Powers of the Szegö Kernel and Hankel Operators on Hardy Spaces

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In this paper we study the action of certain integral operators on spaces of holomorphic functions on some domains in $\mathbb{C}^n$. These integral operators are defined by using powers of the Szegö kernel as integral kernel. We show that they act like differential operators, or like pseudo-differential operators of not necessarily integral order. These operators may be used to give equivalent norms for the Besov spaces B_p of holomorphic functions. As a consequence we prove that, when $1 \le p < \infty$, the small Hankel operators h_f on Hardy and weighted Bergman spaces are in the Schatten class $\mathcal{S}_p$ if and only if the symbol f belongs to B_p.

The type of domains we deal with are the smoothly bounded strictly pseudo-convex domains in $\mathbb{C}^n$ and a class of complex ellipsoids in $\mathbb{C}^n$. Our results for strictly pseudo-convex domains depend on Fefferman's expansion of the Szegö kernel. In this case, its powers act like a power of the derivation in the normal direction. The ellipsoids we consider are the simplest examples of domains of finite type. In this case, the symmetries of the domains can be exploited to use methods of harmonic analysis and describe the pseudo-differential operators involved.

1. Basic Notation and Statement of the Main Results

Let $\mathcal{D} = \{ z : \rho(z) < 0 \}$ be a smoothly bounded domain in $\mathbb{C}^n$, with $\rho \in \mathcal{C}^\infty(\bar{\mathcal{D}})$ and $\nabla\rho \neq 0$ on $\partial\mathcal{D}$. For $p > 0$ let $H^p(\mathcal{D})$ denote the Hardy space of holomorphic functions on $\mathcal{D}$, with norm given by

$$\|f\|^p_{H^p(\mathcal{D})} := \sup_{\varepsilon>0} \int_{\delta(w)=\varepsilon} |f(w)|^p \, d\sigma(w),$$

where $\delta(w) := -\rho(w)$ is equivalent to the distance to the boundary and $d\sigma$ is the surface measure. Let $P_S^{(\sigma)}$ and S_σ denote the Szegö projection and the Szegö kernel respectively:

$$P_S^{(\sigma)} g(z) := \int_{\partial\mathcal{D}} S_\sigma(z, \zeta) g(\zeta) \, d\sigma(\zeta)$$

for $g \in L^2(\partial\mathcal{D})$. We are interested in the (small) Hankel operator $h_f^{(\sigma)}$ with symbol f, defined for $g \in L^2(\partial\mathcal{D})$ as

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$$h_f^{(\sigma)}(g) := P_S^{(\sigma)}\big(f\overline{P_S^{(\sigma)}(g)}\big). \tag{1}$$

Moreover, for $p > 0$, we define the Besov spaces of holomorphic functions $B_p^{p,n/p}(\mathcal{D})$ by setting

$$B_p^{p,n/p}(\mathcal{D}) := \left\{ g \in L^p(\mathcal{D}) : \int_{\mathcal{D}} |\delta^l \nabla^l g|^p B(z,z)\, dV(z) < \infty \right\},$$

where $B(z,w)$ denotes the Bergman kernel on $\mathcal{D}$, dV is the Lebesgue measure on $\mathcal{D}$, and l is some integer such that $lp > 2$. For simplicity of notation, we shall write B_p for $B_p^{p,n/p}(\mathcal{D})$. We also set

$$\|g\|_{B_p} := \|g\|_{L^p} + \left[\int_{\mathcal{D}} |\delta^l \nabla^l g|^p B(z,z)\, dV(z) \right]^{1/p}.$$

When $\mathcal{D}$ is the unit ball in $\mathbb{C}^n$, the Szegö kernel S is known explicitly and $S(z,\zeta) = c_n(1 - z\cdot\bar{\zeta})^{-n}$. In this case, it is quite often useful to study the action of integral operators defined by using powers of the Szegö kernel. The action of these operators is usually expressed in terms of Besov norms, and it is easily understood via the relation

$$\left(I + \frac{N}{n}\right) S(z,\zeta) = S^{(n+1)/n}(z,\zeta), \tag{2}$$

where N is the differential operator $\sum_{j=1}^n z_j \partial_{z_j}$. Thus, equation (2) shows a link between powers of the Szegö kernel and differential operators.

Using identity (2) and iterations of it, Feldman and Rochberg [FR] proved that in the case of the unit ball, when $1 \le p < \infty$, the Hankel operator h_f belongs to the Schatten class $\mathcal{S}_p$ if and only if f is in B_p. Recall that for $p > 0$, given a compact operator T on a Hilbert space, we say that T belongs to the Schatten class $\mathcal{S}_p$ if $\sum_j s_j^p < \infty$, where

$$s_j := \{ \inf \|T - E\| : \operatorname{rank} E \le j \}.$$

Such results generalized the now classical results of Peller [Pe] and Coifman and Rochberg [CR] for the unit disc. It is natural to ask whether these results are valid in a more general setting.

In order to present the results in this paper we need to introduce some more notation. We begin with the case of the ellipsoids.

Let q be a positive integer, and let Ω_q be the ellipsoid in $\mathbb{C}^2$ given by

$$\Omega_q := \{ z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2q} < 1 \}.$$

On these ellipsoids, we first replace the surface measure $d\sigma$ by the measure $d\mu$ defined next, for which explicit computations are available. Precisely, let $d\mu$ be the unique measure on $\partial\Omega_q$ such that, for positive F,

$$\int_{\mathbb{C}^2} F(z)\, dV(z) = \int_0^\infty \int_{\partial\Omega_q} F(rz_1, r^{1/q} z_2)\, d\mu(z) r^{1+2/q}\, dr. \tag{3}$$

The measure $d\mu$ is equivalent to the surface measure $d\sigma$, and the density $d\mu/d\sigma$ is a $\mathcal{C}^\infty$ strictly positive function. We shall denote by P_S^μ and S_μ the Szegö projection and the Szegö kernel related to the measure $d\mu$, respectively. The Hankel operator $h_f^{(\mu)}$ is defined as in (1), using P_S^μ instead of P_S^σ. Notice that, in the case of the unit ball, the measures $d\mu$ and $d\sigma$ coincide.

The identity (2) is no longer valid in the case of ellipsoids, even in an approximate way. But S_μ is known explicitly, and we shall prove an identity for the powers of $S_\mu(z,\zeta)$ that involves some kind of pseudo-differential operators. Specifically, we prove the following results.

THEOREM 1.1. *Let $p > 0$ and let λ be a real number such that $\lambda > 1/p$. Then there exists $c = c(p,\lambda) > 0$ such that, for all $g \in B_p$, we have*

$$\frac{1}{c}\|g\|_{B_p}^p \leq \int_{\Omega_q}\left|\int_{\partial\Omega_q}\frac{S_\mu^{\lambda+1}(z,\zeta)}{S_\mu^\lambda(z,z)}g(\zeta)\,d\mu(\zeta)\right|^p B(z,z)\,dV(z) \leq c\|g\|_{B_p}^p.$$

Notice that the condition on λ ensures that the weight is an integrable function, and that λ can be taken to be 1 if $p > 1$. Thus, the middle term in the foregoing display defines an equivalent norm on B_p. We shall also prove the analog of Theorem 1.1 for integral powers of the Szegö kernel S_σ.

As an application of Theorem 1.1, we shall obtain a necessary and sufficient condition for $h_f^{(\mu)}$ and $h_f^{(\sigma)}$ to belong to the Schatten class $\mathcal{S}_p$ when $1 \leq p < \infty$.

In [Sy1] and [Sy3] it was proved that the condition $f \in B_p$ is a sufficient condition for h_f to belong to $\mathcal{S}_p$ ($1 \leq p < \infty$) for finite-type domains in $\mathbb{C}^2$, for strictly pseudo-convex domains in $\mathbb{C}^n$, and also for ellipsoids in $\mathbb{C}^n$. The necessity of the condition was left open. Here we show that the condition is necessary when we restrict to the class of ellipsoids that we have defined, and also for strictly pseudo-convex domains.

Using the method of [FR], it was also mentioned in [Sy1] that, for $p > 1$, $h_f^{(\mu)} \in \mathcal{S}_p$ implies

$$\int_{\Omega_q}\left|S_\mu^{-1}(z,z)\int_{\partial\Omega_q}S_\mu^2(z,\zeta)f(\zeta)\,d\mu(\zeta)\right|^p B(z,z)\,dV(z) < \infty. \tag{4}$$

In the case of the unit ball, condition (4) is immediately seen to be equivalent to the fact that g is in the space B_p. The reason for this is the link between S_σ^2 and a derivative of S_σ given by (2).

Our result in the case of the ellipsoids is as follows, where h_f stands for both $h_f^{(\mu)}$ and $h_f^{(\sigma)}$.

THEOREM 1.2. *Let $1 \leq p < \infty$. Then $h_f \in \mathcal{S}_p$ is equivalent to $f \in B_p$.*

In the case of $h_f^{(\mu)}$, Theorem 1.2 follows directly from Theorem 1.1 and [Sy1] for $p > 1$. We give a new proof, which extends to the case $p = 1$. We also prove that Theorem 1.2 is still valid for the Hankel operators based on weighted Bergman projections.

We next turn to the case of a strictly pseudo-convex domain. We prove the analogs of Theorems 1.1 and 1.2 in this case. In this context the identity (2), which we wrote for the ball, holds in an approximate way. This is an easy consequence of Fefferman's expansion for the Szegö kernel. Although these results on strictly pseudo-convex domains all follow from somewhat standard techniques, it seems that they never appeared in print before. The idea of approximate identities is also used to deduce the two main theorems in the case of the surface measure $d\sigma$ on the ellipsoids.

We mention that the charaterization of bounded and compact Hankel operator is known in the case of a strictly pseudo-convex domain. In the case of the unit ball [CRW] and in general [KL] it has been shown that h_f is bounded if and only if $f \in$ BMO and is compact if and only if $f \in H^2 \cap \mathrm{VMO}$. We also mention that characterizations of symbols of big Hankel operators have been obtained in [KLR1] and [BeLi], and that related results appear in [KLR2].

The paper is organized as follows. Sections 2, 3, and 4 are devoted to the problem on ellipsoids with measure $d\mu$. In Section 2 we study the powers of the Szegö kernel, and in Section 3 we prove Theorem 1.1 on equivalence of norms in $B_p(\Omega_q)$. We believe that these results, and the techniques involved, may have applications beyond what is offered here. In Section 4 we prove Theorem 1.2. In Section 5, we prove the corresponding results for weighted Bergman spaces on Ω_q. In Section 6 we consider the case of a smoothly bounded strictly pseudo-convex domain, and in Section 7 we conclude by indicating how to translate the results proved on the ellipsoids to the case of the surface measure.

Finally, we mention that in an upcoming paper [BPS] we study the question of factorization of Hardy spaces as well as characterization of bounded and compact Hankel operators on a class of finite type domains in $\mathbb{C}^n$ that includes the ellipsoids.

2. Powers of the Szegö Kernel

The next three sections deal with the Szegö kernel and the Hankel operator related to the measure $d\mu$, so we shall omit all indices or exponents and write simply S (resp. h_f) instead of S_μ (resp. $h_f^{(\mu)}$). We shall also write Ω instead of Ω_q. With respect to the surface measure $d\mu$ on the boundary $\partial\Omega$ (see (3)), the Szegö kernel $S(z,\zeta)$ has expression

$$
\begin{aligned}
S(z,\zeta) &= c\sum_m \frac{\Gamma\left(m_1 + \frac{m_2}{q} + 1 + \frac{1}{q}\right)}{\Gamma(m_1+1)\Gamma\left(\frac{m_2}{q} + \frac{1}{q}\right)} z^m \bar{\zeta}^m \\
&= c(1 - z_1\bar{\zeta}_1)^{-(1+1/q)}\left(1 - \frac{z_2\bar{\zeta}_2}{(1 - z_1\bar{\zeta}_1)^{1/q}}\right)^{-2}.
\end{aligned}
$$

(see [BoLo]). We recall that the Bergman kernel also has an explicit expression of this type, which allows us to consider $d\mu$ as a natural measure on $\partial\Omega$.

We want to study the integral operator M_λ $(\lambda > 0)$, given by

$$
f \mapsto \int_{\partial\Omega} S^{\lambda+1}(\cdot,\zeta) f(\zeta)\, d\mu(\zeta),
$$

when acting on holomorphic functions. We can write

$$S(z,\zeta)=\sum_m \gamma_m^{-1} z^m \bar{\zeta}^m,$$

with $\gamma_m=\int_{\partial\Omega}|\zeta^m|^2\,d\mu(\zeta)$. We are going to write $S^{\lambda+1}(z,\zeta)$ in the same way.

LEMMA 2.1. *For $l>0$ we have*

$$S^l(z,\zeta)=c_l\sum_m \gamma_m^{-1}A_m^{(l)} z^m\bar{\zeta}^m,$$

where the sum is taken over $m\in\mathbb{Z}^2$, $m_1,m_2\geq 0$, and

$$A_m^{(l)}=\frac{\Gamma\big(m_1+\frac{m_2}{q}+\big(1+\frac{1}{q}\big)l\big)}{\Gamma\big(m_1+\frac{m_2}{q}+1+\frac{1}{q}\big)}\cdot\frac{\Gamma(m_2+2l)}{\Gamma(m_2+1)}\cdot\frac{\Gamma\big(\frac{m_2}{q}+\frac{1}{q}\big)}{\Gamma\big(\frac{m_2}{q}+\big(1+\frac{1}{q}\big)l\big)}.$$

Proof. We begin by setting $S^l(z,\zeta)=\sum_m c_m z^m\bar{\zeta}^m$, and we wish to compute the coefficients c_m. Notice that

$$\begin{aligned}\int_{\partial\Omega}S^l(z,\zeta)\zeta^m\,d\mu(\zeta)&=c_m z^m\int_{\partial\Omega}|\zeta^m|^2\,d\mu(\zeta)\\&=c_m\gamma_m z^m.\end{aligned}$$

Recalling that

$$S^l(z,\zeta)=c^l(1-z_1\bar{\zeta}_1)^{-l(1+1/q)}\left(1-\frac{z_2\bar{\zeta}_2}{(1-z_1\bar{\zeta}_1)^{1/q}}\right)^{-2l}$$

and using Lemma 1.6 in [BoLo], it is easy to see that

$$\begin{aligned}A_m^{(l)}&=c_m\gamma_m\\&=c_l'\int_{\partial\Omega}(1-w_1)^{-l(1+1/q)}\left(1-\frac{w_2}{(1-w_1)^{1/q}}\right)^{-2l}\bar{w}_1^{m_1}\bar{w}_2^{m_2}\,d\mu(w)\\&=c_l'\frac{\Gamma(m_2+2l)}{\Gamma(m_2+1)\Gamma(2l)}\cdot\frac{\Gamma\big(m_1+\frac{m_2}{q}+\big(1+\frac{1}{q}\big)l\big)}{\Gamma\big(\frac{m_2}{q}+\big(1+\frac{1}{q}\big)l\big)\Gamma(m_1+1)}\\&\quad\times\frac{\Gamma(m_1+1)\Gamma\big(\frac{m_2}{q}+\frac{1}{q}\big)}{\Gamma\big(m_1+\frac{m_2}{q}+1+\frac{1}{q}\big)}.\end{aligned}$$

This proves the lemma. □

Thus we have shown that, if $f(z)=\sum_m a_m z^m$ is in $H^2(\Omega)$, then

$$\int_{\partial\Omega}S^{\lambda+1}(z,\zeta)f(\zeta)\,d\mu(\zeta)=c_\lambda\sum_m a_m A_m^{(\lambda+1)}z^m. \tag{5}$$

We now define the operators that are our main technical tools.

DEFINITION 2.2. For $\lambda\geq 0$ we define the operators M_λ and $\tilde{M}^\lambda$ acting on holomorphic functions as follows. Let $f(z)=\sum_m a_m z^m$ in $H^2(\Omega)$. Then we set

$$M_\lambda f(z) := \int_{\partial\Omega} S^{\lambda+1}(z,\zeta) f(\zeta)\, d\mu(\zeta) = c_\lambda \sum_m a_m A_m^{(\lambda+1)} z^m \tag{6}$$

and, with some abuse of notation,

$$\tilde{M}^\lambda f(z) := \sum_m \left(m_1 + \frac{m_2}{q} + 1\right)^{(1+1/q)\lambda} \left(\frac{m_2}{q} + 1\right)^{(1-1/q)\lambda} a_m z^m. \tag{7}$$

Notice that, indeed, $\tilde{M}^\lambda \circ \tilde{M}^{\lambda'} = \tilde{M}^{\lambda+\lambda'}$.

We have the following result relating the operators M_λ and $\tilde{M}^\lambda$.

LEMMA 2.3. *Let $\lambda \ge 0$ and let $d\nu$ be a finite measure on Ω that is invariant under the action of $\mathbb{T}^2$ on Ω given by*

$$(w_1, w_2) \mapsto (e^{i\theta_1} w_1, e^{i\theta_2} w_2).$$

Then, for all $p > 0$ there exists $c_p > 0$ such that, for all holomorphic functions $f \in H^2(\Omega)$,

$$\frac{1}{c_p} \int_\Omega |M_\lambda f|^p\, d\nu \le \int_\Omega |\tilde{M}^\lambda f|^p\, d\nu \le c_p \int_\Omega |M_\lambda f|^p\, d\nu.$$

Proof. We set $l := \lambda + 1$ and define

$$\alpha_m := \alpha_m^{(l)} := \frac{A_m^{(l)}}{\left(m_1 + \frac{m_2}{q} + 1\right)^{(1+1/q)(l-1)} \left(\frac{m_2}{q} + 1\right)^{(1-1/q)(l-1)}}. \tag{8}$$

From the invariance of $d\nu$, it follows that

$$\int_\Omega |M_\lambda f|^p\, d\nu = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \int_\Omega |M_\lambda f(e^{i\theta_1} w_1, e^{i\theta_2} w_2)|^p\, d\nu(w_1, w_2)\, d\theta_1\, d\theta_1;$$

the same holds with $\tilde{M}^\lambda$ in place of M_λ. If we integrate first in θ_1, θ_2, we see that the inequalities will follow from the fact that the operator involved on double Taylor series is bounded on $H^p(\mathbb{T}^2)$. It therefore suffices to show that the two sequences $(\alpha_m)_{m\in\mathbb{N}^2}$ and $(\alpha_m^{-1})_{m\in\mathbb{N}^2}$ define two bounded Fourier multipliers of the spaces $H^p(\mathbb{T}^2)$. We shall prove it for the first sequence (the proof for the second is identical).

If we write α_m as a product then we are led to consider sequences of the type

$$\beta_m^{(1)} = \frac{\Gamma(m_2 + 1 + s)}{\Gamma(m_2 + 1)(m_2 + q)^s},$$

$$\beta_m^{(2)} = \frac{\Gamma\left(\frac{m_2}{q} + \frac{1}{q}\right)\left(\frac{m_2}{q} + \frac{1}{q}\right)^s}{\Gamma\left(\frac{m_2}{q} + \frac{1}{q} + s\right)},$$

$$\beta_m^{(3)} = \frac{\Gamma\left(m_1 + \frac{m_2}{q} + 1 + \frac{1}{q} + s\right)}{\Gamma\left(m_1 + \frac{m_2}{q} + 1 + \frac{1}{q}\right)\left(m_1 + \frac{m_2}{q} + 1\right)^s},$$

where in each case s is a positive number related to l and q. It suffices to show that each of these three sequences $(\beta_m^{(j)})$ $(j = 1, 2, 3)$ gives rise to a bounded Fourier multiplier. Let us first look at $(\beta_m^{(j)})$ for $j = 1, 2$. As these sequences depend only on m_2, we may restrict to problems of multipliers on the torus of dimension 1.

We now recall the sufficient condition of Mihlin type, which ensures that the sequence $(\beta_n)_{n\in\mathbb{N}}$ is a Fourier multiplier of $H^p(\mathbb{T})$. We define the difference operators Δ^k by induction, setting $\Delta^0 = \mathrm{Id}$ and $\Delta\beta_n = \beta_n - \beta_{n+1}$. Then the Mihlin condition may be written as

$$|\Delta^k \beta_n| \le C_l (n+1)^{-k}, \quad k = 0, 1, 2, \ldots. \tag{9}$$

If the sequence $(\beta_n)_{n\in\mathbb{N}}$ satisfies (9), then it is a bounded Fourier multiplier of $H^p(\mathbb{T})$ for all $p > 0$ (see [St, pp. 115, 245]). It is easy to prove that the sequence on $\mathbb{N}$ which gives rise to $\beta_m^{(1)}$ satisfies (9), so that it defines a bounded multiplier.

In order to analyze the sequence $\beta_m^{(2)}$, we shall use the following elementary lemma.

Lemma 2.4. *Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence such that, for each $k = 0, 1, \ldots, q-1$, the sequence $(\beta_{qn+k})_{n\in\mathbb{N}}$ is a Fourier multiplier of $H^p(\mathbb{T})$. Then$(\beta_n)_{n\in\mathbb{N}}$ is a Fourier multiplier of $H^p(\mathbb{T})$.*

Proof. Let ω_j $(j = 0, 1, \ldots, q-1)$ denote the qth roots of unity, and for $f \in H^p(\mathbb{T})$ define

$$f_k(z) := \frac{1}{q} \sum_{j=0}^{q-1} \bar{\omega}_j^k f(\omega_j z).$$

Then each f_k is in $H^p(\mathbb{T})$ with norm bounded by the norm of f, and $f = \sum_{k=0}^{q-1} f_k$. Moreover, f_k may be written as $z^k g_k(z^q)$, where g_k has Fourier coefficients given by

$$\hat{g}_k(n) = \hat{f}(nq + k).$$

Let T be the operator given by the multiplier $(\beta_n)_{n\in\mathbb{N}}$. Then Tf_k is given by the action of the multiplier $(\beta_{qn+k})_{n\in\mathbb{N}}$ on g_k. It follows from the hypothesis that Tf_k belongs to $H^p(\mathbb{T})$. Hence, $Tf \in H^p(\mathbb{T})$. □

We may now return to the sequence $(\beta_m^{(2)})$. Using Lemma 2.4, it suffices to show that, for each $k = 0, 1, \ldots, q-1$, the sequence

$$\eta_n^{(k)} = \frac{\Gamma\left(n + \frac{k+1}{q}\right)\left(n + 1 + \frac{k}{q}\right)^s}{\Gamma\left(n + \frac{k+1}{q} + s\right)}$$

satisfies (9). Notice that (by the Stirling formula) we obtain the asymptotics

$$\lim_{n\to\infty} \frac{\Gamma\left(n + \frac{k+1}{q}\right)\left(n + 1 + \frac{k}{q}\right)^s}{\Gamma\left(n + \frac{k+1}{q} + s\right)} = 1,$$

and $\Delta\eta_n^{(k)}$ may be written as $\eta_n^{(k)} \cdot \gamma_n^{(k)}$, with

$$\gamma_n^{(k)} = 1 - \frac{n + \frac{k+1}{q}}{n + \frac{k+1}{q} + s} \cdot \frac{\left(n + 2 + \frac{k}{q}\right)^s}{\left(n + 1 + \frac{k}{q}\right)^s}.$$

Using these two facts, it is elementary to prove that

$$|\Delta^j \gamma_n^{(k)}| \leq C_j (n+1)^{-j},$$

which we wanted to prove to conclude for $(\beta_m^{(2)})$.

It remains to consider the sequence $(\beta_m^{(3)})$. Using the same kind of argument as in Lemma 2.4, we are lead to consider separately the sequences

$$\delta_m^{(k)} = \frac{\Gamma\left(m_1 + m_2 + 1 + \frac{k+1}{q} + s\right)}{\left(m_1 + m_2 + 1 + \frac{k}{q}\right)^s \Gamma\left(m_1 + m_2 + 1 + \frac{k+1}{q}\right)}$$

for $k = 0, 1, \ldots, q-1$. Moreover, using the same arguments as before, it is easy to see that $\delta_m^{(k)} = \eta_{m_1+m_2}$, where $(\eta_n)_{n\in\mathbb{N}}$ is a Fourier multiplier of the spaces $H^p(\mathbb{T})$. In order to conclude the proof, we use the following elementary lemma.

LEMMA 2.5. *Assume that the sequence $(\eta_n)_{n\in\mathbb{N}}$ is a Fourier multiplier of the space $H^p(\mathbb{T})$. Then $(\delta_m)_{m\in\mathbb{N}^2}$, defined by $\delta_m = \eta_{m_1+m_2}$, is a Fourier multiplier of $H^p(\mathbb{T}^2)$.*

Proof. Let $F(\theta_1, \theta_2) = \sum_{m_1\geq 0, m_2 \geq 0} a_{m_1,m_2} e^{i(m_1\theta_1 + m_2\theta_2)}$ be a polynomial. Then

$$F(\theta_1, \theta_2) = G(\theta_1, \theta_2 - \theta_1)$$

with

$$G(\theta_1, \theta_2) = \sum_{n_1 \geq 0} \sum_{0 \leq n_2 \leq n_1} b_{n_1,n_2} e^{i(n_1\theta_1 + n_2\theta_2)}$$

if we choose $b_{n_1,n_2} = a_{n_1-n_2,n_2}$. Multiplication by δ_m for a_{m_1,m_2} becomes multiplication by η_{n_1} for b_{n_1,n_2}. Therefore, the muliplier acts on the first variable for G, and the lemma follows from the fact that F and G have the same norm in $H^p(\mathbb{T}^2)$. □

This ends the proof of Lemma 2.3.

Roughly speaking, Lemma 2.3 shows that the operator M_λ acts on holomorphic functions like $\tilde{M}^\lambda$. In order to understand the action of $\tilde{M}^\lambda$, recall that N is the differential operator $N := z_1 \partial_{z_1} + \frac{z_2}{q} \partial_{z_2}$. For a holomorphic function

$$f(z) = \sum_m a_m z^m,$$

we have the equalities

$$(I + N) f(z) = \sum_m \left(m_1 + \frac{m_2}{q} + 1\right) a_m z^m,$$

$$\left(I + \frac{z_2}{q}\partial_{z_2}\right) f(z) = \sum_m \left(\frac{m_2}{q} + 1\right) a_m z^m.$$

Hence, the action of $\tilde{M}^\lambda$ on holomorphic functions may be seen as the product of fractional powers of $I+N$ and $I+\frac{z_2}{q}\partial_{z_2}$. As it is easier to deal with differential operators than fractional powers, we shall make use of $\tilde{M}^{qk}$ $(k\in\mathbb{N},\ k\geq 1)$. A simple calculation now shows that

$$\begin{aligned}\tilde{M}^{qk}f(z) &= \sum_m \left(m_1+\frac{m_2}{q}+1\right)^{(1+q)k}\left(1+\frac{m_2}{q}\right)^{(q-1)k} a_m z^m \\ &= \left(I+\frac{z_2}{q}\partial_{z_2}\right)^{(q-1)k}(I+N)^{(q+1)k}f(z),\end{aligned}$$

which is a classical differential operator.

We conclude this section by recalling some geometrical facts about our domains and kernels.

As is well known, on the boundary $\partial\Omega$ of Ω there exists a non-isotropic pseudometric d_b (as defined in [NRSW, Def. (1.1)]) for all finite type domains of $\mathbb{C}^2$. In our case, one may use a simple expression for d_b (see [BoLo]). Thus, we set

$$\begin{aligned}d_b(z,w) = {}&\left|\Im\big(\bar{w}_1(w_1-z_1)+q\bar{w}_2|w_2|^{2q-2}(w_2-z_2)\big)\right| \\ &+|w_2|^{2q-2}|w-z|^2+|w-z|^{2q}.\end{aligned}\tag{10}$$

Furthermore, there exists a tubular neighborhood of the boundary $\partial\Omega$ such that each $z\in U$ has a unique normal projection $\pi(z)$ on $\partial\Omega$. For $z,w\in U$ we set

$$d(z,w) := \delta(z)+\delta(w)+d_b(\pi(z),\pi(w)),\tag{11}$$

where, we recall, $\delta(z):=1-|z_1|^2-|z_2|^{2q}$. Moreover, we set

$$\tau(w,r) := \min\{r^{1/2}|w_2|^{1-q},\ r^{1/2q}\}.\tag{12}$$

For the estimates for the Szegö and the Bergman kernels on the diagonal, one has:

$$S(z,z)\simeq(1-|z_1|^2)^{-(1+1/q)}\left(1-\frac{|z_2|^2}{(1-|z_1|^2)^{1/q}}\right)^{-2},\tag{13}$$

$$B(z,z)\simeq(1-|z_1|^2)^{-(2+1/q)}\left(1-\frac{|z_2|^2}{(1-|z_1|^2)^{1/q}}\right)^{-3}.\tag{14}$$

These formulas may be found in [BoLo]. They have also been proved in the context of pseudo-convex domains of finite type by Catlin in [Ca]. In order to recover the well-known Catlin's estimates—that is,

$$\begin{aligned}B(z,z)&\simeq\delta(z)^{-2}\cdot\tau(z)^{-2},\\ S(z,z)&\simeq\delta(z)^{-1}\cdot\tau(z)^{-2},\end{aligned}$$

where, for convenience of notation, we write $\tau(z):=\tau(z,\delta(z))$ (see (12))—it suffices to check that on these ellipsoids one has

$$\tau(z)\simeq\big((1-|z_1|^2)^{1/q}-|z_2|^2\big)^{1/2}\quad\text{and}\quad\delta(z)\simeq(1-|z_1|^2)^{1-1/q}\tau(z)^2.\tag{15}$$

3. Equivalence of Norms

In this section we prove the equivalence of norms in the Besov space—that is, Theorem 1.1. We first prove the inequality on the right in Theorem 1.1. We do this by proving a slightly more general fact. We should emphasize the fact that this part of Theorem 1.1 is valid in the larger context of domains of finite type in $\mathbb{C}^2$ (and of convex domains of finite type in $\mathbb{C}^n$) because it uses only size estimates on the Szegö kernel. The results of Section 2 will only be used in the reverse inequality.

We recall the notation introduced in (6):

$$M_\lambda f(z) = \int_{\partial\Omega} S^{\lambda+1}(z,\zeta)\, f(\zeta)\, d\mu(\zeta).$$

On the ellipsoid Ω we define the differential operator

$$N := z_1\partial_{z_1} + \frac{z_2}{q}\partial_{z_2}, \tag{16}$$

and for $\beta \in \mathbb{R}$ we denote by β^* the number $\beta/2q$ for $\beta \ge 0$ and $\beta/2$ for $\beta < 0$. Recall that

$$\tau^\beta \le c\delta^{\beta^*}.$$

PROPOSITION 3.1. *Let l be a nonnegative integer, $\alpha, \beta \in \mathbb{R}$, $\lambda \ge 0$, and $p > 0$. We assume that the inequalities*

$$\alpha + \beta^* + lp + 1 > 0, \qquad \lambda p + \alpha + 1 + (2\lambda p + \beta)^* > 0$$

are satisfied. Then there exists a constant $c > 0$ such that, for all holomorphic functions f,

$$\int_\Omega |M_\lambda f(z)|^p S^{-p\lambda}(z,z)\delta^\alpha(z)\tau^\beta(z)\, dV(z) \le c\int_\Omega \sum_{0\le k\le l} |\nabla^k f(z)|^p \delta^{lp+\alpha}(z)\tau^\beta(z)\, dV(z).$$

Notice that the conditions on $\alpha, \beta, l, \lambda, p$ are equivalent to the integrability of the weight, and that the right-hand side inequality of Theorem 1.1 corresponds to the case $\alpha = \beta = -2$.

Proof. The proof of Proposition 3.1 is given in three steps. We first give a new expression for $M_\lambda f(z)$, which is obtained by integrations by parts. Then we consider the cases $p > 1$ and $p \le 1$ separately. The new expression is based on the following lemma.

LEMMA 3.2. *Let f, g be holomorphic functions in $H^2(\Omega)$. Then if we set $D_n = N + \left(1 + \frac{n+1}{q}\right)I$ for all nonnegative integer n, the following identities hold:*

$$\int_{\partial\Omega} f(u)\overline{g(u)}\, d\mu(u) = c_0' \int_\Omega f(u)\overline{D_0 g(u)}\, dV(u), \tag{17}$$

$$\int_\Omega f(u)\overline{g(u)}\delta^n(u)\, dV(u) = c_n' \int_\Omega f(u)\overline{D_n g(u)}\delta^{n+1}(u)\, dV(u). \tag{18}$$

Proof. We only prove the identity (17), the proof for (18) being analogous. This identity, once given the Taylor series of f and g, is an easy consequence of (3). Indeed, recall that

$$\|u^m\|^2_{L^2(d\mu)} = c_0 \frac{\Gamma(m_1+1)\Gamma\left(\frac{m_2+1}{q}\right)}{\Gamma\left(m_1+1+\frac{m_2+1}{q}\right)},$$

(see [BoLo]). Write $f(u) = \sum_m a_m u^m$ and $g(u) = \sum_m b_m u^m$. Then

$$\int_{\partial\Omega} f(u)\overline{g(u)}\, d\mu(u) = c_0 \sum_m a_m \bar{b}_m \|u^m\|^2_{L^2(d\mu)}.$$

Moreover, $\|u^m\|^2_{L^2(dV)} = c_1 \Gamma(m_1+1)\Gamma\left(\frac{m_2+1}{q}\right)\left[\Gamma\left(m_1+1+\frac{m_2+1}{q}+1\right)\right]^{-1}$, and

$$\int_{\Omega} f(u)\overline{D_0 g(u)}\, dV(u) = c_0'' \sum_m a_m \bar{b}_m \left(m_1+1+\frac{m_2+1}{q}\right)\|u^m\|^2_{L^2(dV)}.$$

Recalling that $\Gamma(z+1) = z\Gamma(z)$, we obtain (17). $\square$

If we use Lemma 3.2 $k+l$ times, starting from the definition of $M_\lambda f(z)$, we obtain the following.

LEMMA 3.3. *Let $k, l \in \mathbb{N}$. Then there exists $C_{k,l}$ such that, for $f \in H^2(\Omega)$, the following identity holds*:

$$M_\lambda f(z) = C_{k,l} \int_\Omega \bar{D}_{k,\zeta} \cdots \bar{D}_{0,\zeta} S^{\lambda+1}(z,\zeta) D_{k+l} \cdots D_{k+1} f(\zeta)\delta^{k+l}(\zeta)\, dV(\zeta). \quad (19)$$

We now prove Proposition 3.1 for $1 < p < \infty$. Denote by $T_k^{(\lambda)}$ the integral operator defined by

$$T_k^{(\lambda)} g(z) = \delta^\lambda(z)\tau^{2\lambda}(z) \int_\Omega \bar{D}_{k,\zeta} \cdots \bar{D}_{0,\zeta} S^{\lambda+1}(z,\zeta) g(\zeta)\delta^k(\zeta)\, dV(\zeta).$$

It is sufficient to show that, for k large enough and for some constant c, we have

$$\int_\Omega |T_k^{(\lambda)} g(z)|^p \delta^\alpha(z)\tau^\beta(z)\, dV(z) \le c \int_\Omega |g(z)|^p \delta^\alpha(z)\tau^\beta(z)\, dV(z).$$

for all (not necessarily holomorphic) functions g. In order to prove such an estimate, it suffices to show that the integral kernel of the operator $T_k^{(\lambda)}$ on $L^p(\delta^\alpha\tau^\beta dV)$ satisfies the assumptions of Schur's lemma. Notice that the kernel $K_k(z,\zeta)$ has expression

$$K_k(z,\zeta) = \delta^\lambda(z)\tau^{2\lambda}(z)\bar{D}_{k,\zeta} \cdots \bar{D}_{0,\zeta} S^{\lambda+1}(z,\zeta)\delta^{k-\alpha}(\zeta)\tau^{-\beta}(\zeta). \quad (20)$$

From the estimates in [NRSW], we know that

$$|\bar{D}_{k,\zeta} \cdots \bar{D}_{0,\zeta} S^{\lambda+1}(z,\zeta)| \le c d(z,\zeta)^{-2-\lambda-k}\tau(z, d(z,\zeta))^{-2-2\lambda}, \quad (21)$$

where $d(z,u)$ is defined as in (11). We can apply Schur's lemma with the function $\delta^a\tau^b$ by using the following lemma, which relies on the estimates (21).

LEMMA 3.4. *Under the condition*

$$\lambda p + \alpha + 1 + (2\lambda p + \beta)^* > 0,$$

for k large enough one can find a, b such that

$$\int_\Omega |K_k(z,\zeta)|\delta^{ap'+\alpha}(\zeta)\tau^{bp'+\beta}(\zeta)\,dV(\zeta) \le c\delta^{ap'}(z)\tau^{bp'}(z),$$

$$\int_\Omega |K_k(z,\zeta)|\delta^{ap+\alpha}(z)\tau^{bp+\beta}(z)\,dV(z) \le c\delta^{ap}(\zeta)\tau^{bp}(\zeta).$$

Proof. The lemma is an elementary consequence of the following inequality, whose proof may be found in [Sy2]:

$$\int_\Omega d(z,\zeta)^a\tau(z,d(z,\zeta))^b\delta^\alpha(\zeta)\tau^\beta(\zeta)\,dV(\zeta) \le c\delta^{\alpha+a+2}(z)\tau^{\beta+b+2}(z) \tag{22}$$

under the conditions

$$\alpha + \beta^* + 1 > 0 \quad \text{and} \quad -\alpha - a - 2 + (-\beta - b - 2)^* > 0.$$

We recall that from (20) and (21) we have the estimate

$$|K_k(z,\zeta)| \le c\delta^\lambda(z)\tau^{2\lambda}(z)d(z,\zeta)^{-2-\lambda-k}\tau(z,d(z,\zeta))^{-2-2\lambda}\delta^{k-\alpha}(\zeta)\tau^{-\beta}(\zeta).$$

Thus, according to (22), we must show that there exists (a,b) such that

$$-ap' + \lambda + (-bp' + 2\lambda)^* > 0,$$

$$ap + \alpha + \lambda + (bp + \beta + 2\lambda)^* + 1 > 0.$$

If these two conditions are satisfied then the previous two are also satisfied for k large enough.

Now (a,b) satisfies the first condition if it is in a convex cone whose vertex is the point $(\lambda/p', 2\lambda/p')$. Analogously, the second condition is satisfied if (a,b) is inside a convex cone having vertex in (a_0, b_0), where

$$a_0 = -\frac{\alpha+\lambda+1}{p}, \qquad b_0 = -\frac{\beta+2\lambda}{p}.$$

In order for these two regions to intersect, it suffices that the vertex of first cone belongs to the second one:

$$\frac{\alpha}{p'}p + \alpha + \lambda + \left(\frac{2\lambda}{p'}p + \beta + 2\lambda\right)^* + 1 > 0;$$

that is,

$$\lambda p + \alpha + (2\lambda p + \beta)^* + 1 > 0,$$

which is what we wished to show. □

This finishes the proof of Proposition 3.1 in the case $1 < p < \infty$.

Now let $0 < p \le 1$. Let $(w^{(j)})$ denote a sequence of points such that polydiscs of type $Q_j = Q(w^{(j)}, \eta\delta(w^{(j)}))$ give a Whitney covering of Ω as well as

$2Q_j := Q(w^{(j)}, 2\eta\delta(w^{(j)}))$. If we write $g = D_{k+l} \cdots D_{k+1} f(\zeta)$ then we see that $|M_\lambda f(z)|$ is equivalent to the sum over j of

$$d(z, w^{(j)})^{-2-\lambda-k} \tau(z, d(z, w^{(j)}))^{-2-2\lambda} \delta^{k+l}(w^{(j)}) \int_{Q_j} |g(\zeta)|\, dV(\zeta).$$

We have used the fact that all functions $d(z, \cdot), \delta, \tau$ are essentially constant inside Q_j. It follows from the subharmonicity of g (and the fact that $p \leq 1$) that $|M_\lambda f(z)|^p$ can be majorized by a constant times

$$\sum_j d(z, w^{(j)})^{(-2-\lambda-k)p} \tau(z, d(z, w^{(j)}))^{(-2-2\lambda)p} \delta^{(k+l)p}(w^{(j)}) \\ \times \left(\mathrm{Vol}(Q_j)\right)^{p-1} \int_{2Q_j} |g(\zeta)|^p\, dV(\zeta).$$

Now, for k large enough, we can apply inequality (22) to obtain the bound

$$\int_\Omega d(z, w^{(j)})^{(-2-\lambda-k)p} \tau(z, d(z, w^{(j)}))^{(-2-2\lambda)p} \delta^{\lambda p+\alpha}(z) \tau^{2\lambda p+\beta}(z)\, dV(z) \\ \leq c\delta^{-(k+2)p+\alpha+2}(w^{(j)}) \tau^{-2p+\beta+2}(w^{(j)}).$$

From these two last inequalities and using the Whitney property of the covering $2Q_j$, we see that

$$\begin{aligned} &\int_\Omega |M_\lambda f(z)|^p \delta^{\lambda p+\alpha}(z) \tau^{2\lambda p+\beta}(z)\, dV(z) \\ &\quad \leq c \sum_j \delta^{lp+\alpha}(w^{(j)}) \tau^\beta(w^{(j)}) \int_{2Q_j} |g(\zeta)|^p\, dV(\zeta) \\ &\quad \leq c \int_\Omega |\delta(z) g(z)|^p \delta^\alpha(z) \tau^\beta(z)\, dV(z). \end{aligned}$$

This finishes the proof of Proposition 3.1. □

Now we make use of our work in Section 2 relating the operators M_λ and $\tilde{M}^\lambda$. The proof of the next result is an immediate consequence of Lemma 2.3 and Proposition 3.1.

COROLLARY 3.5. *Let l be a nonnegative integer, $\alpha, \beta \in \mathbb{R}$, $\lambda \geq 0$, and $p > 0$. Suppose that the inequalities*

$$\alpha + \beta^* + lp + 1 > 0 \quad \textit{and} \quad \lambda p + \alpha + 1 + (2\lambda p + \beta)^* > 0$$

are satisfied. Then there exists a constant $c > 0$ such that, for all holomorphic functions f,

$$\int_\Omega |\tilde{M}^\lambda f(z)|^p S^{-p\lambda}(z, z) \delta^\alpha(z) \tau^\beta(z)\, dV(z) \\ \leq c \int_\Omega \sum_{0 \leq k \leq l} |\nabla^k f(z)|^p \delta^{lp+\alpha}(z) \tau^\beta(z)\, dV(z).$$

We now prove the left inequality in Theorem 1.1. Assume that

$$\int_{\Omega_q} |M_\lambda g(z)|^p S^{-\lambda p}(z,z)\, B(z,z)\, dV(z) < \infty.$$

Then the same is valid with $\tilde{M}^\lambda$ instead of M_λ. We recall that

$$\tilde{M}^\lambda \circ \tilde{M}^{\lambda'} = \tilde{M}^{\lambda+\lambda'}.$$

Using Corollary 3.5 with λ replaced by $kq - \lambda$, with f replaced by $\tilde{M}^\lambda g$, and with $l = 0$ and α, β suitably chosen, for k large enough we find that

$$\int_{\Omega_q} |\tilde{M}^{kq} g(z)|^p \delta^{kqp-2}(z) \tau^{2kqp-2}\, dV(z) < \infty.$$

Remember that $\tilde{M}^{kq} = \left(I + \frac{z_2}{q}\partial_{z_2}\right)^{(q-1)k}(I+N)^{(q+1)k}$. In order to be able to prove that g is in the Besov space B_p with control of the norm, it suffices to prove the following lemma.

LEMMA 3.6. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta^* + 1 > 0$. Given any $p > 0$, there exists a constant $c > 0$ such that, for all holomorphic functions g,*

$$\int_\Omega |g(z)|^p \delta^\alpha(z) \tau^\beta(z)\, dV(z) \le c \int_\Omega |(I + z_2\partial_{z_2})^k g(z)|^p \delta^\alpha(z) \tau^{\beta+2kp}(z)\, dV(z).$$

Proof. It suffices to prove the lemma for $k = 1$. Using the estimate (15), we have

$$\begin{aligned}\int_\Omega &|g(z)|^p \delta^\alpha(z)\tau^\beta(z)\, dV(z)\\ &\le c \int_{|z_1|<1} (1-|z_1|^2)^{\alpha(1-1/q)}\\ &\qquad \times \int_{|z_2|^2<(1-|z_1|^2)^{1/q}} \left((1-|z_1|^2)^{1/q} - |z_2|^2\right)^{\alpha+\beta/2} |g(z)|^p\, dV(z_2)\, dV(z_1).\end{aligned}$$

Then, by applying Hardy's lemma in the inner integral (see e.g. [AFJP, Thm. 6]), we see that the integral

$$\int_{|z_2|^2<(1-|z_1|^2)^{1/q}} \left((1-|z_1|^2)^{1/q} - |z_2|^2\right)^{\alpha+\beta/2} |g(z)|^p\, dV(z_2)$$

is bounded by a constant times

$$\int_{|z_2|^2<(1-|z_1|^2)^{1/q}} \left((1-|z_1|^2)^{1/q} - |z_2|^2\right)^{\alpha+\beta/2+p} |(I + z_2\partial_{z_2}) g(z)|^p\, dV(z_2).$$

Therefore,

$$\int_\Omega |g(z)|^p \delta^\alpha(z)\tau^\beta(z)\, dV(z) \le c \int_\Omega |(I + z_2\partial_{z_2}) g(z)|^p \delta^\alpha(z) \tau^{\beta+2p}(z)\, dV(z),$$

as follows again from the estimate (15). □

4. Necessary Conditions for Schatten Class Hankel Operators

We now prove Theorem 1.2 for the operator $h_f = h_f^{(\mu)}$. For $1 < p < \infty$ it follows immediately from results in [Sy1, Sec. 4]. We now give another proof, which holds also for $p = 1$. We shall prove that

$$\|h_f\|_{\mathcal{S}_p}^p \geq c \int_\Omega \left| S^{-3}(z,z) \int_{\partial\Omega} S^4(z,\zeta) f(\zeta)\, d\mu(\zeta) \right|^p B(z,z)\, dV(z).$$

Recall that, for $1 \leq p < +\infty$, any bounded operator T, and any orthonormal sequence (a_j), we have that

$$\sum_j |\langle Ta_j, a_j\rangle|^p \leq \|T\|_{\mathcal{S}_p}^p. \tag{23}$$

Moreover, it holds that

$$\|XTY\|_{\mathcal{S}_p} \leq \|X\| \|T\|_{\mathcal{S}_p} \|Y\|, \tag{24}$$

for any bounded operators X, Y.

As before, let $(w^{(j)})$ be a sequence of points such that $Q_j := Q(w^{(j)}, \eta\delta(w^{(j)}))$ is a Whitney covering of Ω and $\tilde{Q}_j = Q(w^{(j)}, \eta\delta(w^{(j)})/C_0)$ are pairwise disjoint. The size of Q_j in the complex transverse direction $N_{w^{(j)}}$ is $\eta\delta(w^{(j)})$, and it is $\eta\tau(w^{(j)})$ in the complex tangential direction (see [Sy1]). Let $B_j = \pi(Q_j)$, where π denotes the normal projection of a tubular neighborhood of the boundary onto the boundary itself. Inside Q_j, the quantity $S^{-3}(z,z)B(z,z)$ is of the order $\mu(B_j)^3 \cdot \mathrm{Vol}(Q_j)^{-1}$. Moreover, using the mean value property, from standard techniques it follows that—for η small enough and α, β fixed in such a way that $\alpha + \beta^* + 1 > 0$—for F holomorphic one has the equivalence

$$\int_\Omega |F(z)|^p \delta^\alpha(z) \tau^\beta(z)\, dV(z) \simeq \sum_j |F(w^{(j)})|^p \delta^\alpha(w^{(j)}) \tau^\beta(w^{(j)})\, \mathrm{Vol}(Q_j) \tag{25}$$

(see [CRW] in the case of the unit disk and [Sy2] for its generalization in our context). In particular, we have

$$\begin{aligned} &\int_\Omega \left| S^{-3}(z,z) \int_{\partial\Omega} S^4(z,\zeta) f(\zeta)\, d\mu(\zeta) \right|^p B(z,z)\, dV(z) \\ &\qquad \leq c \sum_j \mu(B_j)^{3p} \left| \int_{\partial\Omega} S^4(w^{(j)},\zeta) f(\zeta)\, d\mu(\zeta) \right|^p. \end{aligned} \tag{26}$$

If we set $e_j := \mu(B_j)^{3/2} S^2(\cdot, w^{(j)})$, the right-hand side of (26) is equal to

$$c \sum_j \left| \int_{\partial\Omega} h_f(e_j) \overline{e_j}\, d\mu \right|^p = c \sum_j |\langle h_f(e_j), e_j\rangle|^p.$$

Now we claim that the sequence (e_j) is the image under a bounded operator $Y\colon L^2(dV) \to L^2(\partial\Omega)$ of an orthogonal system (a_j) in $L^2(dV)$, with $\|a_j\|_{L^2(dV)} \simeq 1$. Assuming the claim for the moment, we finish the proof.

By Theorem 1.1, (23), (24), and (26), we have that

$$\begin{aligned}\|f\|_{B_p}^p &\le c\int_\Omega \left|S^{-3}(z,z)\int_{\partial\Omega} S^4(z,\zeta)f(\zeta)\,d\mu(\zeta)\right|^p B(z,z)\,dV(z)\\ &\le c\sum_j |\langle Y^* h_f Y(a_j), a_j\rangle|^p\\ &\le c\|Y^* h_f Y\|_{\mathcal{S}_p}^p\\ &\le c\|h_f\|_{\mathcal{S}_p}^p.\end{aligned}$$

Thus, we need only prove the claim. For $g \in L^2(dV)$ we set

$$Yg(\zeta) := \int_\Omega \delta^{1/2}(w)\tau^2(w)S^2(\zeta,w)g(w)\,dV(w).$$

Notice that

$$\begin{aligned}e_j &:= \mu(B_j)^{3/2}S^2(\cdot,w^{(j)})\\ &= \frac{\mu(B_j)^{3/2}}{\mathrm{Vol}(\tilde{Q}_j)}\int_{\tilde{Q}_j} S^2(\cdot,w)\,dV(w)\\ &= \frac{\mu(B_j)^{3/2}}{\mathrm{Vol}(\tilde{Q}_j)}Y(\delta^{-1/2}\tau^{-2}\chi_{\tilde{Q}_j}).\end{aligned}$$

If we define

$$a_j(w) := \frac{\mu(B_j)^{3/2}}{\mathrm{Vol}(\tilde{Q}_j)}\delta^{-1/2}(w)\tau^{-2}(w)\chi_{\tilde{Q}_j}(w),$$

then (a_j) has the required properties to be an orthogonal sequence such that the norms $\|a_j\|_{L^2(dV)} \simeq 1$, since δ and τ are almost constant on $\tilde{Q}_j$.

It remains to show that $Y\colon L^2(dV) \to H^2(\Omega)$ is a bounded operator. Let Y^* be given by

$$Y^*\phi(z) = \delta^{1/2}(z)\tau^2(z)\int_{\partial\Omega} S^2(z,\zeta)\phi(\zeta)\,d\mu(\zeta) = \delta^{1/2}(z)\tau^2(z)M_1\phi(z).$$

It suffices to show that $Y^*\colon H^2(\Omega) \to L^2(dV)$ is bounded, since its Hilbert space adjoint is Y.

It is well known (see [Be, Thm. 1.4]) that $H^2(\Omega)$ can be identified with the space of holomorphic functions ϕ such that

$$\int_\Omega |\nabla\phi(z)|^2\delta(z)\,dV(z) < \infty.$$

The fact that Y^* is bounded follows from Proposition 3.1 with $\alpha = -1$ and $\beta = 0$.

This finishes the proof of Theorem 1.2. □

5. Hankel Operators on Weighted Bergman Spaces

In this section we study the case of weighted Bergman spaces on the complex ellipsoids Ω_q, which we denote by Ω as before, and we prove the analog of Theorems 1.1 and 1.2 in the present context.

Let $\alpha > -1$. We denote by $A^2(\delta^\alpha dV)$ the weighted Bergman space, that is, the closed subspace of $L^2(\delta^\alpha dV)$ consisting of the holomorphic functions. We denote by P_α the weighted Hilbert space orthogonal projection of $L^2(\delta^\alpha dV)$ onto $A^2(\delta^\alpha dV)$. The small Hankel operator with symbol $f \in A^2(\delta^\alpha dV)$ is defined for $g \in L^2(\delta^\alpha dV)$ by setting

$$h_f^\alpha g = P_\alpha\big(f\overline{P_\alpha g}\big).$$

Mutatis mutandis, for the weighted Bergman spaces we have the following analog of Theorem 1.1.

THEOREM 5.1. *Let $p > 0$ and $\alpha > -1$, and let λ be a real number such that $(\lambda - \alpha - 1)p + (2\lambda p)^* + 1 > 0$. Then, for all $g \in A^2(\delta^\alpha dV)$, we have*

$$\|g\|_{B_p}^p \simeq \int_\Omega \left| \int_\Omega \frac{S^{\lambda+1}(z,\zeta)}{S^\lambda(z,z)} \delta(z)^{-\alpha-1} g(\zeta)\delta(\zeta)^\alpha \, dV(z) \right|^p B(z,z)\, dV(z).$$

We now have the following result.

THEOREM 5.2. *Let $1 \le p < \infty$. Then $h_f^\alpha \in \mathcal{S}_p$ is equivalent to $f \in B_p$.*

Proof. For the sufficient condition, the proof given in [Sy3] for finite-type domains in $\mathbb{C}^2$ can be extended to this context.

For the necessary condition we use the relations (23) and (24). We consider the family of holomorphic functions in $A^2(\delta^\alpha dV)$,

$$e_j(z) = \mu(B_j)^{k-1/2}\delta(w^{(j)})^{-(1+\alpha)/2} S^k(z, w^{(j)}),$$

where $k \in \mathbb{N}$ is large enough and $Q(w^{(j)}, \eta\delta(w^{(j)}))$ a Whitney covering of Ω. The function e_j is the image under the operator Y_α of the almost orthonormal family

$$a_j(w) = \frac{\mu(B_j)^{k-1/2}}{\mathrm{Vol}(\tilde{Q}_j)} \delta(w^{(j)})^{-(\alpha+1)/2}\delta(w)^{2-k}\tau(w)^{2(1-k)}\chi_{\tilde{Q}_j}(w).$$

The operator Y_α is defined by

$$Y_\alpha g(z) = \int_\Omega K_\alpha(z,w) g(w)\delta(w)^\alpha \, dV(w),$$

where $K_\alpha(z,w) = \delta(w)^{k-2-\alpha}\tau(w)^{2(k-1)}S^k(z,w)$. We remark that there exists a $c > 0$ such that

$$|K_\alpha(z,w)| \le \frac{c}{d(z,w)^{2+\alpha}\tau(z,d(z,w))^2}.$$

Therefore, Y_α is a bounded operator in $L^2(\delta^\alpha dV)$. As for the proof of Theorem 1.2, we have

$$
\begin{aligned}
\|h_f^\alpha\|_{\mathcal{S}_p}^p &\geq c \sum_j |\langle h_f^\alpha e_j, e_j\rangle_\alpha|^p = \sum_j |\langle f, e_j^2\rangle_\alpha|^p \\
&\geq c \int_\Omega \left| \int_\Omega \frac{S^{\lambda+1}(z,\zeta)}{S^\lambda(z,z)} \delta(z)^{-\alpha-1} g(\zeta)\delta(\zeta)^\alpha \, dV(z) \right|^p B(z,z)\, dV(z) \\
&\geq c\|f\|_{B_p}^p.
\end{aligned}
$$

Hence f is in B_p. □

6. The Case of a Strictly Pseudo-Convex Domain

In this section we denote by $\mathcal{D} \subset \mathbb{C}^n$ a smoothly bounded strictly pseudo-convex domain defined by $\mathcal{D} = \{z \in \mathbb{C}^n, \ \rho(z) < 0\}$, where ρ is a $\mathcal{C}^\infty(\bar{\mathcal{D}})$ strictly plurisubharmonic function and $|\nabla\rho(z)| = 2$ on $\partial\mathcal{D}$. We define as before the surface measure $d\sigma$, the Szegö kernel S, and the Hankel operator h_f. We claim that, in this context, analogous theorems hold true.

Theorem 6.1. *Let $\mathcal{D} \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudo-convex domain. Let $0 < p < +\infty$ and $m \in \mathbb{N}$ such that $m > 1/p$. Then, there exists a constant $c > 0$ so that, for $f \in B_p$,*

$$
\begin{aligned}
\frac{1}{c}\|f\|_{B_p}^p &\leq \int_{\mathcal{D}} \left| \int_{\partial\mathcal{D}} \frac{S^{m+1}(z,w)}{S^m(z,z)} f(w)\, d\sigma(w) \right|^p B(z,z)\, dV(z) + \|f\|_{L^p} \\
&\leq c\|f\|_{B_p}^p.
\end{aligned}
$$

Theorem 6.2. *Let $\mathcal{D} \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudo-convex domain. Let $1 \leq p < +\infty$. Then $h_f \in \mathcal{S}_p$ is equivalent to $f \in B_p$.*

Let us first fix some notation. Let $N := \sum_{i=1}^n \partial_{\bar{z}_i}\rho(z)\partial_{z_i}$ be the complex normal direction and let

$$
\partial^\gamma := \frac{\partial^{|\gamma|}}{\partial w_1^{\gamma_1} \cdots \partial w_n^{\gamma_n}} \quad \text{where } \gamma = (\gamma_1, \dots, \gamma_n).
$$

Notice that, by our normalization, $N\rho = 1$ on $\partial\mathcal{D}$. As on the ellipsoids, we define $\delta := -\rho$, we denote by d_b the Koranyi distance on the boundary, and, as in (11), we write

$$
d(z,w) := \delta(z) + \delta(w) + d_b(\pi(z), \pi(w)) \tag{27}
$$

for z, w in a neighborhood of $\partial\mathcal{D}$. The proof of Theorem 6.1 is based on the asymptotic expansion of the Szegö kernel, as obtained by Fefferman [Fe]. We recall that there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that, for $\rho(w) < \delta_0$ and $|z - w| < \varepsilon_0/2$,

$$
S(z,w) = a(z)\Psi(z,w)^{-n} + E_0(z,w), \tag{28}
$$

where $a := C_{\mathcal{D}} \det L_\rho$ and

$$\Psi(z,w)$$
$$= -\rho(w) - \sum_{i=1}^{n} \partial_{w_i}\rho(w)(z_i - w_i) - \frac{1}{2}\sum_{i,k} \partial^2_{w_i w_k}\rho(w)(z_i - w_i)(z_k - w_k).$$

Moreover, if Δ is the boundary diagonal, then E_0 is in $\mathcal{C}^\infty(\bar{\mathcal{D}} \times \bar{\mathcal{D}} \setminus \Delta)$ and, for every multi-index γ, satisfies the estimates

$$|\nabla^\gamma E_0(z,w)| \le c_\gamma d(z,w)^{-n+1/2-|\gamma|}. \tag{29}$$

Here derivatives are taken in the z or w variable.

We use identity (28) to give a similar description for the kernel $S^{m+1}(z,w)$, $m \ge 1$. For m a positive integer we define the function $a_m(z)$ by setting

$$a_m(z) := \frac{(nm-1)!}{(n-1)!} a(z)^{n-1}.$$

It is immediate that a_m does not vanish in $\bar{\mathcal{D}}$. We define the kernel $E_m(z,w)$ by setting

$$E_m(z,w) = S^{m+1}(z,w) - a_m(z) N_z^{nm} S(z,w).$$

The following proposition gives a pointwise estimate for $E_m(z,w)$ and its derivatives.

PROPOSITION 6.3. *Let m be a positive integer. Then $E_m \in \mathcal{C}^\infty(\bar{\mathcal{D}} \times \bar{\mathcal{D}} \setminus \Delta)$. Moreover, for every multi-index γ, there exists $c_\gamma > 0$ such that for $(z,w) \in \bar{\mathcal{D}} \times \bar{\mathcal{D}} \setminus \Delta$,*

$$|\bar{\partial}_w^\gamma E_m(z,w)| \le c_\gamma d(z,w)^{-n(m+1)+1/2-|\gamma|}.$$

Proof. We prove the estimate for $\gamma = 0$. It suffices to prove that, for $\rho(w) < \delta_0$ and $|z-w| < \varepsilon_0/2$,

$$N_z^{nm} S(z,w) = a_m(z)^{-1} a(z)\Psi(z,w)^{-(m+1)n} + E_m^{(1)}(z,w),$$

with

$$|E_m^{(1)}(z,w)| \le c d(z,w)^{-n(m+1)+1/2}.$$

When computing $N_z^{nm} S(z,w)$, we find derivatives of the error term $E_0(z,w)$, which are directly majorized using (29) and are part of $E_m^{(1)}(z,w)$, as well as derivatives of the main term. Differentiating each time, the denominator gives

$$(-1)^{nm} a_m(z)^{-1} a(z)(N_z\Psi(z,w))^{nm}\Psi(z,w)^{-n(m+1)}$$

while the other derivatives are also majorized by $cd(z,w)^{-n(m+1)+1/2}$. It remains to show that

$$N_z\Psi(z,w) = -1 + O(d(z,w)^{1/2}).$$

This follows from the fact that $N_z\Psi$ is a smooth function that is identically -1 on Δ. Indeed, $N_z\Psi(z,w) + 1$ is bounded, up to a constant, by the distance of (z,w) to Δ, which in turn is bounded by $c(\delta(z) + \delta(w) + |\pi(z) - \pi(w)|)$. Then we use the definition of $d(z,w)$ and the well-known fact that, on the boundary,

$$|z - w| \leq c d_b(z, w). \tag{30}$$

This concludes the proof for $\gamma = 0$. The same proof holds in the general case. □

Let us now give the idea of the proof of Theorem 6.1 in the context of strictly pseudo-convex domains. Let us define the operator M_m as before,

$$M_m f(z) = \int_{\partial\mathcal{D}} S^{m+1}(z, w) f(w)\, d\sigma(w).$$

From the definition of E_m, it follows that

$$M_m f(z) = a_m(z) N^{nm} f(z) + E_m f(z), \tag{31}$$

where the operator E_m defined by

$$E_m f(z) = \int_{\partial\mathcal{D}} E_m(z, w) f(w)\, d\sigma(w)$$

for f in $L^1(\partial\mathcal{D})$. It is well known [Be, Thm. 1.1] that

$$\|f\|_{B_p}^p \simeq \int_{\mathcal{D}} |N^{nm} f(z)|^p \delta(z)^{nmp} B(z, z)\, dV(z) + \|f\|_{L^p}^p. \tag{32}$$

We shall show that the error term $E_m f$ is small compared to $N^{nm} f$. More precisely, the following proposition holds true.

PROPOSITION 6.4. *Let $0 < p < +\infty$, $m \in \mathbb{N}$, and $\alpha \in \mathbb{R}$ such that $\alpha + pmn + 1 > 0$. Then there exists $c > 0$ such that, for $f \in B_p$,*

$$\int_{\mathcal{D}} |E_m f(z)|^p \delta(z)^{pmn+\alpha}\, dV(z) \leq c \int_{\mathcal{D}} \sum_{|\gamma| \leq nm} |\partial^\gamma f(z)|^p \delta(z)^{pmn+p/2+\alpha}\, dV(z).$$

Let us take this proposition for granted, and prove Theorem 6.1. The right inequality is obtained directly, using equality (31) and the estimate (32).

In order to prove the bound from below, we use (31), (32), Proposition 6.4, and the fact that a_m is bounded below to see that

$$\begin{aligned}\|f\|_{B_p}^p \leq{}& c \int_{\mathcal{D}} |M_m f(z)|^p S(z, z)^{-pm} B(z, z)\, dV(z) + \|f\|_{L^p}^p \\ &+ c' \int_{\mathcal{D}} \sum_{|\gamma| \leq nm} |\partial^\gamma f(z)|^p \delta(z)^{nmp+p/2} B(z, z)\, dV(z).\end{aligned}$$

Let $\mathcal{D}_\varepsilon := \{ z \in \mathcal{D} : \delta(z) > \varepsilon \}$. If we choose ε so that $c'\delta(z)^{p/2}$ is small enough in $\mathcal{D} \setminus \mathcal{D}_\varepsilon$, then

$$c' \int_{\mathcal{D}\setminus\mathcal{D}_\varepsilon} \sum_{|\gamma| \leq nm} |\partial^\gamma f(z)|^p \delta(z)^{nmp} B(z, z)\, dV(z) \leq \frac{1}{2} \|f\|_{B_p}^p,$$

while

$$\int_{\mathcal{D}_\varepsilon} \sum_{|\gamma| \leq nm} |\partial^\gamma f(z)|^p \delta(z)^{nmp+p/2} B(z, z)\, dV(z) \leq c \|f\|_{L^p}^p.$$

We then have

$$\|f\|_{B_p}^p \le c \int_{\mathcal{D}} |S^m f(z)|^p S(z,z)^{-mp} B(z,z)\, dV(z) + \|f\|_{L^p}^p + \frac{1}{2}\|f\|_{B_p}^p.$$

In order to finish the proof of Theorem 6.1, we need only prove Proposition 6.4.

Proof of Proposition 6.4. The method is the same as for the proof of Proposition 3.1. It is also given in three steps. We first give a new expression for $E_m f(z)$ which is obtained by integrations by parts. Then we consider the cases $p > 1$ and $p \le 1$ separately. The new expression is based on the following lemma.

LEMMA 6.5. *Let $z \in \mathcal{D}$ and $k, l \in \mathbb{N}$ with $l > 0$. Then there exist $b_{\gamma,\gamma'} \in \mathcal{C}^\infty(\bar{\mathcal{D}})$ such that, for $f \in H^2(\mathcal{D})$, one has*

$$E_m f(z) = \sum_{|\gamma| \le k} \sum_{|\gamma'| \le l} \int_{\mathcal{D}} \bar{\partial}^\gamma E_m(z,w) b_{\gamma,\gamma'}(w) \partial^{\gamma'} f(w) \delta(w)^{k+l-1}\, dV(w).$$

Proof. Since $N\rho = 1$ on $\partial\mathcal{D}$, we have that

$$E_m f(z) = \sum_{i=1}^n \int_{\partial\mathcal{D}} E_m(z,w) f(w) \partial_{\bar{w}_i} \rho(w) \partial_{w_i} \rho(w)\, d\sigma(w).$$

The function $w \mapsto E_m(z,w)$ is an anti-holomorphic function, so Stokes's formula gives

$$E_m(z,w) = \int_{\mathcal{D}} E_m(z,w) \left(\frac{\Delta\rho(w) f(w)}{4} + Nf(w) \right) dV(w).$$

Now we use the fact that there exist $a, b \in \mathcal{C}^\infty(\bar{\mathcal{D}})$ such that

$$1 = a(w) N\rho(w) + (-\rho(w)) b(w) = a(w) \bar{N}\rho(w) + (-\rho(w)) b(w).$$

The lemma is obtained after $k-1$ integrations by parts with respect to w and l integrations by parts with respect to $\bar{w}$ (see [Sy2] for details). □

Let $1 < p < +\infty$. To prove Proposition 6.4, we use Lemma 6.5 with $l = mn$ and estimate each term. In order to do this, it suffices to prove that the operator K_k defined by

$$K_k g(z) = \delta(z)^{mn} \int_{\mathcal{D}} d(z,w)^{-(m+1)n-k+1/2} \delta(w)^{k-3/2} g(w)\, dV(w)$$

is bounded on $L^p(\delta^\alpha dV)$ for k large enough. As usual, we use Schur's lemma with the function δ^{-a}. We shall not give the details, which rely on the analog of (22) in this context (i.e.,

$$\int_{\mathcal{D}} d(z,w)^a \delta^\alpha(w)\, dV(w) \le c\delta^{a+\alpha+n+1}(z)$$

under the conditions $\alpha > -1$ and $a + \alpha + n + 1 < 0$).

For $0 < p \le 1$, as for Proposition 3.1 on ellipsoids, we consider a sequence of points $(w^{(j)})$ in $\mathcal{D}$ such that the polydiscs $Q_j = Q(w^{(j)}, \eta\delta(w^{(j)}))$ give a Whitney covering of $\mathcal{D}$ and then proceed in the same way.

This finishes the proof of the proposition, and therefore also the proof of Theorem 6.1.

Proof of Theorem 6.2. We proceed as in Section 4. We need only prove that the integral operator $Y\colon L^2(dV) \to H^2(\mathcal{D})$ defined by

$$Y\psi(z) = \int_{\mathcal{D}} \delta(z)^{n-1/2} S^2(z,w)\psi(w)\,dV(w)$$

is bounded, or that its adjoint Y^* (obtained formally), given by

$$\begin{aligned} Y^*\phi(z) &= \delta(z)^{n-1/2} \int_{\partial\mathcal{D}} S^2(z,\zeta)\phi(\zeta)\,d\sigma(\zeta) \\ &= a_1(z)\delta(z)^{n-1/2} N^n\phi(z) + \delta(z)^{n-1/2} E_1\phi(\zeta), \end{aligned}$$

is bounded from $H^2(\mathcal{D})$ to $L^2(dV)$. It is well known that, for holomorphic functions (see [Be, Thm. 1.4]),

$$\int_{\mathcal{D}} (\delta(z)^n |N^n\phi(z)|)^2 \frac{dV(z)}{\delta(z)} \le c\|\phi\|^2_{L^2(\partial\mathcal{D})}.$$

For the second term we use Proposition 6.4 with $\alpha = -1$. □

We could also generalize these results to the case of weighted Bergman spaces. We shall not go into details.

7. Hankel Operators Related to the Surface Measure on Ellipsoids

In this section we go back to the complex ellipsoids Ω. We prove the analog of Theorem 1.1 when the measure $d\mu$ is replaced by the surface measure $d\sigma$, as well as the analog of Theorem 1.2 in this context. We now give the new statement.

THEOREM 7.1. *Let $p > 0$ and let m be an integer such that $m > 1/p$. Then there exists $c = c(p,m) > 0$ such that, for all $g \in B_p$, we have*

$$\frac{1}{c}\|g\|^p_{B_p} \le \int_{\Omega} \left| \int_{\partial\Omega} \frac{S^{m+1}_\sigma(z,\zeta)}{S^m_\sigma(z,z)} g(\zeta)\,d\sigma(\zeta) \right|^p B(z,z)\,dV(z) \le c\|g\|^p_{B_p}.$$

We denote by λ the $\mathcal{C}^\infty$ function that gives the density $d\mu/d\sigma$. We shall use the same method as in the previous section, Fefferman's asymptotic expansion being replaced by the fact that the projection P^σ_S can be approximated by P^μ_S. For this we use the Kerzman–Stein trick (as used in [BoLo] in this context, or in [NRSW]). From now on, the scalar product in $H^2(\Omega)$ is defined using the surface measure, and we also refer to the surface measure when we speak of the kernel of an operator. For instance, the kernel of $P^{(\mu)}_S$ is $S_\mu(z,w)\lambda(w)$, while the kernel of $(P^{(\mu)}_S)^*$ is $\lambda(z)S_\mu(z,w)$. From elementary properties of projections, it follows that

$$P^{(\sigma)}_S = (P^{(\mu)}_S)^* + \big(P^{(\mu)}_S - (P^{(\mu)}_S)^*\big) \circ P^{(\sigma)}_S.$$

It follows from the theory of non-isotropic smoothing operators in [NRSW] that the second term is a smoothing operator. More precisely, one has

$$S_\sigma(z,w) = \lambda(z) S_\mu(z,w) + E_0^{(1)}(z,w) \tag{33}$$

with $E_0^{(1)} \in \mathcal{C}^\infty(\bar\Omega \times \bar\Omega \setminus \Delta)$, which is anti-holomorphic in w and satisfies the following estimates for every multi-index γ:

$$|\bar\partial_w^\gamma E_0^{(1)}(z,w)| \le c_\gamma d(z,w)^{-1-|\gamma|} \tau(z, d(z,w))^{-1}. \tag{34}$$

We can likewise write

$$S_\sigma(z,w) = S_\mu(z,w)\lambda(w) + E_0^{(2)}(z,w) \tag{35}$$

with $E_0^{(2)} \in \mathcal{C}^\infty(\bar\Omega \times \bar\Omega \setminus \Delta)$, which satisfies the following estimates for all multi-indices γ, γ':

$$|\bar\partial_w^\gamma \partial_w^{\gamma'} E_0^{(2)}(z,w)| \le c d(z,w)^{-1-|\gamma|} \tau(z, d(z,w))^{-1}. \tag{36}$$

We define

$$E_m(z,w) = S_\sigma(z,w)^{m+1} - \lambda(z)^m S_\mu(z,w)^{m+1} \lambda(w).$$

We then obtain the following proposition, which is the analog of Proposition 6.3 in this context.

PROPOSITION 7.2. *Let m be a positive integer. Then, $E_m \in \mathcal{C}^\infty(\bar\Omega \times \bar\Omega \setminus \Delta)$. Moreover, for all multi-indices γ, γ' there exists a $c > 0$ such that, for $(z,w) \in \bar\Omega \times \bar\Omega \setminus \Delta$,*

$$|\bar\partial_w^\gamma \partial_w^{\gamma'} E_m(z,w)| \le c d(z,w)^{-(m+1)-|\gamma|} \tau(z, d(z,w))^{-m}.$$

We remark that the only difference with Proposition 6.3 is the fact that this time the kernel $E_m(z,w)$ is no longer anti-holomorphic in w. The important point here is that, nevertheless, the estimates do not depend on the multi-index γ'.

We consider the operator $M_m^{(\sigma)}$ defined on holomorphic funtions by

$$M_m^{(\sigma)} f(z) = \int_{\partial\Omega} S_\sigma(z,w)^{m+1} f(w)\, d\sigma(w).$$

Then

$$M_m^{(\sigma)} f(z) = \lambda(z)^m M_m f(z) + E_m f(z),$$

where M_m is the operator defined in Section 2 in relation with the measure $d\mu$, and

$$E_m f(z) = \int_{\partial\Omega} E_m(z,w) f(w)\, d\sigma(w).$$

In order to prove Theorem 7.1 it suffices to estimate the remainder.

PROPOSITION 7.3. *Let l be a nonnegative integer, $\alpha, \beta \in \mathbb{R}$, $m \in \mathbb{N}$, and $p > 0$. We assume that the inequalities*

$$\alpha + \beta^* + lp + 1 > 0, \qquad mp + \alpha + 1 + (2mp + \beta)^* > 0$$

are satisfied. Then there exists a constant $c > 0$ such that, for all holomorphic functions f,

$$\int_\Omega |E_m f(z)|^p S^{-pm}(z,z)\delta^\alpha(z)\tau^\beta(z)\,dV(z) \le c\int_\Omega \sum_{0\le k\le l} |\nabla^k f(z)|^p \delta^{mp+\alpha}(z)\tau^{\beta+1}(z)\,dV(z).$$

Proof. We use integrations by parts as for strictly pseudo-convex domains. Consider the function

$$\rho(z) = \frac{|z_1|^2 + |z_2|^{2q} - 1}{(|z_1|^2 + q^2|z_2|^{4q-2})^{1/2}},$$

so that $|\nabla\rho| = 1$ on $\partial\Omega$, and define a new complex normal vector field by $N^{(\sigma)} := \sum_{i=1}^n \partial_{\bar z_i}\rho(z)\partial_{z_i}$. Then, if we integrate by parts as in the proof of Lemma 6.5, we obtain its analog as follows.

LEMMA 7.4. *Let $z \in \Omega$ and $k, l \in \mathbb{N}$ with $l > 0$. Then there exist $b_{\gamma,\gamma',\gamma''} \in \mathcal{C}^\infty(\bar\Omega)$ such that, for every $f \in H^2(\Omega)$, one has*

$$E_m f(z) = \sum_{|\gamma|\le k}\sum_{|\gamma'|\le l}\sum_{|\gamma''|\le l}\int_\Omega \bar\partial_w^\gamma \partial_w^{\gamma'} E_m(z,w) b_{\gamma,\gamma',\gamma''}(w)\partial^{\gamma''} f(w)\delta(w)^{k+l-1}\,dV(w).$$

Once this lemma is given, we proceed as in Section 3 (Proposition 3.1). We write $E_m f$ using Lemma 7.4 and obtain control of each term by the same method. We shall not give the details. □

The proof of Theorem 7.1 follows from Proposition 7.3 as in Section 6.

It remains to prove Theorem 1.2 when $h_f = h_f^{(\sigma)}$. We use the same proof as before. We are led to consider the operator Y^* given by

$$\begin{aligned} Y^*\phi(z) &= \delta^{1/2}(z)\tau^2(z)\int_{\partial\Omega} S_\sigma^2(z,\zeta)\phi(\zeta)\,d\sigma(\zeta) \\ &= \lambda(z)^m\delta^{1/2}(z)\tau^2(z)M_1\phi(z) + \delta(z)^{n-1/2}E_1\phi(\zeta). \end{aligned}$$

Here M_1 is the operator related to the measure $d\mu$, and we already know (from Section 4) that it gives a bounded operator. The rest of it is a consequence of Proposition 7.3.

FINAL REMARKS. We point out that our results are also valid for Hankel operators on Hardy and Bergman spaces on the ellipsoids in $\mathbb{C}^n$ of the form

$$\Omega = \{z = (z', z_n) \in \mathbb{C}^{n-1}\times\mathbb{C} : |z'|^2 + |z_n|^{2q} < 1\}.$$

The main point is that in this case there is also an explicit formula for the Szegö kernel, which allows the same kind of computations. As we said, in all these cases, the powers of the Szegö kernel act as fractional pseudo-differential operators. The structure of the points of non–strict pseudo-convexity, and the symmetries of the domain, play a fundamental role to etablish this point. It is clearly very difficult to have a conjecture for more general domains.

We use other methods to characterize the boundedness and the compactness of Hankel operators in the forthcoming paper [BPS].

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