setting

$$\hat{W}_t(\omega, \omega') = W_t(\omega), \qquad \hat{\mu}(\omega, \omega', dt\, da) = \mu(\omega', dt\, da),$$

for every $t \geq 0$, $\omega \in \Omega$, $\omega' \in \Omega'$. Then it can be easily checked that, under $\hat{\mathbb{P}}$, $\hat{W}$ is a standard Wiener process and $\hat{\mu}$ is a random Poisson measure on $(0, \infty) \times A$ with the same intensity λ, independent of $\hat{W}$. So we see that $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{W}, \hat{\mu})$ is a setting for a randomized control problem as formulated before, that we call *product extension* of the setting $(\Omega, \mathcal{F}, \mathbb{P}, W)$ for the initial optimal switching problem. This construction will be used again for the proofs of several results below and will be further studied.
We note that by a classical result, see for instance [43] Theorem 2.3.1, we may take $\Omega' = [0, 1]$, $\mathcal{F}'$ the corresponding Borel sets and $\mathbb{P}'$ the Lebesgue measure. This shows that the extended setting is rather "economical" in the loose sense that it does not introduce much randomness with respect to the original setting.
We also note that the initial formulation of a randomized setting was more general, since it was not required that $\hat{\Omega}$ should be a product space $\Omega \times \Omega'$ and, even if it were the case, it was not required that the process $\hat{W}$ should depend only on $\omega \in \Omega$ while the random measure $\hat{\mu}$ should depend only on $\omega' \in \Omega'$.
□

3.2 Equivalence of the optimal switching and the randomized control problems

We can now state one of the main results of the paper.

Theorem 3.1 *Assume that* **(A1)** *and* **(A2)** *are satisfied. Then the values of the optimal switching problem and of the randomized control problem are equal:*

$$\upsilon_0 \;=\; \upsilon_0^{\mathcal{R}}, \tag{3.14}$$

where υ_0 and $\upsilon_0^{\mathcal{R}}$ are defined by (2.10) *and* (3.12) *respectively. This common value only depends on the objects $A, b, \sigma, f, g, c, x_0, \xi_0$ appearing in assumption* **(A1)**.

The last sentence follows immediately from Remark 3.3, from the equality $\upsilon_0 = \upsilon_0^{\mathcal{R}}$ and from the obvious fact that υ_0 cannot depend on the measure λ introduced in assumption **(A2)**. The following section is entirely devoted to the proof of the equality.

4 Proof of Theorem 3.1

4.1 Preliminaries

In this section, **(A1)** and **(A2)** are always assumed to hold. We will prove separately the two inequalities $\upsilon_0^{\mathcal{R}} \leq \upsilon_0$ and $\upsilon_0 \leq \upsilon_0^{\mathcal{R}}$. In both cases, we need similar constructions, which consist in starting with a given setting $(\Omega, \mathcal{F}, \mathbb{P}, W)$ for the optimal switching problem formulated in section 2.2, building a product space by adding another suitable probability space as an independent factor and thus arriving at a suitable setting $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{W}, \hat{\mu})$ for a randomized control problem as formulated before. In this paragraph we present this construction and its main properties needed later.

Let us start with a setting $(\Omega, \mathcal{F}, \mathbb{P}, W)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and W a d-dimensional standard Wiener process and let $(\Omega', \mathcal{F}', \mathbb{P}')$ be another arbitrary probability space. We finally set $\hat{\Omega} = \Omega \times \Omega'$ and denote by $\hat{\mathcal{F}}$ the completion of the product σ-algebra $\mathcal{F} \otimes \mathcal{F}'$ with respect to $\mathbb{P} \otimes \mathbb{P}'$ and by $\hat{\mathbb{P}}$ the extension of $\mathbb{P} \otimes \mathbb{P}'$ to $\hat{\mathcal{F}}$. One can introduce a canonical extension of W to $\hat{\Omega}$ setting $\hat{W}_t(\omega, \omega') = W_t(\omega)$ for every $t \geq 0$, $\omega \in \Omega$, $\omega' \in \Omega'$. Then $\hat{W}$ is a standard Wiener process under $\hat{\mathbb{P}}$, as it can be easily checked. More generally, any random element defined in Ω or Ω' has an extension defined by similar formulae, whose law under $\hat{\mathbb{P}}$ is the same as the law under the original probability.

One can formulate an optimal switching problem in the new setting $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{W})$ in the same way as before: we let $\mathbb{F}^{\hat{W}} = (\mathcal{F}_t^{\hat{W}})_{t\geq 0}$ denote the right-continuous and $\hat{\mathbb{P}}$-complete filtration generated by $\hat{W}$, and we define the set of admissible strategies $\hat{\mathcal{A}}$ as the elements of the form $\hat{\alpha} = (\hat{\tau}_n, \hat{\xi}_n)_{n\geq 1}$ satisfying properties analogous to $(i) - (v)$ in section 2.2, but with the filtration $\mathbb{F}^{\hat{W}}$ instead of $\mathbb{F}^W$. For any $\hat{\alpha} \in \hat{\mathcal{A}}$ one finds the corresponding trajectory $\hat{X}^{\hat{\alpha}}$ solving the controlled equation

$$d\hat{X}_t^{\hat{\alpha}} \;=\; b_t(\hat{X}^{\hat{\alpha}}, \hat{\alpha}(t))\, dt + \sigma_t(\hat{X}^{\hat{\alpha}}, \hat{\alpha}(t))\, d\hat{W}_t, \qquad \hat{X}_0^{\hat{\alpha}} = x_0, \tag{4.1}$$

where $\hat{\alpha}(\cdot)$ is the piecewise constant process associated to $\hat{\alpha}$, and computes the corresponding reward:

$$\hat{J}(\hat{\alpha}) := \hat{\mathbb{E}}\Big[\int_0^T f_t(\hat{X}^{\hat{\alpha}}, \hat{\alpha}(t))\, dt + g(\hat{X}^{\hat{\alpha}}, \hat{\alpha}(T))\Big] - \hat{\mathbb{E}}\Big[\sum_{n\geq 1} 1_{\hat{\tau}_n < T}\, c_{\hat{\tau}_n}(\hat{X}^{\hat{\alpha}}, \hat{\xi}_{n-1}, \hat{\xi}_n)\Big]. \tag{4.2}$$

Finally, the value is defined as

$$\hat{\upsilon}_0 := \sup_{\hat{\alpha} \in \hat{\mathcal{A}}} \hat{J}(\hat{\alpha}). \tag{4.3}$$

One may wish to compare this value with the value of the switching problem formulated in the original setting $(\Omega, \mathcal{F}, \mathbb{P}, W)$. To this end, let us recall that $\mathbb{F}^W = (\mathcal{F}_t^W)_{t\geq 0}$ denotes the right-continuous and $\mathbb{P}$-complete filtration in Ω generated by W. Every σ-algebra $\mathcal{F}_t^W$ gives rise to a

σ-algebra in $\hat{\Omega}$ defined as

$$\mathcal{F}_t^W \times \Omega' := \{A \times \Omega' \ : \ A \in \mathcal{F}_t^W\}.$$

This way one obtains a new filtration in $\hat{\Omega}$ (which is right-continuous but not $\hat{\mathbb{P}}$-complete in general). Recalling that $\mathbb{F}^{\hat{W}} = (\mathcal{F}_t^{\hat{W}})_{t\geq 0}$ denotes the right-continuous and $\hat{\mathbb{P}}$-complete filtration generated by $\hat{W}$, and letting $\mathcal{N}$ denote the family of $\hat{\mathbb{P}}$-null sets in $\hat{\mathcal{F}}$, one arrives at the equality

$$\mathcal{F}_t^{\hat{W}} = (\mathcal{F}_t^W \times \Omega') \vee \mathcal{N}, \qquad t \geq 0, \tag{4.4}$$

which can be verified by lengthy but standard arguments.

If τ is an $\mathbb{F}^W$-stopping time then its canonical extension defined by $\hat{\tau}(\omega,\omega') = \tau(\omega)$ is a $\mathbb{F}^{\hat{W}}$-stopping time; indeed, for every $t \geq 0$, $\{\hat{\tau} \leq t\} = \{\tau \leq t\} \times \Omega'$ belongs to $\mathcal{F}_t^W \times \Omega'$ and so to $\mathcal{F}_t^{\hat{W}}$. Now suppose that $A \in \mathcal{F}_\tau^W$; then for every $t \geq 0$,

$$(A \times \Omega') \cap \{\hat{\tau} \leq t\} = (A \cap \{\tau \leq t\}) \times \Omega' \in \mathcal{F}_t^W \times \Omega' \subset \mathcal{F}_t^{\hat{W}}.$$

This shows that if $A \in \mathcal{F}_\tau^W$ then $A \times \Omega' \in \mathcal{F}_{\hat{\tau}}^{\hat{W}}$. This property implies that for any $\mathcal{F}_\tau^W$-measurable random variable ξ, its canonical extension $\hat{\xi}(\omega,\omega') = \xi(\omega)$ is $\mathcal{F}_{\hat{\tau}}^{\hat{W}}$-measurable.

It follows that if we start from an admissible control strategy $\alpha \in \mathcal{A}$ of the form $\alpha = (\tau_n, \xi_n)_{n\geq 1}$ and denote $\hat{\tau}_n, \hat{\xi}_n$ the canonical extensions of τ_n, ξ_n respectively, then $\hat{\alpha} := (\hat{\tau}_n, \hat{\xi}_n)_{n\geq 1}$ is an admissible strategy for the optimal switching problem formulated in the setting $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \hat{W})$, hence an element of $\hat{\mathcal{A}}$. Moreover, it is easy to realize that in this case the process $\hat{X}^{\hat{\alpha}}$ solution to (4.1) is the same as the canonical extension of the process X^α defined as the solution to the controlled equation (2.5) ($\hat{X}_t^{\hat{\alpha}}(\omega,\omega') = X_t^\alpha(\omega)$) and, moreover, the reward (4.2) is the same as the original one: $\hat{J}(\hat{\alpha}) = J(\alpha)$. We deduce that the two values satisfy the inequality

$$\upsilon_0 = \sup_{\alpha\in\mathcal{A}} J(\alpha) \leq \sup_{\hat{\alpha}\in\hat{\mathcal{A}}} \hat{J}(\hat{\alpha}) = \hat{\upsilon}_0.$$

Following [2], we next introduce a variant of the optimal switching problem formulated in the new setting $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{W})$. We define a new filtration, denoted $\mathbb{F}^{\hat{W},\infty} = (\mathcal{F}_t^{\hat{W},\infty})_{t\geq 0}$, as follows: we first introduce

$$\Omega \times \mathcal{F}' := \{\Omega \times B \ : \ B \in \mathcal{F}'\},$$

which is a σ-algebra in $\hat{\Omega}$, and then set

$$\mathcal{F}_t^{\hat{W},\infty} := \mathcal{F}_t^{\hat{W}} \vee (\Omega \times \mathcal{F}') = (\mathcal{F}_t^W \times \Omega') \vee \mathcal{N} \vee (\Omega \times \mathcal{F}'), \qquad t \geq 0.$$

Next we define a new set of admissible strategies, denoted $\hat{\mathcal{A}}^\infty$, consisting of the elements of the form $\hat{\alpha} = (\hat{\tau}_n, \hat{\xi}_n)_{n\geq 1}$ satisfying properties analogous to $(i) - (v)$ in section 2.2, but with the filtration $\mathbb{F}^{\hat{W},\infty}$ instead of $\mathbb{F}^W$. For any such $\hat{\alpha}$ one finds the corresponding trajectory $\hat{X}^{\hat{\alpha}}$ solving the controlled equation (4.1) and computes the corresponding reward $\hat{J}(\hat{\alpha})$ by (4.2) as before. The corresponding value is defined as

$$\hat{\upsilon}_0^\infty := \sup_{\hat{\alpha}\in\hat{\mathcal{A}}^\infty} \hat{J}(\hat{\alpha}). \tag{4.5}$$

Since $\mathbb{F}^{\hat{W}}$ is a smaller filtration than $\mathbb{F}^{\hat{W},\infty}$, we have $\hat{\mathcal{A}} \subset \hat{\mathcal{A}}^\infty$ and we conclude that $\upsilon_0 \leq \hat{\upsilon}_0 \leq \hat{\upsilon}_0^\infty$. Actually, it turns out that the three values in fact coincide:

Lemma 4.1 *With the previous notations we have* $\upsilon_0 = \hat{\upsilon}_0 = \hat{\upsilon}_0^\infty$.

The intuitive explanation is that in the optimal switching problem for $\hat{\upsilon}_0^\infty$ the controller has access to the information coming from the Wiener filtration as well as the one represented by the σ-algebra $\Omega \times \mathcal{F}'$; however, under $\hat{\mathbb{P}}$ the latter is independent of W and so it has no use in getting a better performance. We do not write down the proof of this Lemma, since it is entirely analogous to Lemma 4.1 of [2] (there the notation $\mathbb{F}^{W,\mu'_\infty}$ and $\mathcal{A}^{W,\mu'}$ was used instead of our notation $\mathbb{F}^{\hat{W},\infty}$ and $\hat{\mathcal{A}}^\infty$). The conclusion of this lemma will be used in the proof of the inequality $\upsilon_0^{\mathcal{R}} \le \upsilon_0$ below.

4.2 Proof of the inequality $\upsilon_0^{\mathcal{R}} \le \upsilon_0$

We follow closely [2], making use in particular of the basic Proposition 4.2 in that paper.

Let $(\Omega, \mathcal{F}, \mathbb{P}, W)$ be a setting for the optimal switching problem formulated in section 2.2. We construct a setting for a randomized control problem in the form of an appropriate product extension as described in Remark 3.4.

Let λ be a Borel measure on A satisfying **(A2)**. As a first step, we construct a suitable surjective measurable map $\pi : \mathbb{R} \to A$ and a measure λ' on the Borel subsets of the real line satisfying the condition $\lambda = \lambda' \circ \pi^{-1}$ (the image measure of λ' under π) and such that $\lambda'(\{r\}) = 0$ for every $r \in \mathbb{R}$.

We do not report the details of the construction of π and λ', for which we refer the reader to paragraph 4.1 of [2]. We just mention that it is a very simple consequence of the well known fact that the space of modes A, being a Borel space, is known to be either finite or countable (with the discrete topology) or isomorphic, as a measurable space, to the real line: see e.g. [7], Corollary 7.16.1.

Next, we choose $(\Omega', \mathcal{F}', \mathbb{P}')$ to be the canonical probability space of a non-explosive Poisson point process on $(0, \infty) \times \mathbb{R}$ with intensity λ'. Thus, Ω' is the set of sequences $\omega' = (t_n, r_n)_{n \ge 1} \subset (0, \infty) \times \mathbb{R}$ with $t_n < t_{n+1} \nearrow \infty$, $(\sigma_n, \rho_n)_{n \ge 1}$ is the canonical marked point process (i.e. $\sigma_n(\omega') = t_n$, $\rho_n(\omega') = r_n$), and $\mu' = \sum_{n \ge 1} \delta_{(\sigma_n, \rho_n)}$ is the corresponding random measure. Let $\mathcal{F}'$ denote the smallest σ-algebra such that all the maps σ_n, ρ_n are measurable, and $\mathbb{P}'$ the unique probability on $\mathcal{F}'$ such that μ' is a Poisson random measure with intensity λ' (since λ' is a finite measure, this probability actually exists). We will also use the completion of the space $(\Omega', \mathcal{F}', \mathbb{P}')$, still denoted by the same symbol by abuse of notation. Setting

$$\eta_n = \pi(\rho_n), \qquad \mu = \sum_{n \ge 1} \delta_{(\sigma_n, \eta_n)},$$

it is easy to verify that μ is a Poisson random measure on $(0, \infty) \times A$ with intensity λ, defined in $(\Omega', \mathcal{F}', \mathbb{P}')$.

Then we perform the construction described in section 4.1: we define $\hat{\Omega} = \Omega \times \Omega'$, we denote by $\hat{\mathcal{F}}$ the completion of $\mathcal{F} \otimes \mathcal{F}'$ with respect to $\mathbb{P} \otimes \mathbb{P}'$ and by $\hat{\mathbb{P}}$ the extension of $\mathbb{P} \otimes \mathbb{P}'$ to $\hat{\mathcal{F}}$. As explained before, W has a canonical extension to a $\hat{\mathbb{P}}$-standard Wiener process $\hat{W}$ in $\hat{\Omega}$. The Poisson random measure μ also has a canonical extension to a random measure $\hat{\mu}$ on $(0, \infty) \times A$ defined on $\hat{\Omega}$ setting $\hat{\mu} = \sum_{n \ge 1} \delta_{(\hat{\sigma}_n, \hat{\eta}_n)}$, where $\hat{\sigma}_n(\omega, \omega') := \sigma_n(\omega')$ and $\hat{\eta}_n(\omega, \omega') := \eta_n(\omega')$. It is immediate to verify that $\hat{\mu}$ is also a Poisson random measure with intensity λ, independent of $\hat{W}$. We may summarize this construction saying that $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{W}, \hat{\mu})$ is a setting for a randomized control problem.

We can them formulate the corresponding randomized optimization problem as in section 3.1: we define the $\hat{\mathbb{P}}$-completed filtration $\mathbb{F}^{\hat{W}, \hat{\mu}} = (\mathcal{F}_t^{\hat{W}, \hat{\mu}})_{t \ge 0}$ generated by $\hat{W}$ and $\hat{\mu}$ as in formula (3.3), we introduce the classes $\hat{\mathcal{V}}, \hat{\mathcal{V}}_{\inf > 0}$ and, for any admissible control $\hat{\nu} \in \hat{\mathcal{V}}$, the corresponding martingale $\kappa^{\hat{\nu}}$, the probability $\hat{\mathbb{P}}^{\hat{\nu}}(d\omega\, d\omega') = \kappa_T^{\hat{\nu}}(\omega, \omega')\,\hat{\mathbb{P}}(d\omega\, d\omega')$, the processes $\hat{I}$, given by formula (3.1) and solution to (3.2) respectively, the reward $J^{\mathcal{R}}(\hat{\nu})$ given by (3.7)-(3.8)-(3.9) and the value $\upsilon_0^{\mathcal{R}}$ defined in (3.12). We recall that this value does not depend on the specific setting chosen above for the randomized optimal control problem, as noticed in Remark 3.3.

We mention that we have the following alternative description of the filtration $\mathbb{F}^{\hat{W},\hat{\mu}} = (\mathcal{F}_t^{\hat{W},\hat{\mu}})_{t \geq 0}$. We first introduce in $(\Omega', \mathcal{F}')$ the $\mathbb{P}'$-complete right-continuous filtration $\mathbb{F}^\mu = (\mathcal{F}_t^\mu)_{t\geq 0}$, generated by μ and defined by

$$\mathcal{F}_t^\mu = \sigma\big(\mu((0,s] \times C)\ :\ s \in [0,t],\, C \in \mathcal{B}(A)\big) \vee \mathcal{N}',$$

where $\mathcal{N}'$ denotes the family of $\mathbb{P}'$-null sets of $\mathcal{F}'$. Next we introduce the σ-algebra in $\hat{\Omega}$ defined as

$$\Omega \times \mathcal{F}_t^\mu := \{\Omega \times B\ :\ B \in \mathcal{F}_t^\mu\}.$$

Then we have the equality

$$\mathcal{F}_t^{\hat{W},\hat{\mu}} = (\mathcal{F}_t^W \times \Omega') \vee (\Omega \times \mathcal{F}_t^\mu) \vee \mathcal{N}, \qquad t \geq 0, \tag{4.6}$$

which is analogous to formula (4.4) and can be proved by similar arguments.

At this point we make use of the following technical result, which is a special case of Proposition 4.2 in [2]:

Proposition 4.1 *For every $\hat{\nu} \in \hat{\mathcal{V}}_{\inf > 0}$ there exists $\hat{\alpha}^{\hat{\nu}} \in \mathcal{A}^{\hat{W},\infty}$ such that*

$$\mathscr{L}_{\hat{\mathbb{P}}^{\hat{\nu}}}(\hat{W}, \hat{I}) \ = \ \mathscr{L}_{\hat{\mathbb{P}}}(\hat{W}, \hat{\alpha}^{\hat{\nu}}), \tag{4.7}$$

i.e., the law of $(\hat{W}, \hat{I})$ under $\hat{\mathbb{P}}^{\hat{\nu}}$ is the same as the law of $(\hat{W}, \hat{\alpha}^{\hat{\nu}})$ under $\hat{\mathbb{P}}$.

The proof of the inequality $\upsilon_0^{\mathcal{R}} \leq \upsilon_0$ is now finished as follows. Take $\hat{\nu} \in \hat{\mathcal{V}}_{\inf > 0}$ and construct $\hat{\alpha}^{\hat{\nu}} \in \mathcal{A}^{\hat{W},\infty}$ as in Proposition 4.1. Since $\hat{X}$ is obtained solving equation (3.2) and $\hat{X}^{\hat{\alpha}^{\hat{\nu}}}$ is obtained solving equation (4.1) (with $\hat{\alpha}^{\hat{\nu}}$ instead of $\hat{\alpha}$) it is a well-known fact that under the conditions in Assumption **(A1)** the equality (4.7) implies that

$$\mathscr{L}_{\hat{\mathbb{P}}^{\hat{\nu}}}(\hat{X}, \hat{I}) = \mathscr{L}_{\hat{\mathbb{P}}}(\hat{X}^{\hat{\alpha}^{\nu}}, \hat{\alpha}^{\hat{\nu}}). \tag{4.8}$$

This immediately entails that $J^{\mathcal{R}}(\hat{\nu}) = \hat{J}(\hat{\alpha}^{\hat{\nu}})$. It follows that $J^{\mathcal{R}}(\hat{\nu}) \leq \hat{\upsilon}_0^\infty$, where the latter was defined in (4.5). From the arbitrariness of $\hat{\nu}$ we deduce that $\sup_{\hat{\nu} \in \hat{\mathcal{V}}_{\inf > 0}} J^{\mathcal{R}}(\hat{\nu}) \leq \hat{\upsilon}_0^\infty$. From (3.13) it follows that $\upsilon_0^{\mathcal{R}} \leq \hat{\upsilon}_0^\infty$. Since by Lemma 4.1 we have $\upsilon_0 = \hat{\upsilon}_0^\infty$ we arrive at the desired conclusion $\upsilon_0^{\mathcal{R}} \leq \upsilon_0$. □

4.3 Proof of the inequality $\upsilon_0 \leq \upsilon_0^{\mathcal{R}}$

In this proof we borrow some constructions from [21] and [2], but an entirely new proof is needed in order to take into account the occurrence of the switching costs that were not considered in those papers.

Suppose we are given a setting $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W)$ for the optimal switching problem as described in section 2.2, and consider the controlled equation (2.5) and the reward (2.7).

Lemma 4.2 *For any $\delta > 0$ there exists an admissible switching strategy $\alpha = (\tau_n, \xi_n)_{n \geq 1} \in \mathcal{A}$ such that*

$$J(\alpha) \geq \upsilon_0 - \delta$$

and moreover

(i) there exists an integer $N \geq 1$ such that $\tau_n = +\infty$ as soon as $n > N$,

(ii) the set $\{\xi_n(\omega) \ : \ \omega \in \Omega, n = 1, \ldots, N\}$ is finite.

For the proof of this Lemma, we need the following stability result, that will be used several times below. Following [36], for any pair $\alpha^1, \alpha^2 : \Omega \times [0,T] \to A$ of measurable processes we define a distance $\tilde{\rho}(\alpha^1, \alpha^2)$ setting

$$\tilde{\rho}(\alpha^1, \alpha^2) \quad = \quad \mathbb{E}\Big[\int_0^T \rho(\alpha^1_t, \alpha^2_t)\, dt\Big].$$

where ρ is an arbitrary metric compatible with the topology of A and satisfying $\rho < 1$. Using in particular the continuity condition **(A1)**-(iii) one can show the following.

Lemma 4.3 *Suppose we have a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{Q})$ with filtrations $\mathbb{G}^k = (\mathcal{G}^k_t)_{t \geq 0}$ $(k \geq 0)$ and a process B which is a Wiener process with respect to each $\mathbb{G}^k$. Consider the equations*

$$dY^k_t = b_t(Y^k, \gamma^k(t))\, dt + \sigma_t(Y^k, \gamma^k(t))\, dB_t, \qquad \hat{Y}^k_0 = x_0,$$

where each γ^k is an admissible switching strategy with respect to $\mathbb{G}^k$ (i.e., satisfying properties $(i) - (v)$ in section 2.2, but with the filtration $\mathbb{G}^k$ instead of $\mathbb{F}^W$). Suppose that

$$\tilde{\rho}(\gamma^k, \gamma^0) \to 0, \quad \text{and} \quad \gamma^k(T) \to \gamma^0(T)\ \mathbb{Q} - a.s. \tag{4.9}$$

as $k \to \infty$. Then for every $p \in [1, \infty)$,

$$\mathbb{E}^{\mathbb{Q}} \sup_{t \in [0,T]} |Y^k_t - Y^0_t|^p \to 0, \quad \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T f_t(Y^k, \gamma^k(t))\, dt\Big] \to \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T f_t(Y^0, \gamma^0(t))\, dt\Big]. \tag{4.10}$$

$$\mathbb{E}^{\mathbb{Q}}\Big[g(Y^k, \gamma^k(T))\Big] \to \mathbb{E}^{\mathbb{Q}}\Big[g(Y^0, \gamma^0(T))\Big], \tag{4.11}$$

so that in particular $J_1(\gamma^k) \to J_1(\gamma^0)$.

Proof. The convergence result (4.10) was first proved in [36] in the standard diffusion case. The simple extension to the non-Markovian case is presented in [21], Lemma 4.1 and Remark 4.1. This holds under the condition $\tilde{\rho}(\gamma^k, \gamma^0) \to 0$ alone. Using the second assumption in (4.9), the continuity assumption **(A1)**-(iii) and the growth conditions (2.4), the convergence (4.11) follows easily. □

Proof of Lemma 4.2. By the definition of υ_0, for any $\delta > 0$ there exists an admissible switching strategy $\alpha = (\tau_n, \xi_n)_{n \geq 1} \in \mathcal{A}$ such that $J(\alpha) \geq \upsilon_0 - \delta/3$. Next we modify α in two steps, in order to satisfy the additional requirements in the statement of the Lemma.

In a first step we consider the strategy obtained by taking only the first N switchings in α, that we denote $\alpha^N = (\tau_n, \xi_n)_{n=1}^N$. Formally, we use this notation to indicate the strategy where we have modified the pairs (τ_n, ξ_n) for $n > N$ setting them equal to $(\infty, \bar{\xi})$ where $\bar{\xi} \in A$ is fixed arbitrarily. We claim that $J(\alpha^N) \geq J(\alpha) - 2\delta/3$ for N sufficiently large.

To verify the claim we first note that, for the piecewise constant processes $\alpha^N(\cdot)$, $\alpha(\cdot)$ associated to α^N and α we have $\alpha^N(t) = \alpha(t)$ for $t \in [0, T \wedge \tau_N]$ and so

$$\tilde{\rho}(\alpha^N(\cdot), \alpha(\cdot)) = \mathbb{E}\Big[\int_0^T \rho(\alpha^N(t), \alpha(t))\, dt\Big] = \mathbb{E}\Big[\int_{\tau_N \wedge T}^T \rho(\alpha^N(t), \alpha(t))\, dt\Big] \leq \mathbb{E}\,[T - (\tau_N \wedge T)] \to 0,$$

since $\tau_N \to \infty$. Since $\{\tau_N > T\} \subset \{\alpha^N(T) = \alpha(T)\}$ we also have $\mathbb{P}(\alpha^N(T) = \alpha(T)) \geq \mathbb{P}(\tau_N > T) \to 1$, so that $\alpha^N(T) \to \alpha(T)$ in $\mathbb{P}$-probability and, passing to a subsequence if necessary, we may assume $\alpha^N(T) \to \alpha(T)$ $\mathbb{P}$-a.s. Applying Lemma 4.3 to the controlled equations satisfied by X^{α^N} and X^α and setting $B = W$, $Y^k = X^{\alpha^k}$, $\gamma^k(\cdot) = \alpha^k(\cdot)$ and $Y^0 = X^\alpha$, $\gamma^0(\cdot) = \alpha(\cdot)$, we conclude that $J_1(\alpha^N) \to J_1(\alpha)$.

Since $\alpha^N(t) = \alpha(t)$ for $t \in [0, T \wedge \tau_N]$ we also have $X_t^{\alpha^N} = X_t^\alpha$ for $t \in [0, T \wedge \tau_N]$ and therefore for $n = 1, \ldots, N$ we have

$$1_{\tau_n < T}\, c_{\tau_n}(X^{\alpha^N}, \xi_{n-1}, \xi_n) = 1_{\tau_n < T}\, c_{\tau_n}(X^\alpha, \xi_{n-1}, \xi_n).$$

If N is chosen so large that $|J_1(\alpha^N) - J_1(\alpha)| < \delta/3$ then, taking into account the fact that costs are nonnegative, we obtain

$$\begin{aligned} J(\alpha) = J_1(\alpha) - J_2(\alpha) &\leq J_1(\alpha) - \mathbb{E}\Big[\sum_{n=1}^N 1_{\tau_n < T}\, c_{\tau_n}(X^\alpha, \xi_{n-1}, \xi_n)\Big] \\ &= J_1(\alpha) - \mathbb{E}\Big[\sum_{n=1}^N 1_{\tau_n < T}\, c_{\tau_n}(X^{\alpha^N}, \xi_{n-1}, \xi_n)\Big] \\ &\leq J_1(\alpha^N) + \delta/3 - \mathbb{E}\Big[\sum_{n=1}^N 1_{\tau_n < T}\, c_{\tau_n}(X^{\alpha^N}, \xi_{n-1}, \xi_n)\Big] \\ &= J(\alpha^N) + \delta/3, \end{aligned}$$

and since we have $J(\alpha) \geq \mathcal{V}_0 - \delta/3$ we obtain $J(\alpha^N) \geq \mathcal{V}_0 - 2\delta/3$ as claimed.

As a second step we fix N and we further modify α^N in the following way. Since A is a Borel space, it is separable. Let us fix a dense sequence $(a_i)_{i \geq 1}$ and define, for each integer $k \geq 1$, a map $\Pi_k : A \to A$ that assigns to each $b \in A$ its nearest point in $\{a_1, \ldots, a_k\}$, more precisely

$$\Pi_k(b) = a_{i(b)}, \quad \text{where} \quad i(b) := \min\{j \in \{1, \ldots, k\} \ :\ \rho(b, a_j) \leq \rho(b, a_i) \text{ for all } i \in \{1, \ldots, k\}\}.$$

It is easy to see that $\Pi_k : A \to A$ is Borel measurable and $\rho(\Pi_k(a), a) \downarrow 0$ as $k \to \infty$.

Starting from the strategy $\alpha^N = (\tau_n, \xi_n)_{n=1}^N$ constructed above we define $\alpha^{N,k} = (\tau_n, \Pi_k(\xi_n))_{n=1}^N$. We note that each strategy $\alpha^{N,k}$ satisfies the conditions stated in the Lemma. To finish the proof it is therefore enough to prove that $J(\alpha^{N,k}) \to J(\alpha^N)$ as $k \to \infty$: indeed, taking any k sufficiently large we have $J(\alpha^{N,k}) \geq J(\alpha^N) - \delta/3$ so that any such strategy satisfies $J(\alpha^{N,k}) \geq \mathcal{V}_0 - \delta$.

In order to prove that $J(\alpha^{N,k}) \to J(\alpha^N)$ we start noting that $\rho(\alpha^{N,k}(t), \alpha^N(t)) \to 0$ $\mathbb{P}$-a.s. for every $t \in [0, T]$. In particular $\alpha^{N,k}(T) \to \alpha^N(T)$ $\mathbb{P}$-a.s. and we also have $\tilde{\rho}(\alpha^{N,k}(\cdot), \alpha^N(\cdot)) \to 0$. Another application of Lemma 4.3 shows that $J_1(\alpha^{N,k}) \to J_1(\alpha^N)$ and we also have

$$\forall\, p \in [1, \infty), \qquad \mathbb{E} \sup_{t \in [0,T]} |X_t^{\alpha^{N,k}} - X_t^{\alpha^N}|^p \to 0.$$

Passing to a subsequence if necessary, we may assume that $\sup_{t\in[0,T]}|X_t^{\alpha^{N,k}} - X_t^{\alpha^N}| \to 0$ $\mathbb{P}$-a.s. and for every $n = 1, \dots, N$ we have, by the continuity assumptions in **(A1)**-(iii),

$$1_{\tau_n<T}\, c_{\tau_n}(X^{\alpha^{N,k}}, \Pi_k(\xi_{n-1}), \Pi_k(\xi_n)) \to 1_{\tau_n<T}\, c_{\tau_n}(X^{\alpha^N}, \xi_{n-1}, \xi_n), \qquad \mathbb{P}-a.s.$$

From the growth condition (2.4) we obtain the inequality

$$0 \le 1_{\tau_n<T}\, c_{\tau_n}(X^{\alpha^{N,k}}, \Pi_k(\xi_{n-1}), \Pi_k(\xi_n)) \le 1_{\tau_n<T}\, L\,(1 + \sup_{t\in[0,T]} |X^{\alpha^{N,k}}|)^r$$

and by (2.6) we conclude that the right-hand side is bounded in $L^p(\mathbb{P})$ for every $p \in [1, \infty)$. It follows that

$$\mathbb{E}\Big[1_{\tau_n<T}\, c_{\tau_n}(X^{\alpha^{N,k}}, \Pi_k(\xi_{n-1}), \Pi_k(\xi_n))\Big] \to \mathbb{E}\Big[1_{\tau_n<T}\, c_{\tau_n}(X^{\alpha^N}, \xi_{n-1}, \xi_n)\Big],$$

and we conclude that $J_2(\alpha^{N,k}) \to J_2(\alpha^N)$ since the number of switchings is bounded by N. This way we have proved that $J(\alpha^{N,k}) \to J(\alpha^N)$, which ends the proof of the Lemma. □

In order to proceed further we need to construct a product probability space as explained in section 4.1, making use of a properly chosen auxiliary probability space denoted $(\Omega', \mathcal{F}', \mathbb{P}')$. This can be taken as an arbitrary probability space where appropriate random objects are defined. For integers $m, n, k \ge 1$, we assume that real random variables U_n^m, S_n^m and random measures π^k are defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ and satisfy the following conditions:

1. every U_n^m is uniformly distributed on $(0, 1)$;
2. every S_n^m admits a density (denoted $f_n^m(t)$) with respect to the Lebesgue measure, and we have $0 < S_1^m < S_2^m < S_3^m < \dots$ for every m, and $S_n^m \to 0$ as $m \to \infty$ for every n;
3. every π^k is a Poisson random measure on $(0, \infty) \times A$, admitting compensator $k^{-1}\lambda(da)\,dt$ with respect to its natural filtration;
4. the random elements U_n^m, S_j^h, π^k are all independent.

The inequalities required in point 2. above can be satisfied for instance by choosing the support of each density f_n^m inside the interval $((1 - 2^{-n})/m, (1 - 2^{-n-1})/m)$. The role of these random elements will become clear in the constructions that follow. Notice that for the construction of the space $(\Omega', \mathcal{F}', \mathbb{P}')$ only the knowledge of the measure λ is required. Moreover by a classical result, see [43] Theorem 2.3.1, we may take $\Omega' = [0, 1]$, $\mathcal{F}'$ the corresponding Borel sets and $\mathbb{P}'$ the Lebesgue measure.

Next we perform the construction described in section 4.1. Let us define $\hat\Omega = \Omega \times \Omega'$, let us denote by $\hat{\mathcal{F}}$ the completion of the product σ-algebra $\mathcal{F} \otimes \mathcal{F}'$ with respect to $\mathbb{P} \otimes \mathbb{P}'$ and by $\mathbb{Q}$ the extension of $\mathbb{P} \otimes \mathbb{P}'$ to $\hat{\mathcal{F}}$ (the notation $\hat{\mathbb{P}}$ will be used for a different probability introduced below). As before we denote $\hat W_t$, $\hat U_n^m$, $\hat S_j^h$, $\hat\pi^k$ the canonical extensions of W, U_n^m, S_j^h, π^k to $\hat\Omega$.

Since $\hat W$ is a standard Wiener process under $\hat{\mathbb{P}}$ we can consider the optimal switching problem in the setting $(\hat\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat W)$ as in section 4.1: we define the set of admissible strategies $\hat{\mathcal{A}}$ as the elements of the form $\hat\alpha = (\hat\tau_n, \hat\xi_n)_{n\ge1}$ satisfying properties analogous to $(i) - (v)$ in section 2.2, but with the filtration $\mathbb{F}^{\hat W}$ instead of $\mathbb{F}^W$. For any $\hat\alpha \in \hat{\mathcal{A}}$ one finds the corresponding trajectory $\hat X^{\hat\alpha}$

solving the controlled equation (4.1) and computes the corresponding reward $\hat{J}(\hat{\alpha})$ given in (4.2), namely

$$\hat{J}(\hat{\alpha}) = \hat{J}_1(\hat{\alpha}) - \hat{J}_2(\hat{\alpha}) = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T f_t(\hat{X}^{\hat{\alpha}}, \hat{\alpha}(t))\,dt + g(\hat{X}^{\hat{\alpha}}, \hat{\alpha}(T))\Big] - \mathbb{E}^{\mathbb{Q}}\Big[\sum_{n\geq 1} 1_{\hat{\tau}_n<T}\, c_{\hat{\tau}_n}(\hat{X}^{\hat{\alpha}}, \hat{\xi}_{n-1}, \hat{\xi}_n)\Big], \tag{4.12}$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation under $\mathbb{Q}$. It was explained in Remark 2.3 that any switching strategy can be viewed as a marked point process in A. In the following it will be convenient to identify any $\hat{\alpha} \in \hat{\mathcal{A}}$ of the form $\hat{\alpha} = (\hat{\tau}_n, \hat{\xi}_n)_{n\geq 1}$ with the corresponding random measure on $(0,\infty)\times A$ defined as

$$\hat{\alpha} = \sum_{n\geq 1} \delta_{(\hat{\tau}_n, \hat{\xi}_n)}\, 1_{\hat{\tau}_n<\infty}$$

where δ denotes the Dirac measure. We will use the same symbol to denote the strategy and the corresponding measure. We will also need the corresponding natural filtration $\mathbb{F}^{\hat{\alpha}} = (\mathcal{F}_t^{\hat{\alpha}})_{t\geq 0}$ in $(\hat{\Omega}, \hat{\mathcal{F}})$ defined by the formula:

$$\mathcal{F}_t^{\hat{\alpha}} \quad = \quad \sigma(\hat{\alpha}((0,s]\times C)\ :\ s\in[0,t],\ C\in\mathcal{B}(A)), \tag{4.13}$$

and also the filtration $\mathbb{F}^{\hat{W}} \vee \mathbb{F}^{\hat{\alpha}} := (\mathcal{F}_t^{\hat{W}} \vee \mathcal{F}_t^{\hat{\alpha}})_{t\geq 0}$. We denote $\mathcal{P}(\mathbb{F}^{\hat{\alpha}})$, $\mathcal{P}(\mathbb{F}^{\hat{W}} \vee \mathbb{F}^{\hat{\alpha}})$, the corresponding predictable σ-algebras.

A basic role in the arguments below will be played by the concept of compensator (or dual predictable projection) of this random measure, as presented for instance in [30].

Lemma 4.4 *For any $\delta > 0$ there exists an admissible switching strategy $\hat{\beta} \in \hat{\mathcal{A}}$ such that*

$$\hat{J}(\hat{\beta}) \geq \upsilon_0 - 2\delta$$

and moreover the $\mathbb{Q}$-compensator of the corresponding random measure on $(0,T]\times A$ with respect to $\mathbb{F}^{\hat{W}} \vee \mathbb{F}^{\hat{\beta}}$ is absolutely continuous with respect to the measure $\lambda(da)\,dt$ and it has the form

$$\hat{\nu}_t^{\hat{\beta}}(\omega, \omega', a)\,\lambda(da)\,dt$$

where $\hat{\nu}^{\hat{\beta}} : \hat{\Omega}\times[0,T]\times A \to [0,\infty)$ is a $\mathcal{P}(\mathbb{F}^{\hat{W}} \vee \mathbb{F}^{\hat{\beta}}) \otimes \mathcal{B}(A)$-measurable function.

Proof. Given $\delta > 0$, let us consider the strategy α constructed in Lemma 4.2 and let us denote $\hat{\alpha} = (\hat{\tau}_n, \hat{\xi}_n)_{n\geq 1}$ its canonical extension. We have seen in section 4.1 that $\hat{\alpha} \in \hat{\mathcal{A}}$ and $J(\alpha) = \hat{J}(\hat{\alpha})$. By construction of $\hat{\alpha}$ it holds that $\hat{J}(\hat{\alpha}) \geq \upsilon_0 - \delta$, $\hat{\tau}_n = \infty$ as soon as $n > N$, and the set $\{\hat{\xi}_n(\omega)\ :\ \omega\in\Omega, n = 1,\ldots,N\}$ is finite. The corresponding random measure and piecewise constant process are

$$\hat{\alpha} = \sum_{n=1}^N \delta_{(\hat{\tau}_n, \hat{\xi}_n)}\, 1_{\hat{\tau}_n<\infty}, \qquad \hat{\alpha}(t) \ = \ \xi_0 1_{[0,\hat{\tau}_1)}(t) + \sum_{n=1}^N \hat{\xi}_n 1_{[\hat{\tau}_n, \hat{\tau}_{n+1})}(t),$$

where $\xi_0 \in A$ is the given starting mode.

The idea of the proof is to perturb this random measure slightly in such a way that the corresponding reward will not be changed too much and at the same time its compensator will have the desired properties.

Let ρ be a metric inducing the topology of A and satisfying $\rho < 1$. For every $m \geq 1$, let $\mathbf{B}(b, 1/m)$ denote the open ball of radius $1/m$, with respect to the metric ρ, centered at $b \in A$. Since $\lambda(da)$ has full support, we have $\lambda(\mathbf{B}(b, 1/m)) > 0$ and we can define a transition kernel $q^m(b, da)$ in A setting

$$q^m(b, da) \quad = \quad \frac{1}{\lambda(\mathbf{B}(b, 1/m))} 1_{\mathbf{B}(b,1/m)}(a)\lambda(da).$$

We recall that we require A to be a Borel space, and we denote by $\mathcal{B}(A)$ its Borel σ-algebra. There exists a Borel measurable function $q^m : A \times [0, 1] \to A$ such that for every $b \in A$ the measure $B \mapsto q^m(b, B)$ $(B \in \mathcal{B}(A))$ is the image of the Lebesgue measure on $[0, 1]$ under the mapping $u \mapsto q^m(b, u)$. Thus, if U is a random variable defined on some probability space and having uniform law on $[0, 1]$ then, for fixed $b \in A$, the A-valued random variable $q^m(b, U)$ has law $q^m(b, da)$. The use of the same symbol q^m should not generate confusion. The existence of the function q^m (even for a general transition kernel on A) is well known when A is a separable complete metric space, in particular, when A is the unit interval $[0, 1]$, (see e.g. [43], Theorem 3.1.1) and the general case reduces to this one, since it is known that any Borel space is either finite or countable (with the discrete topology) or isomorphic, as a measurable space, to the interval $[0, 1]$: see e.g. [7], Corollary 7.16.1.

For fixed $m \geq 1$, define $\hat{R}_0^m = 0$ and

$$\hat{R}_n^m = \hat{\tau}_n + \hat{S}_n^m, \quad \hat{\beta}_n^m = q^m(\hat{\xi}_n, \hat{U}_n^m), \qquad n \geq 1.$$

Since $\hat{\tau}_n < \hat{\tau}_{n+1}$ and since $\hat{S}_n^m > 0$ we see that $\hat{\alpha}^m := (\hat{R}_n^m, \hat{\beta}_n^m)_{n\geq1}$ is an admissible strategy (the property that $\mathbb{Q}(\hat{R}_n^m = T \text{ for some } n) = 0$ comes from the fact that $\hat{S}_n^m$ have absolutely continuous laws and are independent of $\hat{\tau}_n$). Let

$$\hat{\alpha}^m = \sum_{n=1}^{N} \delta_{(\hat{R}_n^m, \hat{\beta}_n^m)}, \qquad \hat{\alpha}^m(t) \ = \ \xi_0 1_{[0,\hat{R}_1^m)}(t) + \sum_{n=1}^{N} \hat{\beta}_n^m 1_{[\hat{R}_n^m, \hat{R}_{n+1}^m)}(t),$$

denote the corresponding random measure and the associated piecewise constant process.

It is possible to compute explicitly the $\mathbb{Q}$-compensator of these random measures with respect to $\mathbb{F}^{\hat{W}} \vee \mathbb{F}^{\hat{\alpha}^m}$, which is given by the formula

$$\sum_{n=1}^{N} 1_{(\hat{\tau}_n \vee \hat{R}_{n-1}^m, \hat{R}_n^m]}(t)\, q^m(\hat{\xi}_n, da) \frac{f_n^m(t - \hat{\tau}_n)}{1 - F_n^m(t - \hat{\tau}_n)}\, dt,$$

where we denote by $F_n^m(s) = \int_{-\infty}^{s} f_n^m(t)dt$ the cumulative distribution function of S_n^m, with the convention that $\frac{f_n^m(s)}{1-F_n^m(s)} = 0$ if $F_n^m(s) = 1$. The proof of this result is given in Lemma A.11 in [21]. We can write this formula in the form

$$\left[\sum_{n=1}^{N} 1_{(\hat{\tau}_n \vee \hat{R}_{n-1}^m, \hat{R}_n^m]}(t)\, \frac{1}{\lambda(\mathbf{B}(\hat{\xi}_n, 1/m))}\, 1_{\mathbf{B}(\hat{\xi}_n,1/m)}(a) \frac{f_n^m(t - \hat{\tau}_n)}{1 - F_n^m(t - \hat{\tau}_n)}\right] \lambda(da)\, dt$$

where the function in square brackets is a nonnegative $\mathcal{P}(\mathbb{F}^{\hat{W}} \vee \mathbb{F}^{\hat{\alpha}^m}) \otimes \mathcal{B}(A)$-measurable function.

To finish the proof it is enough to show that $\hat{J}(\hat{\alpha}^m) \to \hat{J}(\hat{\alpha})$ as $m \to \infty$ (or at least for a subsequence m_k). Indeed, since $\hat{J}(\hat{\alpha}) \geq \mathcal{V}_0 - \delta$, for large m we will have $\hat{J}(\hat{\alpha}^m) \geq \mathcal{V}_0 - 2\delta$ and we can take $\hat{\beta} = \hat{\alpha}^m$ for such m in the statement of the Lemma, since its compensator satisfies the required conditions.

To prove the required convergence $\hat{J}(\hat{\alpha}^m) \to \hat{J}(\hat{\alpha})$ we first note that

$$0 < \hat{R}_n^m - \hat{\tau}_n = \hat{S}_n^m \to 0, \qquad \mathbb{Q} - a.s.$$

We deduce that $\mathbb{Q}$-a.s., $\hat{\alpha}^m(t) \to \hat{\alpha}(t)$, except perhaps at points $\hat{\tau}_n$, and so dt-a.s. In particular, since there are no switchings at the terminal time T, we have $\mathbb{Q}(\hat{\tau}_n = T \text{ for some } n) = 0$ and we conclude that $\hat{\alpha}^m(T) \to \hat{\alpha}(T)$ $\mathbb{Q}$-a.s. We also note that by the choice of the kernel $q^m(b, da)$ we have $\rho(\hat{\xi}_n, \hat{\beta}_n^m) < 1/m \to 0$ and therefore for the distance already considered above we have

$$\tilde{\rho}(\hat{\alpha}, \hat{\alpha}^m) = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \rho(\hat{\alpha}(t), \hat{\alpha}^m(t))\, dt\Big] \to 0, \qquad m \to \infty. \tag{4.14}$$

Applying Lemma 4.3 to the controlled equations satisfied by $\hat{X}^{\hat{\alpha}^m}$ and $\hat{X}^{\hat{\alpha}}$ and setting $B = \hat{W}$, $Y^k = \hat{X}^{\hat{\alpha}^k}$, $\gamma^k(\cdot) = \hat{\alpha}^k(\cdot)$ and $Y^0 = \hat{X}^{\hat{\alpha}}$, $\gamma^0(\cdot) = \hat{\alpha}(\cdot)$ we conclude that $\hat{J}_1(\hat{\alpha}^m) \to \hat{J}_1(\hat{\alpha})$.

It remains to study the convergence of $\hat{J}_2(\hat{\alpha}^m)$. Since it is a finite sum, it is enough to check that for every $n = 1, \dots, N$

$$\mathbb{E}^{\mathbb{Q}}\Big[1_{\hat{R}_n^m < T}\, c_{\hat{R}_n^m}(\hat{X}^{\hat{\alpha}^m}, \hat{\beta}_{n-1}^m, \hat{\beta}_n^m)\Big] \to \mathbb{E}^{\mathbb{Q}}\Big[1_{\hat{\tau}_n < T}\, c_{\hat{\tau}_n}(\hat{X}^{\hat{\alpha}}, \hat{\xi}_{n-1}, \hat{\xi}_n)\Big], \tag{4.15}$$

as $m \to \infty$. By the growth condition (2.4) in **(A1)** we have

$$|c_{\hat{R}_n^m}(\hat{X}^{\hat{\alpha}^m}, \hat{\beta}_{n-1}^m, \hat{\beta}_n^m)| \le L\,(1 + \sup_{t \in [0,T]} |X_t^{\hat{\alpha}^m}|)^r$$

and the right-hand side is bounded in all $L^p(\mathbb{Q})$ spaces, by the estimate (2.6). So it is enough to check that we have convergence $\mathbb{Q}$-almost surely for the terms in right brackets in (4.15). Once again, since $\tilde{\rho}(\hat{\alpha}, \hat{\alpha}^m) \to 0$, the application of Lemma 4.3 gives that, for any $p \in [1, \infty)$

$$\mathbb{E}^{\mathbb{Q}}[\sup_{t \in [0,T]} |X_t^{\hat{\alpha}^m} - X_t^{\hat{\alpha}}|^p \to 0,$$

and so, at least for a subsequence, we have $\|X^{\hat{\alpha}^m} - X^{\hat{\alpha}}\|_\infty \to 0$ $\mathbb{Q}$-a.s. We have already checked above that, $\mathbb{Q}$-a.s., $\hat{R}_n^m \to \hat{\tau}_n$, $\hat{\beta}_n^m \to \hat{\xi}_n$ and so we have $1_{\hat{R}_n^m < T} \to 1_{\hat{\tau}_n < T}$ and finally

$$1_{\hat{R}_n^m < T}\, c_{\hat{R}_n^m}(\hat{X}^{\hat{\alpha}^m}, \hat{\beta}_{n-1}^m, \hat{\beta}_n^m) \to 1_{\hat{\tau}_n < T}\, c_{\hat{\tau}_n}(\hat{X}^{\hat{\alpha}}, \hat{\xi}_{n-1}, \hat{\xi}_n),$$

by the continuity properties of the coefficient c stated in Assumption **(A1)**. The required convergence (4.15) is proved and the proof of the Lemma is finished. □

Lemma 4.5 *For any $\delta > 0$ there exists an admissible switching strategy $\hat{\alpha} \in \hat{\mathcal{A}}$ such that*

$$\hat{J}(\hat{\alpha}) \ge \upsilon_0 - 3\delta$$

and moreover the $\mathbb{Q}$-compensator of the corresponding random measure on $(0, T] \times A$ with respect to $\mathbb{F}^{\hat{W}} \vee \mathbb{F}^{\hat{\alpha}}$ is absolutely continuous with respect to the measure $\lambda(da)\, dt$ and it has the form

$$\hat{\nu}_t(\omega, \omega', a)\, \lambda(da)\, dt$$

where $\nu : \hat{\Omega} \times [0, T] \times A \to [0, \infty)$ is a $\mathcal{P}(\mathbb{F}^{\hat{W}} \vee \mathbb{F}^{\hat{\alpha}}) \otimes \mathcal{B}(A)$-measurable function satisfying $\inf \nu > 0$. *Moreover, denoting by N_T the number of jump times of $\hat{\alpha}$ in $[0, T]$, we have $N_T \in L^p(\mathbb{Q})$ for every $p \in [1, \infty)$.*

Proof. Let $\hat\beta \in \hat{\mathcal{A}}$ be the switching strategy constructed in Lemma 4.4, that we write in the form of a random measure $\hat\beta = \sum_{n=1}^N 1_{\hat R_n<\infty}\delta_{(\hat R_n,\hat\beta_n)}$ having at most N summands. The idea of the proof is to modify the associated random measure by adding an independent Poisson process with "small" intensity. This will not affect the reward too much and will produce a random measure whose compensator remains absolutely continuous with respect to the measure $\lambda(da)\,dt$ with a bounded density which, in addition, is bounded away from zero.

Recall that on the space $(\Omega',\mathcal{F}',\mathbb{P}')$ we assumed that for every integer $k\geq 1$ there existed a Poisson random measure π^k on $(0,\infty)\times A$, admitting compensator $k^{-1}\lambda(da)\,dt$ with respect to its natural filtration. We denoted $\hat\pi^k$ its canonical extension to $(\hat\Omega,\hat{\mathcal{F}})$, that we write in the form of a random measure on $(0,\infty)\times A$:

$$\hat\pi^k \;=\; \sum_{n\geq 1}\delta_{(\hat\sigma_n^k,\hat\eta_n^k)},$$

for a marked point process $(\hat\sigma_n^k,\hat\eta_n^k)_{n\geq 1}$. Let us define other random measures setting

$$\hat\mu^k = \hat\beta + \hat\pi^k.$$

Note that the jumps times $(\hat R_n)_{n\geq 1}$ are independent of the jump times $(\hat\sigma_n^k)_{n\geq 1}$, and the latter have absolutely continuous laws. It follows that, except possibly on a set of $\mathbb{Q}$ probability zero, their graphs are disjoint, i.e. $\hat\beta$ and $\hat\pi^k$ have no common jumps, and $\hat\mu^k$ do not charge the terminal time T. Therefore, the random measures $\hat\mu^k$ can be identified with admissible switching strategies (they belong to $\hat{\mathcal{A}}$) and, together with their associated piecewise constant processes (denoted $\mu^k(\cdot)$) admit a representation of the form

$$\hat\mu^k = \sum_{n\geq 1}\delta_{(\hat\tau_n^k,\hat\xi_n^k)}, \qquad \hat\mu^k(t) = \xi_0 1_{[0,\hat\tau_1^k)}(t) + \sum_{n\geq 1}\xi_n^k 1_{[\hat\tau_n^k,\hat\tau_{n+1}^k)}(t), \quad t\in[0,T],$$

where ξ_0 is the starting mode, $(\hat\tau_n^k,\hat\xi_n^k)_{n\geq 1}$ is a marked point process, each $\hat\tau_n^k$ coincides with one of the times $\hat R_n$ or one of the times $\hat\sigma_n^k$, and each $\hat\xi_n^k$ coincides with one of the random variables $\hat\eta_n^k$ or one of the random variables $\hat\beta_n$.

We recall that $\hat\beta$ had at most N switchings, and we define $N_T^k := \sum_{n\geq 1} 1_{\hat\sigma_n^k\leq T}$ which has Poisson law with parameter $\lambda(A)T/k$. It follows that the number of jump times $\hat\tau_n^k$ in $[0,T]$ of each $\hat\mu^k$ cannot exceed $N+N_T^k$ and therefore it belongs to $L^p(\mathbb{Q})$ for every $p\in[1,\infty)$.

Let us verify that the $\mathbb{Q}$-compensator of each $\hat\mu^k$ with respect to $\mathbb{F}^{\hat W}\vee\mathbb{F}^{\hat\mu^k}$ satisfies the properties in the statement of the Lemma. We first note that, since $\hat\beta$ and $\hat\pi^k$ are independent, it is easy to prove that $\hat\mu^k=\hat\beta+\hat\pi^k$ has compensator $(\hat\nu_t^{\hat\beta}(\omega,\omega',a)+k^{-1})\,\lambda(da)\,dt$ with respect to the filtration $\mathbb{F}^{\hat W}\vee\mathbb{F}^{\hat\beta}\vee\mathbb{F}^{\hat\pi^k}$=$(\mathcal{F}_t^{\hat W}\vee\mathcal{F}_t^{\hat\beta}\vee\mathcal{F}_t^{\hat\pi^k})_{t\geq 0}$. Let us denote $\mathbb{F}^{\hat\pi^k}=(\mathcal{F}_t^{\hat\pi^k})$ the natural filtration of $\hat\pi^k$ defined as in (4.13). We wish to compute the $\mathbb{Q}$-compensator of $\hat\mu^k$ with respect to the filtration $\mathbb{F}^{\hat W}\vee\mathbb{F}^{\hat\mu^k}$ $=(\mathcal{F}_t^{\hat W}\vee\mathcal{F}_t^{\hat\mu^k})_{t\geq 0}$, which is smaller than $\mathbb{F}^{\hat W}\vee\mathbb{F}^{\hat\beta}\vee\mathbb{F}^{\hat\pi^k}$. To this end, consider the measure space $([0,\infty)\times\Omega\times A,\mathcal{B}([0,\infty))\otimes\mathcal{F}\otimes\mathcal{B}(A),dt\otimes\mathbb{Q}(d\omega)\otimes\lambda(da))$. Although this is not a probability space, one can define in a standard way the conditional expectation of any positive measurable function, given an arbitrary sub-σ-algebra. Let us denote by $\hat\nu_t^k(\omega,\omega',a)$ the conditional expectation of the random field $\hat\nu_t^{\hat\beta}(\omega,\omega',a)+k^{-1}$ with respect to the σ-algebra $\mathcal{P}(\mathbb{F}^{\hat W}\vee\mathbb{F}^{\hat\mu^k})\otimes\mathcal{B}(A)$. It is then easy to verify that the compensator of $\hat\mu^k$ with respect to $\mathbb{F}^{\hat W}\vee\mathbb{F}^{\hat\mu^k}$ coincides with $\hat\nu^k$. Moreover, since $\hat\nu_t^{\hat\beta}$ is nonnegative, we can take a version of $\hat\nu^k$ satisfying

$$\inf_{\hat\Omega\times[0,T]\times A}\hat\nu^k \geq k^{-1} > 0.$$

To finish the proof of Lemma 4.5 it is enough to show that $\hat{J}(\hat{\mu}^k) \to \hat{J}(\hat{\beta})$ as $k \to \infty$ (or at least for a subsequence). Indeed, since $\hat{J}(\hat{\beta}) \ge \upsilon_0 - 2\delta$, for large k we will have $\hat{J}(\hat{\mu}^k) \ge \upsilon_0 - 3\delta$ and we can take $\hat{\alpha} = \hat{\mu}^k$ for such k in the statement of the Lemma, since its compensator satisfies the required conditions.

We first claim that, for large k, $\hat{\mu}^k(\cdot)$ is close to $\hat{\beta}(\cdot)$ with respect to the metric $\tilde{\rho}$, namely that

$$\tilde{\rho}(\hat{\mu}^k(\cdot), \hat{\beta}(\cdot)) := \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \rho(\hat{\mu}^k(t), \hat{\beta}(t))\, dt\Big] \to 0, \qquad k \to \infty. \tag{4.16}$$

Recall that the jump times of $\hat{\pi}^k$ are denoted $\hat{\sigma}_n^k$. Since $\hat{\sigma}_1^k$ has exponential law with parameter $\lambda(A)/k$ the event $B_k = \{\hat{\sigma}_1^k > T\}$ has probability $e^{-\lambda(A)T/k}$, so that $\mathbb{Q}(B_k) \to 1$ as $k \to \infty$. We note that, on the set B_k, we have $\hat{\mu}^k(t) = \hat{\beta}(t)$ for all $t \in [0, T]$. Since we assume $\rho < 1$, we have $\tilde{\rho}(\mu^k(\cdot), \hat{\beta}(\cdot)) \le T(1 - \mathbb{Q}(B_k))$ and the claim (4.16) follows immediately.

Similarly, since $\hat{\mu}^k(T) = \hat{\beta}(T)$ on B_k, we have $\hat{\mu}^k(T) \to \hat{\beta}(T)$ in $\mathbb{Q}$-probability, and passing to a subsequence (denoted by the same symbol) if necessary we can assume $\hat{\mu}^k(T) \to \hat{\beta}(T)$ $\mathbb{Q}$-a.s.

Applying Lemma 4.3 to the controlled equations satisfied by $\hat{X}^{\hat{\mu}^k}$ and $\hat{X}^{\hat{\beta}}$ and setting $B = \hat{W}$, $Y^k = \hat{X}^{\hat{\alpha}^k}$, $\gamma^k(\cdot) = \hat{\alpha}^k(\cdot)$ and $Y^0 = \hat{X}^{\hat{\alpha}}$, $\gamma^0(\cdot) = \hat{\alpha}(\cdot))$ we conclude that $\hat{J}_1(\hat{\mu}^k) \to \hat{J}_1(\hat{\beta})$. It remains to study the convergence of

$$\hat{J}_2(\hat{\mu}^k) = \mathbb{E}^{\mathbb{Q}}\Big[\sum_{n\ge1} 1_{\hat{\tau}_n^k < T}\, c_{\hat{\tau}_n^k}(\hat{X}^{\hat{\mu}^k}, \hat{\xi}_{n-1}^k, \hat{\xi}_n^k)\Big].$$

We recall that $\hat{\beta}$ had at most N switchings, and we defined $N_T^k = \sum_{n\ge1} 1_{\hat{\sigma}_n^k \le T}$ which has Poisson law with parameter $\lambda(A)T/k$. By the growth conditions in Assumption **(A1)** we have

$$\sum_{n\ge1} 1_{\hat{\tau}_n^k < T}\, c_{\hat{\tau}_n^k}(\hat{X}^{\hat{\mu}^k}, \hat{\xi}_{n-1}^k, \hat{\xi}_n^k) \le (N + N_T^k)\, L\, (1 + \sup_{t\in[0,T]} |\hat{X}_t^{\hat{\mu}^k}|)^r$$

and recalling (2.6) we see that for every $p \in [1, \infty)$ the right-hand side is bounded in $L^p(\mathbb{Q})$ by a constant independent of k. Setting again $B_k = \{\hat{\sigma}_1^k > T\}$ and recalling that $\mathbb{Q}(B_k) \to 1$, by the Hölder inequality we conclude that

$$\mathbb{E}^{\mathbb{Q}}\Big[1_{B_k^c} \sum_{n\ge1} 1_{\hat{\tau}_n^k < T}\, c_{\hat{\tau}_n^k}(\hat{X}^{\hat{\mu}^k}, \hat{\xi}_{n-1}^k, \hat{\xi}_n^k)\Big] \to 0, \qquad k \to \infty.$$

Next we note that on the event B_k the measures $\hat{\mu}^k$ and $\hat{\beta}$ coincide on $(0, T] \times A$ and therefore on $B_k \times [0, T]$ we also have $\hat{\mu}^k(\cdot) = \hat{\beta}(\cdot)$ and $\hat{X}^{\hat{\mu}^k} = \hat{X}^{\hat{\beta}}$ $\mathbb{Q}$-a.s. It follows that

$$\begin{aligned}
\hat{J}_2(\hat{\mu}^k) &= \mathbb{E}^{\mathbb{Q}}\Big[1_{B_k} \sum_{n\ge1} 1_{\hat{\tau}_n^k < T}\, c_{\hat{\tau}_n^k}(\hat{X}^{\hat{\mu}^k}, \hat{\xi}_{n-1}^k, \hat{\xi}_n^k)\Big] + \mathbb{E}^{\mathbb{Q}}\Big[1_{B_k^c} \sum_{n\ge1} 1_{\hat{\tau}_n^k < T}\, c_{\hat{\tau}_n^k}(\hat{X}^{\hat{\mu}^k}, \hat{\xi}_{n-1}^k, \hat{\xi}_n^k)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[1_{B_k} \sum_{n=1}^{N} 1_{\hat{R}_n < T}\, c_{\hat{R}_n}(\hat{X}^{\hat{\beta}}, \hat{\beta}_{n-1}, \hat{\beta}_n)\Big] + \mathbb{E}^{\mathbb{Q}}\Big[1_{B_k^c} \sum_{n\ge1} 1_{\hat{\tau}_n^k < T}\, c_{\hat{\tau}_n^k}(\hat{X}^{\hat{\mu}^k}, \hat{\xi}_{n-1}^k, \hat{\xi}_n^k)\Big] \\
&\le \hat{J}_2(\hat{\beta}) + \mathbb{E}^{\mathbb{Q}}\Big[1_{B_k^c} \sum_{n\ge1} 1_{\hat{\tau}_n^k < T}\, c_{\hat{\tau}_n^k}(\hat{X}^{\hat{\mu}^k}, \hat{\xi}_{n-1}^k, \hat{\xi}_n^k)\Big].
\end{aligned}$$

Since we clearly have $\hat{J}_2(\hat{\beta}) \le \hat{J}_2(\hat{\mu}^k)$ it follows that $\hat{J}_2(\hat{\mu}^k) \to \hat{J}_2(\hat{\beta})$. Now we have verified that $\hat{J}(\hat{\mu}^k) \to \hat{J}(\hat{\beta})$ and the proof of Lemma 4.5 is finished. □

We are now able to end the proof of the inequality $\upsilon_0 \leq \upsilon_0^{\mathcal{R}}$.

Let $\delta > 0$ be given and denote $\hat{\alpha} = \sum_{n\geq 1} \delta_{(\hat{\sigma}_n,\hat{\eta}_n)} 1_{\hat{\sigma}_n<\infty}$ the random measure corresponding to the strategy $\hat{\alpha}$ given by Lemma 4.5.

Let $\mathcal{N}$ denote the family of $\mathbb{Q}$-null sets of $(\hat{\Omega}, \hat{\mathcal{F}})$. Then the filtration $(\mathcal{F}_t^{\hat{W}} \vee \mathcal{F}_t^{\hat{\alpha}} \vee \mathcal{N})_{t\geq 0}$ coincides with the filtration previously denoted by $\mathbb{F}^{\hat{W},\hat{\alpha}} = (\mathcal{F}_t^{\hat{W},\hat{\alpha}})_{t\geq 0}$ (compare with formula (3.3) or (4.6)). It is easy to see that $\hat{\nu}_t(\omega, \omega', a)\, \lambda(da)\, dt$ is the $\mathbb{Q}$-compensator of $\hat{\alpha}$ with respect to $\mathbb{F}^{\hat{W},\hat{\alpha}}$ as well.

Using the Girsanov theorem for point processes (see e.g. [30]) we next construct an equivalent probability under which $\hat{\alpha}$ becomes a Poisson random measure with intensity λ. Since the function $\hat{\nu}$ occurring in Lemma 4.5 is a strictly positive $\mathcal{P}(\mathbb{F}^{\hat{W},\hat{\alpha}}) \otimes \mathcal{B}(A)$-measurable random field with bounded inverse, the Doléans exponential process

$$M_t \;:=\; \exp\Big(\int_0^t \int_A (1 - \hat{\nu}_s(a)^{-1})\, \hat{\nu}_t(a)\lambda(da)\, ds\Big) \prod_{\hat{\sigma}_n \leq t} \hat{\nu}_{\hat{\sigma}_n}(\hat{\eta}_n)^{-1}, \qquad t \in [0, T], \tag{4.17}$$

is a strictly positive martingale (with respect to $\mathbb{F}^{\hat{W},\hat{\alpha}}$ and $\mathbb{Q}$), and we can define an equivalent probability $\hat{\mathbb{P}}$ on the space $(\hat{\Omega}, \hat{\mathcal{F}})$ setting $\hat{\mathbb{P}}(d\omega\, d\omega') = M_T(\omega, \omega')\mathbb{Q}(d\omega\, d\omega')$. The expectation under $\hat{\mathbb{P}}$ will be denoted $\hat{\mathbb{E}}$. By the Girsanov theorem, the restriction of $\hat{\alpha}$ to $(0, T] \times A$ has $(\hat{\mathbb{P}}, \mathbb{F}^{\hat{W},\hat{\alpha}})$-compensator $\lambda(da)\, dt$, so that in particular it is a Poisson random measure. It can also be proved by standard arguments (see e.g. [21], page 2155, for detailed verifications in a similar framework) that $\hat{W}$ remains a $(\hat{\mathbb{P}}, \mathbb{F}^{\hat{W},\hat{\alpha}})$-Wiener process and that $\hat{W}$ and $\hat{\alpha}$ are independent under $\hat{\mathbb{P}}$. We have thus constructed a setting $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{W}, \hat{\alpha})$ for a randomized control problem as in section 3.1.

Although the random field ν is not bounded in general, so in particular it does not belong to the class $\hat{\mathcal{V}}$ of admissible controls for the randomized control problem, we can still introduce the Doléans exponential process $\kappa^{\hat{\nu}}$ corresponding to $\hat{\nu}$ by the formula (3.5), namely:

$$\kappa_t^{\hat{\nu}} = \exp\left(\int_0^t \int_A (1 - \hat{\nu}_s(a))\lambda(da)\, ds\right) \prod_{\hat{\sigma}_n \leq t} \nu_{\hat{\sigma}_n}(\hat{\eta}_n), \qquad t \in [0, T]. \tag{4.18}$$

Comparing (4.17) and (4.18) shows that $\kappa_T^{\hat{\nu}} M_T \equiv 1$. It follows that $\hat{\mathbb{E}}[\kappa_T^{\hat{\nu}}] = \mathbb{E}^{\mathbb{Q}}[M_T \kappa_T^{\hat{\nu}}] = 1$, so that $\kappa^{\hat{\nu}}$ is indeed a $\hat{\mathbb{P}}$-martingale on $[0, T]$ and we can define the corresponding probability $\hat{\mathbb{P}}^{\hat{\nu}}(d\omega) := \kappa_T^{\hat{\nu}}(\omega)\hat{\mathbb{P}}(d\omega)$. Since $\kappa_T^{\hat{\nu}} M_T \equiv 1$, the Girsanov transformation $\hat{\mathbb{P}} \mapsto \hat{\mathbb{P}}^{\hat{\nu}}$ is the inverse of the transformation $\mathbb{Q} \mapsto \hat{\mathbb{P}}$ made above, and changes back the probability $\hat{\mathbb{P}}$ into $\mathbb{Q}$ considered above, so that we have $\hat{\mathbb{P}}^{\hat{\nu}} = \mathbb{Q}$.

Let $\hat{X}$ be the solution to the equation

$$d\hat{X}_t = b_t(\hat{X}, \hat{I}_t)\, dt + \sigma_t(\hat{X}, \hat{I}_t)\, d\hat{W}_t, \qquad \hat{X}_0 = x_0, \tag{4.19}$$

where $\hat{I}$ is the piecewise constant A-valued process associated to $\hat{\alpha}$ and starting at the initial mode ξ_0 (the same as in formula (3.1), and elsewhere indicated $\hat{\alpha}(\cdot)$):

$$\hat{I}_t \;= \xi_0\, 1_{[0,\hat{\sigma}_1)}(t) + \sum_{n\geq 1} \hat{\eta}_n\, 1_{[\hat{\sigma}_n,\hat{\sigma}_{n+1})}(t), \qquad t \geq 0. \tag{4.20}$$

The corresponding reward of the switching problem is then

$$\hat{J}(\hat{\alpha}) \;=\; \mathbb{E}^{\mathbb{Q}}\left[\int_0^T f_t(\hat{X}, \hat{I}_t)\, dt + g(\hat{X}, \hat{I}_T) - \sum_{n\geq 1} 1_{\hat{\sigma}_n<T}\, c_{\hat{\sigma}_n}(\hat{X}, \hat{\eta}_{n-1}, \hat{\eta}_n)\right]$$

$$= \quad \hat{\mathbb{E}}^{\hat{\nu}} \left[\int_0^T f_t(\hat{X}, \hat{I}_t)\, dt + g(\hat{X}, \hat{I}_T) - \sum_{n \ge 1} 1_{\hat{\sigma}_n < T}\, c_{\hat{\sigma}_n}(\hat{X}, \hat{\eta}_{n-1}, \hat{\eta}_n) \right], \tag{4.21}$$

where we have used $\hat{\mathbb{P}}^{\hat{\nu}} = \mathbb{Q}$ in the last equality.

For any integer $k \ge 1$ define $\hat{\nu}_t^k(a) = \hat{\nu}_t(a) \wedge k$. Therefore $\hat{\nu}^k \in \hat{\mathcal{V}}$, we can define the corresponding process $\kappa^{\hat{\nu}^k}$ by formula (3.5), the probability $\hat{\mathbb{P}}^{\hat{\nu}^k}(d\omega) = \kappa_T^{\hat{\nu}^k}(\omega)\, \hat{\mathbb{P}}(d\omega)$, and compute the reward $J^{\mathcal{R}}(\hat{\nu}^k)$ of the corresponding randomized problem. Since equation (4.19) coincides with the randomized equation (3.2), this is given by

$$J^{\mathcal{R}}(\hat{\nu}^k) \quad = \quad \hat{\mathbb{E}}^{\hat{\nu}^k} \left[\int_0^T f_t(\hat{X}, \hat{I}_t)\, dt + g(\hat{X}, \hat{I}_T) - \sum_{n \ge 1} 1_{\hat{\sigma}_n < T}\, c_{\hat{\sigma}_n}(\hat{X}, \hat{\eta}_{n-1}, \hat{\eta}_n) \right], \tag{4.22}$$

where $\hat{\mathbb{E}}^{\hat{\nu}^k}$ denotes the expectation under $\hat{\mathbb{P}}^{\nu^k}$.

We claim that $J^{\mathcal{R}}(\hat{\nu}^k) \to \hat{J}(\hat{\alpha})$ as $k \to \infty$. Assuming this for a moment, since $\hat{J}(\hat{\alpha}) \ge \upsilon_0 - 3\delta$, we will have $J^{\mathcal{R}}(\hat{\nu}^k) \ge \upsilon_0 - 4\delta$ for large k, and since $J^{\mathcal{R}}(\hat{\nu}^k)$ is the reward of a randomized control problem, by Remark 3.3 it can not exceed the value $\upsilon_0^{\mathcal{R}}$ defined in (3.12), whatever the setting where the randomized problem is formulated. It follows that $\upsilon_0^{\mathcal{R}} \ge \upsilon_0 - 4\delta$ and by the arbitrariness of δ we obtain the required inequality $\upsilon_0^{\mathcal{R}} \ge \upsilon_0$.

It remains to prove the claim that $J^{\mathcal{R}}(\hat{\nu}^k) \to \hat{J}(\hat{\alpha})$. Setting

$$\Phi = \int_0^T f_t(\hat{X}, \hat{I}_t)\, dt + g(\hat{X}, \hat{I}_T) - \sum_{n \ge 1} 1_{\hat{\sigma}_n < T}\, c_{\hat{\sigma}_n}(\hat{X}, \hat{\eta}_{n-1}, \hat{\eta}_n)$$

and comparing (4.21) with (4.22), proving the claim amounts to showing that $\hat{\mathbb{E}}^{\hat{\nu}^k}[\Phi] \to \hat{\mathbb{E}}^{\hat{\nu}}[\Phi]$ or equivalently $\hat{\mathbb{E}}[\kappa_T^{\hat{\nu}^k} \Phi] \to \hat{\mathbb{E}}[\kappa_T^{\hat{\nu}} \Phi]$. Using the growth condition (2.4) in Assumption **(A1)** we see that

$$|\Phi| \le c\,(1 + N_T)\,(1 + \sup_{t \in [0,T]} |\hat{X}_t|)^r$$

for a suitable constant c, where N_T denotes the number of jump times $\hat{\sigma}_n$ of $\hat{\alpha}$ in $[0, T]$. From Lemma 4.5 we know that $N_T \in L^p(\mathbb{Q})$ for every $p \in [1, \infty)$ and by (3.6) we conclude that $\Phi \in L^p(\mathbb{Q})$ for every $p \in [1, \infty)$ as well. The required convergence $\hat{\mathbb{E}}[\kappa_T^{\hat{\nu}^k} \Phi] \to \hat{\mathbb{E}}[\kappa_T^{\hat{\nu}} \Phi]$ can now be verified by standard arguments, exactly the same as in [21], pages 2156-2157.

□

5 The randomized BSDE

In this section the assumptions **(A1)** and **(A2)** are assumed to hold. We start from the formulation of the randomized control problem introduced in section 3.1. For simplicity of notation, from now on we drop all superscripts ˆ and start from a setting, denoted $(\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, W is a standard Wiener process in $\mathbb{R}^d$, $\mu = \sum_{n \ge 1} \delta_{(\sigma_n, \eta_n)}$ is a Poisson random measure on A with intensity λ, independent of W. We consider the piecewise constant process I in A associated with μ defined in (3.1), the corresponding trajectory X solution to equation (3.2) and the $\mathbb{P}$-complete right-continuous filtration $\mathbb{F}^{W,\mu} = (\mathcal{F}_t^{W,\mu})_{t \ge 0}$ generated by

W, μ and defined by formula (3.3). We recall the estimate $\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t|^p\Big] < \infty$ for all $p \in [1,\infty)$ (compare (3.4)).

Our aim is to show that the value of the randomized problem can be represented in terms of a constrained BSDE, that we will call *randomized*. From Theorem 3.1 it follows that the randomized BSDE also represents the value of the original switching problem.

On the space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathbb{F}^{W,\mu}$, let us consider the following constrained BSDE on the time interval $[0,T]$:

$$\begin{cases} Y_t \;=\; g(X, I_T) + \displaystyle\int_t^T f_s(X, I_s)\, ds + K_T - K_t - \int_t^T Z_s\, dW_s - \int_{(t,T]}\int_A U_s(a)\, \mu(ds\, da), \\ U_t(a) \;\le\; c_t(X, I_{t-}, a). \end{cases} \tag{5.1}$$

We look for a (minimal) solution to (5.1) in the sense of the following definition.

Definition 5.1 *A quadruple $(Y_t, Z_t, U_t(a), K_t)$ ($t \in [0,T]$, $a \in A$) is called a solution to the BSDE* (5.1) *if*

1. *$Y \in \mathcal{S}^2(\mathbb{F}^{W,\mu})$, the set of real-valued càdlàg $\mathbb{F}^{W,\mu}$-adapted processes satisfying $\|Y\|^2_{\mathcal{S}^2} := \mathbb{E}[\sup_{t\in[0,T]}|Y_t|^2] < \infty$;*
2. *$Z \in L^2_W(\mathbb{F}^{W,\mu})$, the set of $\mathbb{F}^{W,\mu}$-predictable processes with values in $\mathbb{R}^d$ satisfying $\|Z\|^2_{L^2_W} := \mathbb{E}\big[\int_0^T |Z_t|^2 dt\big] < \infty$;*
3. *$U \in L^2_\mu(\mathbb{F}^{W,\mu})$, the set of real-valued $\mathcal{P}(\mathbb{F}^{W,\mu}) \otimes \mathcal{B}(A)$-measurable processes satisfying $\|U\|^2_{L^2_\mu} := \mathbb{E}\big[\int_0^T \int_A |U_t(a)|^2 \lambda(da) dt\big] < \infty$;*
4. *$K \in \mathcal{K}^2(\mathbb{F}^{W,\mu})$, the subset of $\mathcal{S}^2(\mathbb{F}^{W,\mu})$ consisting of $\mathbb{F}^{W,\mu}$-predictable nondecreasing processes with $K_0 = 0$;*
5. *$\mathbb{P}$-a.s. the equality in* (5.1) *holds for every $t \in [0,T]$, and the constraint $U_t(a) \le c_t(X, I_{t-}, a)$ is understood to hold $\mathbb{P}(d\omega)\lambda(da)dt$-almost everywhere.*

A minimal solution (Y, Z, U, K) is a solution to (5.1) *such that for any other solution (Y', Z', U', K'), we have $\mathbb{P}$-a.s., $Y_t \le Y'_t$ for all $t \in [0,T]$.*

We now state the main result of this section.

Theorem 5.1 *There exists a unique minimal solution $(Y, Z, U, K) \in \mathcal{S}^2(\mathbb{F}^{W,\mu}) \times L^2_W(\mathbb{F}^{W,\mu}) \times L^2_\mu(\mathbb{F}^{W,\mu}) \times \mathcal{K}^2(\mathbb{F}^{W,\mu})$ to the randomized BSDE* (5.1). *Moreover, we have $Y_0 = \sup_{\nu\in\mathcal{V}} J^{\mathcal{R}}(\nu)$, and, more generally (setting $\eta_0 = \xi_0$ for convenience)*

$$Y_t \;=\; \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\bigg[\int_t^T f_s(X, I_s)\, ds + g(X, I_T) - \sum_{n\ge1} 1_{t<\sigma_n<T}\, c_{\sigma_n}(X, \eta_{n-1}, \eta_n) \,\bigg|\, \mathcal{F}_t^{W,\mu}\bigg]. \tag{5.2}$$

Remark 5.1 From Theorems 3.1 and 5.1 we deduce the BSDE representation for the original optimal switching problem:

$$Y_0 \;=\; \sup_{\alpha\in\mathcal{A}} J(\alpha).$$

□

We need the following preliminary result.

Lemma 5.1 *For every $\nu \in \mathcal{V}$ and $t \in [0,T]$, we have $\mathbb{P}$-a.s.*

$$\mathbb{E}^\nu\Big[\sum_{n\geq 1} 1_{t<\sigma_n<T}\, c_{\sigma_n}(X,\eta_{n-1},\eta_n)\Big|\mathcal{F}_t^{W,\mu}\Big] = \mathbb{E}^\nu\Big[\sum_{n\geq 1} 1_{t<\sigma_n\leq T}\, c_{\sigma_n}(X,\eta_{n-1},\eta_n)\Big|\mathcal{F}_t^{W,\mu}\Big] \tag{5.3}$$

$$= \mathbb{E}^\nu\Big[\int_t^T\int_A c_s(X,I_{s-},a)\,\nu_s(a)\,\lambda(da)\,ds\Big|\mathcal{F}_t^{W,\mu}\Big] \tag{5.4}$$

In particular, for $t=0$, we have $J_2^{\mathcal{R}}(\nu) = \mathbb{E}^\nu\Big[\int_0^T\int_A c_s(X,I_{s-},a)\,\nu_s(a)\,\lambda(da)\,ds\Big]$.

Proof. The equality in (5.3) is obvious since $\mathbb{P}(\sigma_n = T$ for some $n) = 0$. Since the $\mathbb{P}^\nu$-compensator of $\mu(ds\,da)$ is $\nu_s(a)\,\lambda(da)\,ds$, and by **(A1)**-(ii) the random field $c_s(X,I_{s-},a)$ is $\mathcal{P}(\mathbb{F}^{W,\mu})$ $\otimes$ $\mathcal{B}(A)$-measurable and nonnegative, we obtain the second equality (5.4):

$$\mathbb{E}^\nu\Big[\sum_{n\geq 1} 1_{t<\sigma_n\leq T}\, c_{\sigma_n}(X,\eta_{n-1},\eta_n)\Big|\mathcal{F}_t^{W,\mu}\Big] = \mathbb{E}^\nu\Big[\int_{(t,T]}\int_A c_s(X,I_{s-},a)\;\mu(da\,ds)\Big|\mathcal{F}_t^{W,\mu}\Big].$$

$$= \mathbb{E}^\nu\Big[\int_t^T\int_A c_s(X,I_{s-},a)\,\nu_s(a)\,\lambda(da)\,ds\Big|\mathcal{F}_t^{W,\mu}\Big].$$

□

Remark 5.2 It follows from the Lemma that formula (5.2) can written

$$Y_t = \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\Big[\int_t^T f_s(X,I_s)\,ds + g(X,I_T) - \int_t^T\int_A c_s(X,I_{s-},a)\,\lambda(da)\,ds\Big|\mathcal{F}_t^{W,\mu}\Big].$$

Similar remarks apply to several formulae that follow below.

Proof (of Theorem 5.1) Let us introduce for every $n \in \mathbb{N}$ the following penalized BSDE on $[0,T]$:

$$Y_t^n = g(X,I_T) + \int_t^T f_s(X,I_s)\,ds + K_T^n - K_t^n - \int_t^T Z_s^n\,dW_s - \int_t^T\int_A U_s^n(a)\,\mu(ds\,da), \tag{5.5}$$

where

$$K_t^n = n\int_0^t\int_A \big(U_s^n(a) - c_s(X,I_{s-},a)\big)^+\,\lambda(da)\,ds.$$

By (2.4) and (3.4) we have $\mathbb{E}|g(X,I_T)|^2 < \infty$ and $\mathbb{E}\int_0^T |f_t(X,I_t)|^2\,dt < \infty$, so it follows from Lemma 2.4 in [41] that, for every $n \in \mathbb{N}$, there exists a unique solution $(Y^n,Z^n,U^n) \in \mathcal{S}^2(\mathbb{F}^{W,\mu}) \times L_W^2(\mathbb{F}^{W,\mu}) \times L_\mu^2(\mathbb{F}^{W,\mu})$ to the above penalized BSDE.

Next we claim that for every $t \in [0,T]$ we have, $\mathbb{P}$-a.s.

$$Y_t^n = \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_n} \mathbb{E}^\nu\Big[\int_t^T f_s(X,I_s)\,ds + g(X,I_T) - \sum_{n\geq 1} 1_{t<\sigma_n<T}\, c_{\sigma_n}(X,\eta_{n-1},\eta_n)\Big|\mathcal{F}_t^{W,\mu}\Big], \tag{5.6}$$

where $\mathcal{V}_n = \{\nu \in \mathcal{V}\colon \nu$ takes values in $(0,n]\}$. To prove the claim we take any $\nu \in \mathcal{V}_n$ and we first notice that

$$\mathbb{E}^\nu\Big[\int_{(t,T]}\int_A U_s^n(a)\,\mu(ds\,da)\Big|\mathcal{F}_t^{W,\mu}\Big] = \mathbb{E}^\nu\Big[\int_t^T\int_A U_s^n(a)\,\nu_s(a)\,\lambda(da)\,ds\Big|\mathcal{F}_t^{W,\mu}\Big],$$

because the $\mathbb{P}^\nu$-compensator of $\mu(ds\,da)$ is $\nu_s(a)\lambda(da)\,ds$. Next we note that the process $\int_0^\cdot Z_s^n\,dW_s$ is a $\mathbb{P}^\nu$-local martingale, since W is a Wiener process under $\mathbb{P}^\nu$; recalling that $d\mathbb{P}^\nu = \kappa_T^\nu d\mathbb{P}$ and using the estimates (3.10) it is easy to prove that it is in fact a $\mathbb{P}^\nu$-martingale, so that in particular

$$\mathbb{E}^\nu\bigg[\int_t^T Z_s^n\,dW_s\,\bigg|\,\mathcal{F}_t^{W,\mu}\bigg] = 0.$$

So taking expectation $\mathbb{E}^\nu$ in (5.5), adding and subtracting both sides of equality (5.4) and rearranging terms we obtain

$$\begin{aligned} Y_t^n &= \mathbb{E}^\nu\bigg[\int_t^T f_s(X,I_s)\,ds + g(X,I_T) - \sum_{n\geq 1} 1_{t<\sigma_n<T}\,c_{\sigma_n}(X,\eta_{n-1},\eta_n)\,\bigg|\,\mathcal{F}_t^{W,\mu}\bigg] \qquad (5.7)\\ &+ \mathbb{E}^\nu\bigg[\int_t^T\int_A \Big\{n\big(U_s^n(a)-c_s(X,I_{s-},a)\big)^+ - \big(U_s^n(a)-c_s(X,I_{s-},a)\big)\nu_s(a)\Big\}\,\lambda(da)\,ds\,\bigg|\,\mathcal{F}_t^{W,\mu}\bigg]. \end{aligned}$$

This is sometimes called the fundamental relation for the penalized control problem corresponding to admissible controls $\mathcal{V}_n$. The term in curly brackets $\Big\{\dots\Big\}$ is nonnegative, since $\nu_s(a)$ takes values in $(0,n]$ and we have the numerical inequality $nx^+ \geq x\nu$ for every $x\in\mathbb{R}$ and $\nu\in(0,n]$. It follows that

$$Y_t^n \;\geq\; \mathbb{E}^\nu\bigg[\int_t^T f_s(X,I_s)\,ds + g(X,I_T) - \sum_{n\geq 1} 1_{t<\sigma_n<T}\,c_{\sigma_n}(X,\eta_{n-1},\eta_n)\,\bigg|\,\mathcal{F}_t^{W,\mu}\bigg], \quad \nu\in\mathcal{V}_n. \qquad (5.8)$$

Now we show that the term in curly brackets can be made as small as we wish for an appropriate choice of $\nu\in\mathcal{V}_n$. We note that, given $0<\varepsilon<n$ and $x\in\mathbb{R}$ and choosing

$$\bar\nu = n\,1_{\{x\geq 0\}} + \varepsilon\,1_{\{-1<x<0\}} - (\varepsilon/x)\,1_{\{x\leq -1\}}$$

we have $\bar\nu\in[\varepsilon,n]$ and $nx^+ - x\bar\nu\leq\varepsilon$. So it follows that setting

$$\begin{aligned} \nu_s^{\varepsilon,n}(a) &= n\,1_{\{U_s^n(a)-c_s(X,I_{s-},a)\geq 0\}} + \varepsilon\,1_{\{-1<U_s^n(a)-c_s(X,I_{s-},a)<0\}}\\ &\quad -\varepsilon\,(U_s^n(a)-c_s(X,I_{s-},a))^{-1}\,1_{\{U_s^n(a)-c_s(X,I_{s-},a)\leq -1\}}, \end{aligned}$$

we have $\nu^{\varepsilon,n}\in\mathcal{V}_n$ and

$$\Big\{n\big(U_s^n(a)-c_s(X,I_{s-},a)\big)^+ - \nu_r^{\varepsilon,n}(a)(U_s^n(a)-c_s(X,I_{s-},a))\Big\}\leq\varepsilon$$

($\nu_s^{\varepsilon,n}(a)$ is an approximation of $n\,1_{\{U_s^n(a)-c_s(X,I_{s-},a)\geq 0\}}$ which is not in $\mathcal{V}_n$ since it can take the value zero). From (5.7) it follows that

$$Y_t^n \;\leq\; \mathbb{E}^{\nu^{\varepsilon,n}}\bigg[\int_t^T f_s(X,I_s)\,ds + g(X,I_T) - \sum_{n\geq 1} 1_{t<\sigma_n<T}\,c_{\sigma_n}(X,\eta_{n-1},\eta_n)\,\bigg|\,\mathcal{F}_t^{W,\mu}\bigg] + \varepsilon\,(T-t)\,\lambda(A),$$

which, together with (5.8), proves the claim (5.6).

Recalling that $d\mathbb{P}^\nu = \kappa_T^\nu d\mathbb{P}$, using the estimates (3.10), (3.11) and recalling (2.4) and (3.4), we deduce that

$$\sup_n Y_t^n \;<\; \infty, \qquad \text{for all } 0\leq t\leq T. \qquad (5.9)$$

Let us define $\check{g}$, $\check{Y}$ and $\check{U}$ by the equalities

$$\begin{cases} \check{Y}_t = Y_t - \int_0^t \int_A c_s(X, I_{s-}, a)\, \mu(ds, da), \\ \check{g} = g(X, I_T) - \int_0^T \int_A c_s(X, I_{s-}, a)\, \mu(ds, da) \\ \check{U}_t(a) = U_t(a) - c_t(X, I_{t-}, a), \end{cases}$$

so that equation (5.1) can be written as follows:

$$\begin{cases} \check{Y}_t = \check{g} + \int_t^T \Big(f_s(X, I_s) - \int_A \check{U}_s(a)\, \lambda(da) \Big) ds + K_T - K_t - \int_t^T Z_s\, dW_s - \int_t^T \int_A \check{U}_s(a)\, \tilde{\mu}(ds\, da), \\ \check{U}_t(a) \ \leq \ 0, \end{cases} \tag{5.10}$$

where $\tilde{\mu}(dt\, da) = \mu(dt\, da) - \lambda(da)\, dt$ denotes the compensated Poisson measure. Let us check that (Y, Z, U, K) belongs to the space $\mathcal{S}^2(\mathbb{F}^{W,\mu}) \times L^2_W(\mathbb{F}^{W,\mu}) \times L^2_\mu(\mathbb{F}^{W,\mu}) \times \mathcal{K}^2(\mathbb{F}^{W,\mu})$ if and only if $(\check{Y}, Z, \check{U}, K)$ does. In fact, noting that the process $c_t(X, I_{t-}, a)$ is $\mathcal{P}(\mathbb{F}^{W,\mu}) \otimes \mathcal{B}(A)$-measurable and non-negative, it is enough to verify that

$$\mathbb{E}\left[\left| \int_0^T \int_A c_s(X, I_{s-}, a)\, \mu(ds, da) \right|^2 \right] < \infty,$$

which is equivalent to

$$\mathbb{E}\left[\int_0^T \int_A c_s(X, I_{s-}, a)^2\, \lambda(da)\, ds \right] < \infty.$$

This last inequality follows from the growth assumption (2.4) and the estimate (3.4), taking into account that the random measure $\mu((0,T] \times A)$ has Poisson law with parameter $\lambda(A)T$. It also follows that $\check{g}$ also belongs to L^2 and it is $\mathbb{F}^{W,\mu}$-measurable. We conclude that (Y, Z, U, K) is the minimal solution to (5.1) if and only if $(\check{Y}, Z, \check{U}, K)$ is the minimal solution to (5.10).

Next we also note that equation (5.10) is a particular case of a backward stochastic differential equation studied in a general non-Markovian framework in [35]. In particular, existence and uniqueness of the minimal solution to equation (5.10) (or, equivalently, to equation (5.1)) follow from Theorem 2.1 in [35]. Indeed, Assumption **(H0)** in [35] is clearly satisfied. Concerning Assumption **(H1)**, this is only used in Lemma 2.2 of [35] to prove that the sequence $(Y^n)_n$ satisfies (5.9), a property that in our setting has been proved by different arguments. Finally, from Theorem 2.1 in [35] we also have that $Y^n_t(\omega)$ converges increasingly to $Y_t(\omega)$ as $n \to \infty$, $\mathbb{P}(d\omega)$-a.s. Since $\mathcal{V} = \cup_n \mathcal{V}_n$, letting $n \to \infty$ in (5.6) we obtain (5.2). □

Formula (5.2) shows that the process Y constructed in Theorem 5.1 can be seen as the value of an optimization problem. Our final result shows that it satisfies a version of dynamic programming principle in the randomized context. We omit the proof which is very similar to Lemma 4.8 in [21] or Theorem 5.3 of [2], after obvious changes of notation.

Theorem 5.2 *For all* $0 \leq t \leq T$, *we have*

$$Y_t \ = \ \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^\nu \left[\int_t^\tau f_r(X, I_r)\, dr - \sum_{n \geq 1} 1_{t < \sigma_n < T}\, c_{\sigma_n}(X, \eta_{n-1}, \eta_n) + Y_\tau \,\middle|\, \mathcal{F}^{W,\mu}_t \right]$$

$$= \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_t} \mathbb{E}^\nu\left[\int_t^\tau f_r(X,I_r)\,dr - \sum_{n\geq 1} 1_{t<\sigma_n<T}\, c_{\sigma_n}(X,\eta_{n-1},\eta_n) + Y_\tau \,\middle|\, \mathcal{F}_t^{W,\mu}\right], \quad (5.11)$$

where $\mathcal{T}_t$ denotes the class of $[t,T]$-valued $\mathbb{F}^{W,\mu}$-stopping times.

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