Learning Matters: Reappraising object allocation rules when agents strategically investigate[1]

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Abstract

Individuals form preferences through search, interviews, discussion, and investigation. In a stylized object allocation model, we characterize the equilibrium learning strategies induced by different allocation rules and trace their welfare consequences. Our analysis reveals that top trading cycles rules dominate serial priority rules under inequality-averse measures of social welfare.

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Running Head: Learning Matters.

1 Introduction

Individuals go to great lengths to investigate the value of goods that they may consume: firms interview candidates, students visit colleges, consumers test drive cars, and so on. How do these decisions interact with allocation rules? Toward an answer, we directly model the preference formation stage as a strategic game.[1] We model a stylized object allocation problem in which individuals begin with common expected values and each has an opportunity to learn the true personalized value of one object. Within this framework, we evaluate two prominent families of assignment rules: serial priority rules and top trading cycles (TTC) rules. Our model delivers two emphatic conclusions. First, learning incentives differ markedly across allocation rules and equilibrium strategies depend crucially on their details . Second, all TTC rules dominate all priority rules according to a range of inequality-averse social welfare functions and are equivalent in the utilitarian sense.

[1]Our approach departs from standard practice of the literature on object allocation and moves toward standard practice of the mechanism design literature where information acquisition has become de rigueur (e.g., Bergemann and Välimäki (2002)).

An example: visiting colleges

To motivate our investigation and illustrate our results, we discuss an extended example. Suppose there are three colleges, each with one available seat: an ivy league school, a state school, and a technical school. In expectation, students rank them in that order. The actual value a student receives by attending each school is uncertain but can be learned by visiting. Concretely, a visit reveals whether the school is a *good fit* or a *bad fit* for the student. Unfortunately, time constraints limit the student to one trip, creating a decision problem: which campus should he visit?

Each school is a good or bad fit for each student independently with equal probability. Table 1 summarizes the relevant utility information.

School	Good fit	Bad fit	Expectation
Ivy	14	4	9
State	12	2	7
Technical	10	0	5

Table 1: Utilities from each type of school in expectation and when realized as a good or bad fit.

First consider a single student free to enroll at any school. If visiting a school reveals it to be a good fit for the student, then he will choose it over each of the other schools, about which he knows only the expected value. If visiting a school reveals it to be a bad fit, then the student will prefer either of the remaining schools. The student has three options: visit the ivy, state, or technical school. If he chooses to visit the ivy school, his expected utility is

$$Pr(\text{good fit})u(\text{ivy}|\text{good}) + Pr(\text{bad fit})\mathbf{E}[u(\text{state})]$$

$$= \tfrac{1}{2} \cdot 14 + \tfrac{1}{2} \cdot 7 = 10.5.$$

Similar calculations show that his expected utility from visiting the state school is also 10.5 and from visiting the technical school is 9.5. Comparing these, the student optimally visits either the ivy or state school.

Next, suppose there is a second student who must simultaneously decide where to visit. Visitation decisions are now more complex: each student must consider the visitation decision of the other student, as well as how the schools will resolve potential conflict. To understand these incentives, we compare assignment rules.

Priority rule. Suppose the first student has higher priority at all schools, meaning that he is able to attend his most preferred school regardless of the other student's preferences. This student may proceed as if facing a one-student problem, but the second student's problem is more complicated. How should he respond?

The second student expects the first student to choose one of his optimal visitation strategies. As we saw above, the two strategies of the first student have the same implications for the second student: between the ivy and the state schools precisely one will remain available to him as the first student will take the other. If the second student visits either of these schools and it is unavailable, he will have "wasted" his opportunity to visit. To see this, suppose that he visits the ivy school. It is either available or unavailable to him with equal probability. So his expected utility is

$$Pr(\text{ivy unavailable})\mathbf{E}[u(\text{state})]+$$
$$Pr(\text{ivy available})\big(Pr(\text{good fit})u(\text{ivy}|\text{good}) + Pr(\text{bad fit})\mathbf{E}[u(\text{technical})]\big)$$

$$= \tfrac{1}{2}\cdot 7 + \tfrac{1}{2}\,(\tfrac{1}{2}\cdot 14 + \tfrac{1}{2}\cdot 5) = 8.25.$$

A similar calculation shows that his expected utility from visiting the state school is 8.75. If instead he visits the technical school, his visit always reveals useful information: if it is a good fit, he enrolls; if it is a bad fit, he opts for whichever of the other schools is available. His expected utility is

$$Pr(\text{good fit})u(\text{technical}|\text{good})+$$
$$Pr(\text{bad fit})\big(Pr(\text{ivy available})\mathbf{E}[u(\text{ivy})] + (1 - Pr(\text{ivy available}))\mathbf{E}[u(\text{state})]\big)$$

$$= \tfrac{1}{2}\cdot 10 + \tfrac{1}{2}\,(\tfrac{1}{2}\cdot 9 + \tfrac{1}{2}\cdot 7) = 9.$$

The second student optimally visits the technical school. He prefers to visit a school that is certain to be available, even though it has a lower expected value. Altogether, the students' expected utilities in equilibrium are 10.5 and 9 respectively.

Endowment (TTC) rule. Suppose the first student has higher priority at the ivy school, but the second student has higher priority at the state school. Intuitively, these priorities give each student a right-of-refusal at the school where he has higher priority, effectively endowing him with the seat at that school.

We suppose that the first student visits the ivy school[2] and focus on the second student. Given his priority, the second student knows that the state school will always be available to him. In contrast, the ivy school will be available only if the first student finds it to be a bad fit. This suggests that visiting the state school, the second student's "endowment", is optimal. To confirm this, first suppose he visits the ivy school. If the ivy school is available, he enrolls if he finds it to be a good fit and otherwise chooses the state school. If the ivy school is unavailable, he compares the state and technical schools by expected value and chooses the state school. His expected utility is

$$Pr(\text{ivy available})\big(Pr(\text{good fit})u(\text{ivy}|\text{good}) + Pr(\text{bad fit})\mathbf{E}[u(\text{state})]\big)$$
$$+Pr(\text{ivy unavailable})\mathbf{E}[u(\text{state})])$$

$$= \tfrac{1}{2}\,(\tfrac{1}{2}\cdot 14 + \tfrac{1}{2}\cdot 7) + \tfrac{1}{2}\cdot 7 = 8.75.$$

On the other hand, if he visits the state school, he is able to enroll whenever he finds it to be a good fit. If he finds it to be a bad fit, he chooses the remaining school. His expected utility is then

$$Pr(\text{good fit})u(\text{state}|\text{good})+$$
$$Pr(\text{bad fit})\big(Pr(\text{ivy available})\mathbf{E}[u(\text{ivy})] + Pr(\text{ivy unavailable})\mathbf{E}[u(\text{technical})]\big)$$

$$= \tfrac{1}{2}\cdot 12 + \tfrac{1}{2}\,(\tfrac{1}{2}\cdot 9 + \tfrac{1}{2}\cdot 5) = 9.5.$$

[2]In fact, as further computation shows, this is his dominant strategy.

The technical school is also guaranteed to be available and analogous computation shows the second student's expected utility when visiting it is 9. Thus, the second student optimally visits the state school and obtains an expected utility of 9.5. Reasoning similarly, the first student's equilibrium expected utility is 10.

Comparing the rules, the change in priority at the state school moves it from the first student's effective endowment to the second student's effective endowment. As a result, the second student changes his visitation strategy and is also better off. This illustrates our central conclusion: compared with priority rules, TTC rules lead to more equal distributions of utility.[3]

Preview of results

We elaborate on the features of our model, all of which appear in the example. First, objects have common expected values. This is appropriate when individuals have access to similar information or consult the same source.[4] We model values as the common expected value plus an idiosyncratic private value. With this specification, we are able to capture private values that follow essentially any symmetric distribution, including uniform and normal distributions, as well as binary "good-news/bad-news" distributions as in the example. Beyond symmetry, we assume that private values are independent and identically distributed across individuals and objects. Thus, as in the example, whether one individual is a good or bad fit at one

[3] That the sums of the expected utilities are equal ($9 + 10.5 = 9.5 + 10$) also generalizes; the rules are never Pareto-comparable but instead entail a redistribution of expected utility.

[4] For example, many college-bound seniors use the ranking of *US News and World Reports* as a baseline. More prosaically, we assume a common prior.

school provides no information about whether he is a good or bad fit at another school, nor whether a different student is a good or bad fit at the original school.

Each individual has access to a learning technology that permits him to learn the realization of one private value. We interpret this as a time constraint. Although simple, this technology allows us to distinguish among rules according to the learning incentives they provide. Moreover, by abstracting from cost considerations, we are able to isolate incentives regarding *what* to learn rather than *whether* or *how much* to learn.

After investigation, individuals report their preference rankings over objects to a central authority that then applies a pre-determined rule to allocate the objects. This induces a two-stage game:

- Investigation: each individual chooses an object to investigate.
- Reporting: each individual reports preferences to the central authority.

Choices at each stage are simultaneous. In the first stage, individuals choose strategies to maximize their expected utility given the strategies of others. As our interest is in the investigation stage, we restrict attention to *strategy-proof* rules. As each individual then has a dominant strategy to truthfully report his preferences, we abstract from this stage and assume truthful reports. Finally, while our model includes cardinal utility information, the rules that we study are ordinal, thereby retaining comparability with the standard model. Cardinal utilities permit expected utility computations and allow us to consider measures of welfare that account for investigation.

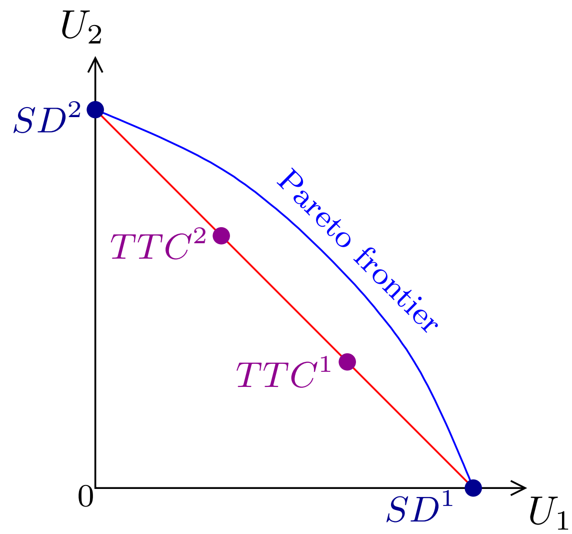

Figure 1: Welfare comparisons. The figure shows each of two agents' ex-ante utility under the two priority rules and the two TTC rules. We mark the ex-ante utility profile under the first priority rule (which favors agent 1) by SD^1 and that under the second one by SD^2. Similarly, we mark the ex-ante utility profile under the first TTC rule (which endows agent 1 with the ex-ante best object) by TTC^1 and that under the second one by TTC^2. The horizontal axis measures the ex-ante utility of agent 1 and the vertical axis measures the ex-ante utility of agent 2. As shown: (1) the utilitarian welfare is the same for each of these rules; (2) the utility profile under each TTC rule Lorenz dominates the utility profile under either priority rule; and (3) the priority rules are Pareto efficient whereas the TTC rules are not.

To demonstrate the importance of learning incentives, we consider two prominent families of rules: priority rules, each parameterized by a priority order; and top trading cycles (TTC) rules, each parameterized by an endowment profile. We first characterize the (Nash) equilibria of the learning game induced by each type of rule (Proposition 1). Importantly, we show that all equilibria under a given rule are unique in welfare terms. With equilibria in hand, we turn to welfare comparisons. Our main result is that each TTC rule Lorenz dominates each priority rule, meaning that TTC rules improve the welfare of those least well off (Theorem 1). Consequently, TTC rules also achieve higher social welfare according to every inequality-averse social welfare function (Corollary 1). In turn, this implies that TTC rules achieve higher welfare under progressive measures including the Rawlsian max-min social welfare function. In terms of utilitarian welfare, the rules are equivalent (Proposition 2). On the other hand, priority rules are Pareto-efficient whereas TTC rules may not be (Proposition 3). Figure 1 illustrates our primary conclusions. Overall, our analysis provides the first formal results confirming the strong intuition that TTC rules are more fair than priority rules.

In discrete allocation problems, it is impossible to treat agents equally, an issue often addressed by randomization. In our setting, randomization over rules occurs *before* agents decide what to learn.[5] Consequently, prior to this randomization, each agent faces the same utility distribution. Comparing this common distribution under priority and TTC rules, Proposition 2 tells us that the mean of this distribution is the same. On the other hand, Theorem 1 tells us that the variance of the distribution is greater under priority rules than under TTC rules.

Related literature

Recent literature underscores the importance of incorporating learning into economic settings. For example, in an auction setting where agents may acquire information at a cost, Bergemann and Välimäki (2002) single out the Vickrey-Clark-Groves mech-

[5]As we elaborate below and formalize in the appendix, the opposite order yields inferior welfare outcomes.

anism by efficiency properties.[6] Similarly, studying two-sided matching, Drummond and Boutilier (2013) and Rastegari et al. (2013) model firms that conduct interviews in order to fill out partial rankings of candidates and seek to minimize the number of interviews required to implement a stable matching.

Bade (2015) is the first to study endogenous information acquisition in the object allocation context. Focusing on costly learning, she proposes a finite state space model with partitional learning technologies and an ex-ante notion of efficiency that accounts for these costs. She shows that, among *group strategy-proof* rules, only priority rules are robustly efficient: Priority rules are efficient for all admissible learning technologies while all other *group strategy-proof* rules are Pareto-dominated by modified priority rules[7] for at least some learning technologies. However, this provocative result relies on the richness of possible learning technologies that cause some agents to invest too little or too much in learning.[8] By contrast, we specialize the learning technology in order to explicitly solve for equilibrium strategies and analyze social welfare across rules in a meaningful economic setting.[9] Our results are decidedly less favorable toward the priority rules: TTC rules are not only undominated by prior-

[6] Rogerson (1992) and Hatfield et al. (2014) use similar intuition to relate strategy-proofness and incentives for investment prior to the allocation of resources.

[7] The modification allows the order of lower priority agents to depend on the objects chosen by higher priority agents.

[8] A corollary in Harless (2015) shows that the characterization of the priority rules in Bade (2015) extends to the case where learning is free, although the dominance relationships no longer apply.

[9] To our knowledge, this is the first paper to explicitly solve for equilibria of a strategic investigation game in the context of object allocation.

ity rules, but achieve equivalent utilitarian welfare and outperform priority rules in terms of equity.[10]

Our inequality comparisons apply the familiar Lorenz order, frequently used to compare wealth and other distribution in terms of equity.[11] While common in other settings, the criterion has only recently been considered in object allocation. Pycia and Ünver (2016) use it to illustrate rules with "brokers." Specifically, they consider the problem of a manager who must assign one task to each worker under the "constraint" that one distinguished task ought not to be assigned to one distinguished agent unless efficiency requires it. They show that, when all agents share the same preference relation, which is drawn uniformly at random, a rule in which the distinguished agent brokers the distinguished object Lorenz dominates all "Hierarchical Exchange" rules that respect the manager's constraint. Harless (2015) also applies the Lorenz order to compare object allocation rules. In a model with initial uncertainty about preferences but no strategic learning, he shows that top trading cycles rules are Lorenz dominant among Trading Cycles rules. Interpreted as a model in which learning is free, these results reinforce the conclusions we draw here which are

[10]At the same time, we also find that TTC rules are inefficient, even with the additional structure of our model and according to an ordinal notion of Pareto-dominance, reinforcing Bade (2015)'s robustness conclusion.

[11]Since introduced by Lorenz (1905), the order has been widely applied to study wealth and income inequality (see, e.g., Kakwani (1977), Seidl and Chakravarty (1992), Sen and Foster (1997), and Cowell (2000)) and underlies the Gini coefficient measurements which appear even in popular introductory textbooks (e.g., Krugman and Wells (2015) and Boyes and Melvin (2012)). It has recently been applied to kidney exchange (Roth et al. 2005), object allocation (Pycia and Ünver 2016), and bankruptcy problems (Thomson 2012).

themselves entirely new.

To focus our analysis, we restrict attention to two prominent classes of rules in the literature: priority[12] and TTC[13] rules.[14] When preferences are known, there is an important sense in which the families are equivalent (Abdulkadiroğlu and Sönmez 1998):[15] Imagine that the central authority may randomly select either one priority order or one endowment profile and then apply the corresponding priority or TTC rule. Holding agents' preferences fixed, their expected allocations are the same whether randomizing over priority orders or endowment profiles.

With strategic learning, however, we observe an important difference: Randomizing over endowments yields a less variable utility distribution and thereby promotes equity. Here we suppose that the results of randomization are announced before learning decisions are finalized so that agents may condition their choices on this information. Randomizing instead after agents investigate dampens their learning incentives and decreases the benefits of learning, an intuition we formalize in Appendix A.6.1. The resulting ex-ante welfare loss carries a clear policy recommendation: Randomize first. More generally, provide agents with as much information about the rule as possible as early as possible.

While both TTC rules and priority rules achieve the same utilitarian social welfare, they fail to maximize utilitarian welfare. In a companion paper (Harless and

[12]See, for instance, Svensson (1994) and Bu (2014).

[13]See, for instance, Ma (1994), Morrill (2013), and Anno (2015).

[14]Both of these are subclasses of Hierarchical Exchange rules (Pápai 2000a) and, in turn, of Trading Cycles rules (Pycia and Ünver 2016).

[15]See Pathak and Sethuraman (2011), Bade (2014), and Carroll (2014) for related equivalences. See Liu and Pycia (2013) for such a result in large markets.

Manjunath 2015), we compute the maximal utilitarian welfare that a social planner could achieve by optimally directing and observing the results of learning. To maximize utilitarian welfare, the planner asks each agent to investigate a distinct object. Since he observes the results of these investigations, whether an agent keeps the object that he investigates depends on whether his private value for it is realized to be positive or negative. Unless there is a single agent with a negative realization, agents with positive realizations keep their objects and those with negative realizations exchange with others. If there is a single agent with a negative realization, he and the agent with the smallest positive realization swap or keep their investigated objects depending on the magnitudes of their realizations. Starkly, depending on the distribution of private values,the common utilitarian welfare under TTC and priority rules may achieve an arbitrarily large or arbitrarily small fraction of the maximal utilitarian surplus beyond arbitrarily assigning objects without learning.

The remainder of the paper is organized as follows: We formalize our model in Section 2. We study the equilibrium welfare under priority and TTC rules in Section 3. We conclude in Section 4. All proofs are in the appendix.

2 Model

A finite set of objects $A \equiv \{a_1, \ldots, a_m\}$ must be assigned to a finite set of agents $N \equiv \{1, \ldots, n\}$ where $2 \leq n = m$ so that each agent receives one object.[16] Each object $a \in A$ has **common value** $v_a \in \mathbb{R}$ and no two objects have the same common

[16]Our analysis generalizes to cases where $2 \leq n$ and $2 \leq m$. To ease exposition, we maintain the assumption $n = m$ in the body of the paper and relegate the full analysis to the appendix.

value. Labeling objects in decreasing order of their common values, for each $a_k \in A$, let $v_k \equiv v_{a_k}$ so that $v_1 > v_2 > \cdots > v_m$. For each agent $i \in N$, each object $a \in A$ also has a **private value** ε_{ia} so the **value** of object a to agent i is

$$v_{ia} \equiv v_a + \varepsilon_{ia}.$$

The private values $(\varepsilon_{ia})_{i \in N, a \in A}$ are random variables which are initially unknown to the agents but potentially discoverable. We assume that $(\varepsilon_{ia})_{i \in N, a \in A}$ are independently and identically distributed according to a probability measure over $\mathbb{R}$, which is summarized by the cumulative distribution function (cdf) F. Additionally, we assume:

1. Symmetry: the distribution is symmetric around zero.

2. Reversal: realizations of ε_{ia} are above $v_1 - v_m$ with positive probability.

3. Almost no ties: for each pair $a, b \in A$, there is no atom at $v_a - v_b$.

Assumption (1) ensures that positive and negative realizations are equally likely. Assumption (2) ensures that all pairwise comparisons can be reversed by sufficiently high or low realizations of the private value. Assumption (3) ensures that, having investigated one object, an agent is almost never indifferent between two objects. It is satisfied, for example, by any distribution with a continuous cumulative distribution

When $n > m$, the last $n - m$ agents play no role and can be safely ignored. When $n < m$, we must extend each TTC rule to an appropriate Hierarchical Exchange rule (Pápai 2000a), but our results are insensitive to the choice of extension.

function. Using the ordering by common values, we simplify notation and write ε_{ij} for ε_{ia_j}.

Returning to our introductory example, we capture this setting formally by labeling the ivy, state, and technical schools a_1, a_2, and a_3 respectively. The common expected utilities from attending each school are $v_1 = 9$, $v_2 = 7$, and $v_3 = 5$. According to the "good-news/bad-news" technology, each school is either a good fit or a bad fit, represented by $\varepsilon_{ia} \in \{-5, +5\}$ with $F(\cdot)$ taking the form of a probability mass distribution with discontinuities placing full and equal weight on these two values. Since $F(\cdot)$ is symmetric and draws are independent, the example fits our formal setting exactly.

2.1 Learning and equilibrium

Let $\mathcal{P}$ be the set of linear orders over A and $P_0 \equiv (a_1, \ldots, a_m)$. Initially, each agent ranks the objects according to P_0, but has the opportunity to learn his private value for one object.[17] Agent i's **investigation strategy** is $\sigma_i \in A$ and an investigation strategy profile is $\sigma \equiv (\sigma_i)_{i \in N}$. Agent i's investigation reveals $\varepsilon_{i\sigma_i}$ and revises his ranking to $P_i(\varepsilon_{i\sigma_i})$. The updated ranking agrees with P_0 on $A \backslash \{\sigma_i\}$, although the

[17] This specification restricts attention to pure strategies, but allowing mixed investigation strategies does not affect our results. In particular, for all rules we consider, there is at most one agent whose best response is not unique. Even in this case, the agent's choice among his best responses has no implications for the strategies or welfare of other agents. See the proof in Appendix A.3 for complete details.

position of σ_i may differ.[18] In particular, for each $a \in A\backslash\{\sigma_i\}$, $\sigma_i\ P_i(\varepsilon_{i\sigma_i})\ a$ if and only if $\varepsilon_{i\sigma_i} > v_a - v_{\sigma_i}$ so that $\mathbf{E}_F[v_{ia}] = v_a < v_{\sigma_i} + \varepsilon_{i\sigma_i}$.[19] By Assumption (3), $P_i(\varepsilon_{i\sigma_i})$ is a linear order with probability 1, so we assume $P_i(\varepsilon_{i\sigma_i}) \in \mathcal{P}$. Let $\varepsilon_\sigma \equiv (\varepsilon_{i\sigma_i})_{i\in N}$ and $P(\varepsilon_\sigma) \equiv (P_i(\varepsilon_{i\sigma_i}))_{i\in N}$.

After learning, each agent i reports $P_i \in \mathcal{P}$, (possibly different from $P_i(\varepsilon_{i\sigma_i})$) to a central authority which then applies a rule to determine the final allocation.[20] The set of (feasible) allocations is $X \equiv \{\nu\colon N \to A : \forall i, j \in N, \text{ if } i \neq j \text{ then } \nu_i \neq \nu_j\}$ so an **allocation** is a function that assigns a distinct object to each agent. A **rule** $\varphi\colon \mathcal{P}^N \to X$ picks an allocation for each profile of reported preferences. A rule is **strategy-proof** if, conditional on investigation, it is a weakly dominant strategy for each agent i to report $P_i(\varepsilon_{i\sigma_i})$. As we are interested in understanding agents' investigation strategies, we restrict attention to *strategy-proof* rules and henceforth assume that they report $P(\varepsilon_\sigma)$.

Let φ be a *strategy-proof* rule and $\sigma \in A^N$. For each ε_σ, let $\nu(\varepsilon_\sigma) \equiv \varphi(P(\varepsilon_\sigma))$. That is, at the realization ε_σ, $\nu(\varepsilon_\sigma)$ is the realized allocation. Agent i's **expected utility** at σ under φ is

$$U_i(\sigma, \varphi) \equiv \mathbf{E}_F[v_{i\nu_i(\varepsilon_\sigma)}] = \int \cdots \int v_{i\nu_i(\varepsilon_\sigma)}\, dF(\varepsilon_{1\sigma_1}) \cdots dF(\varepsilon_{n\sigma_n}).^{21}$$

[18] By Assumption (1), for each pair $a, b \in A\backslash\{\sigma_i\}$, $\mathbf{E}_F[\varepsilon_{ia}] = \mathbf{E}_F[\varepsilon_{ib}] = 0$. By independence $\mathbf{E}_F[v_a|\varepsilon_{i\sigma_i}] = v_a$ and $\mathbf{E}_F[v_b|\varepsilon_{i\sigma_i}] = v_b$. Thus, $a\ P_i(\varepsilon_{i\sigma_i})\ b$ if and only if $v_a > v_b$.

[19] Given a random variable x distributed with cdf F and a function g, we denote the expected value of $g(x)$ by $\mathbf{E}_F[g(x)]$.

[20] Equivalently, each agent reports a pair $(\sigma_i, \varepsilon_i) \in A \times \mathbb{R}$ representing $\varepsilon_{i\sigma_i}$. We define reports as preference rankings because our rules are defined over this domain.

The decomposition follows by independence.

Although we model a two stage game in which agents first choose investigation strategies and *then* submit preferences, we restrict attention to strategy-proof rules and assume that each agent plays his weakly dominant strategy to report truthfully. This allows us to abstract from the second stage and focus on learning incentives and strategies. Our equilibrium notion is thus the standard Nash equilibrium of a one-stage simultaneous move game. An **equilibrium under $\boldsymbol{\varphi}$** is $\sigma \in A^N$ such that for each $i \in N$ and each $\sigma'_i \in A$, $U_i(\sigma, \varphi) \geq U_i(\sigma'_i, \sigma_{-i}, \varphi)$.

2.2 Rules

We study two classes of *strategy-proof* rules: priority rules and top trading cycles rules. Given an order $\prec$ over N, the **priority rule** associated with $\prec$, SD,[22] begins with an order over agents and uses their reported preferences to simulate sequential choice: first, the agent with the highest priority is assigned his most preferred object, then the agent with the second highest priority is assigned his most preferred object

[21]An equivalent way of defining U_i is as follows. For each $i \in N$ and each $a_k \in A$, let $B_k \subseteq \mathbb{R}^n_+$ be the realizations of ε_σ where $\nu_i(\varepsilon_\sigma) = a_k$. Then i's expected utility is $\sum_{k=1}^n Pr(\varepsilon_\sigma \in B_k)E[v_k + \epsilon_{ik}|\varepsilon_\sigma \in B_k]$. If $\sigma_i = a_l$ then for each $l \neq k$, $E[\varepsilon_{ik}|\varepsilon_\sigma \in B_k] = E[\varepsilon_{ik}] = 0$ since ε_σ contains no information about ε_{ik}. Thus, i's expected utility is $Pr(\varepsilon_\sigma \in B_l)E[\varepsilon_{il}|\varepsilon_\sigma \in B_l] + \sum_{k=1}^n Pr(\varepsilon_\sigma \in B_l)v_k$ which is identical to the expression above.

[22]We abbreviate priority rules by SD, suggestive of "serial dictatorship," another common name for these rules. This is to avoid confusion with preferences. Also, although each order in fact defines a different rule, we suppress the dependence on the associated order whenever there is no confusion.

among those that remain, and so on.[23]

Given an allocation $\mu \in X$, the **top trading cycles (TTC) rule** associated with μ, TTC,[24] simulates trade according to Gale's top trading cycles algorithm (Shapley and Scarf 1974) as if μ were an endowment. Following the algorithm, each agent points at the owner of his most preferred object. This forms at least one cycle. Agents in a cycle are assigned the objects at whose owner they point and these agents and objects are removed. Each unassigned agent now points at the owner of his most preferred object among those that remain. This procedure continues until each agent has been assigned an object.

Henceforth, when considering SD, we label the agents so that $1 \prec 2 \prec \cdots \prec n$ and when considering TTC, we label the agents so that for each $i \in N$, $\mu_i = a_i$.

3 Comparison of rules

Our goal is to compare welfare under priority and TTC rules. Since we are interested in equilibrium welfare, we first characterize the equilibria under each type of rule.

Proposition 1. *Given a priority rule or a TTC rule, all equilibria are welfare equivalent.*[25]

[23]The sequential procedure is merely descriptive; in fact, agents *simultaneously* choose investigation strategies *before* submitting preferences *simultaneously*.

[24]Again, we suppress the dependence on the associated endowment.

[25]As the proof relies only on the ordinal rankings of the objects, this result is robust to re-scaling utility functions.

The proof of Proposition 1 is constructive, identifying unique equilibrium strategies for almost all agents. For a priority rule, there is a canonical equilibrium in which each agent follows a "one down" strategy: The highest priority agent investigates the object with the second highest common value, the second highest priority agent investigates the object with the third highest common value, and so on. In fact, except for the agents with highest and lowest priority, these equilibrium strategies are unique. The first agent may alternatively investigate the object with the highest common value. Because the distributions of private values are identical and symmetric, this is just as informative as investigating the object with the second highest common value. The last agent's learning strategy is irrelevant; he receives the final remaining object in each case.[26]

For a TTC rule, there is a canonical equilibrium with particularly simple structure: Each agent investigates his endowment. Except in the special case of two agents and two objects, the equilibrium is unique. The two-agent case has multiple welfare-equivalent equilibria because of symmetry, essentially for the same reasons that the highest priority agent has multiple equilibrium strategies under a priority rule.

Intuitively, under either type of rule, each agent learns about the best object that is certain to be available to him. If he instead investigates an object that may be unavailable, he risks wasting the opportunity to learn. Similarly, if more than one object is certain to be available, investigating a worse object reduces the

[26]When $n < m$, the last agent also has a dominant strategy and investigates a_{n+1}. See the proof in Appendix A.3.1 for details.

potential benefit from investigation. This argument is somewhat subtle, but the key observation is again symmetry: investigating a better object is more likely to lead to a preference reversal with an even better object.[27] The formal proof relies on carefully specifying and comparing each agent's option sets under various possible scenarios. These results depend crucially on the assumption that the distributions of private values are independent, symmetric, and identical.

Somewhat surprisingly, the total utility is the same under either type of rule. From a utilitarian perspective, then, the rules are equivalent. Given an investigation profile $\sigma \in A^N$ and rule φ, the **utilitarian welfare** of the pair (σ, φ) is $\boldsymbol{U(\sigma, \varphi)} \equiv \sum_N U_i(\sigma, \varphi)$. If there is a unique (in welfare terms) equilibrium under φ, say σ, we let the **utilitarian welfare of** $\boldsymbol{\varphi}$ be $U(\sigma, \varphi)$.

Proposition 2. *Each TTC rule and each priority rule achieves the same utilitarian social welfare.*

According to Proposition 2, the difference between rules amounts to a transfer of ex-ante expected utility among the agents. To better understand the equivalence, consider a problem with two agents and two objects and suppose that both agents investigate the first object, a learning profile consistent with an equilibrium under either rule. The rules select different allocations precisely when both agents prefer the second object: under SP, the first agent receives the second object; under TTC, the second agent retains it. Nevertheless, the conditional expected value of each object is the same for both agents; differently resolving this conflict simply transfers

[27] This is also why the argument does not apply to the first agent under a priority rule: there is no "even better object" for comparison.

the "benefit" from one agent to the other. While the general argument is more complicated, the symmetry and independence of distributions allows us to extend this logic to all equilibrium strategies as well as larger problems.

Both priority and TTC rules are members of the broader class of Hierarchical Exchange rules (Pápai 2000a). For many applications, orders (for priority rules), endowments (for TTC rules), or roles (more generally, for exchange rules) are chosen at random (Abdulkadiroğlu and Sönmez 1998; Pathak and Sethuraman 2011; Bade 2014; Carroll 2014; Liu and Pycia 2013). In such contexts, uniform randomization distributes Utilitarian social welfare equally among agents, making the Utilitarian criterion a natural starting point. While Proposition 2 shows that TTC and priority rules fare equally on this count, the comparison does not extend to all Hierarchical Exchange rules. In fact, even very similar rules such as *conditional* priority rules result in loss of welfare (see Appendix A.6.2).

Our main result compares the welfare distributions induced by TTC and priority rules. General sentiment, reflected by "market" and "dictatorship" descriptions, suggests that TTC rules are more equitable. Our model permits a direct comparison, delivering the first formal statement confirming this idea. Moreover, leveraging the structure of our equilibria, we are able to provide new and deep intuition for this result and also its limitations.

Remark 1. Intuition for comparison between TTC and priority rules. As members of the same larger family of Hierarchical Exchange rules, but TTC and priority rules can be represented through inheritance structures (Pápai 2000a). A priority rule begins with the first agent initially endowed with all of the objects and subsequent agents inheriting objects following the priority order. In our model, the first agent always receives an object with one of the two highest common values. The second agent therefore always inherits the remaining objects and may be thought of as part of his endowment. Continuing, the second agent always receives an object with one of the three highest common values, the third agent an object with one of the four highest common values, and so on. Translating, we may represent a priority rule by an endowment structure in which agent 1 is endowed with $\{a_1, a_2\}$ and each agent $i > 1$ is endowed with a_{i+1}, allowing $a_{n+1} = \emptyset$. Comparing this to the initial endowments under TTC,

$$\begin{aligned} SD: &\quad \big(\{a_1, a_2\}, \{a_3\}, \{a_4\}, \ldots, \{a_n\}, \emptyset\big) \\ TTC: &\quad \big(\{a_1\}, \{a_2\}, \{a_3\}, \ldots, \{a_{n-1}\}, \{a_n\}\big) \end{aligned}$$

Two comparisons are clear: Relative to TTC, the first agent's endowment improves under SD while the last agent's worsens. For the remaining agents, the comparison is ambiguous; the inheritance structure weakens their position with respect to preceding agents, but strengthens their position with respect to following agents.

The intuition of Remark 1 holds generally: The first agent is always better off under a priority rule than under a TTC rule, whereas the last agent is always better off under a TTC rule than under a priority rule. In fact, more is true: Each group of agents at the bottom is better off under a TTC rule than under a priority rule. To formalize, we make Lorenz comparisons (Lorenz 1905), adopting a tool commonly used to compare equity of wealth and other distributions. Let $u, u' \in \mathbb{R}^N$ be two utility profiles with the same total utility. Then u **Lorenz dominates** u' if the worst off agent's utility under u is at least as high as that of the worst off agent under u', the sum of the two worst off agents' utilities under u is at least as high as that of the two worst off agents under u', and so on. Extending this definition to rules, one rule **Lorenz dominates** another rule if the equilibrium utility profile under the first Lorenz dominates that under the second. We have seen that all priority and TTC rules are equivalent in terms of utilitarian social welfare (Proposition 2). However, TTC rules are superior according to this more stringent criterion.[28]

Theorem 1. *Each TTC rule Lorenz dominates each priority rule.*

Appealing to Theorem 1 of Dasgupta et al. (1973), our Theorem 1 allows us to unambiguously compare TTC and priority rules according to all strictly Schur-concave social welfare functions.[29] This inequality-averse class includes, for example, the Rawlsian max-min social welfare function (Rawls 1972).[30]

Corollary 1. *Each TTC rule dominates each priority rule according to each strictly Schur-concave social welfare function.*

[28]By examining the difference in equilibrium investigation strategies, the difference of each agent's utility is on the order of one object rank per agent.

[29]A social welfare function $W : \mathbb{R}^N \to \mathbb{R}$ is **Schur-concave** if for each $u \in \mathbb{R}^N$ and each $N \times N$ bistochastic matrix B, $W(u) \leq W(uB)$ with strict inequality whenever B is not a permutation matrix. See Marshall et al. (2011).

[30]By results of Dasgupta et al. (1973) and Rothschild and Stiglitz (1973), a similar result applies for quasi-concave social welfare functions.

None of the social welfare functions that we consider depend on the identities of the agents. That is, each measure is invariant to the renaming of agents or reshuffling of their utilities. Consequently, when we compare two rules of the same kind (either priority or TTC rules), they are equivalent under all of these criteria.

By way of comparison with standard model, we briefly discuss two extreme learning technologies: No learning and complete learning. When there is no learning, agents rank objects according to their common values and therefore submit identical preferences at the reporting stage. Consequently, TTC makes the same assignment as the priority rule which orders agents according to the common values of their endowments; there is no meaningful distinction between TTC and priority rules.

At the other extreme, when agents learn everything before submitting their preferences, the welfare comparisons delivered by our model continue to apply: TTC rules dominate priority rules in terms of equity.[31] For intuition, consider the Rawlsian max-min criterion. Here, our comparison of "option sets" carries through. In particular, the lowest priority agent under a priority rule has an option set that is no larger – and typically smaller – than the option set of any agent under a TTC rule. Consequently, the lowest priority agent is be better off under each TTC rule. So each TTC rule dominates each SD rule according to the Rawlsian criterion.

In addition to our welfare comparisons, we are interested in whether Pareto improvements are possible when we account for learning. In our model, agents report preference rankings, so we introduce a notion of efficiency that respects this informa-

[31]A formal analysis in the case of complete learning requires considerable additional argument. Nevertheless, the conclusions follow from Harless (2015) which studies a (standard) object assignment problem under uncertainty.

tional assumption. Formally, an investigation profile $\sigma \in A^N$ and rule $\varphi \colon \mathcal{P}^N \to X$ are jointly **(Pareto) efficient** if no alterative pair achieves an ex-ante Pareto improvement: for each $\sigma' \in A^N$ and $\varphi' \colon \mathcal{P}^N \to X$, if there is $i \in N$ such that $U_i(\sigma', \varphi') > U_i(\sigma, \varphi)$, then there is $j \in N$ such that $U_j(\sigma', \varphi') < U_j(\sigma, \varphi)$. A rule $\varphi \colon \mathcal{P}^N \to X$ is **(Pareto) efficient** if for each equilibrium $\sigma \in A^N$ under φ, the pair (σ, φ) is ex-ante efficient.[32]

Proposition 3. *Each priority rule is efficient. If there are at least three objects, no TTC rule is efficient. However, no priority rule ever dominates a TTC rule.*

The efficiency of priority rules is intuitive: a priority rule lexicographically favors each agent in turn, so no agent can be made better off without making a higher priority agent worse off.[33] The analysis of TTC rules is more subtle. To provide further intuition, we begin with an example illustrating the potential for ex-ante Pareto improvement.

Example 1. Ex-ante inefficiency of *TTC*. Let and $N \equiv \{1, 2\}$ and $A \equiv \{a, b\}$. The personal value of each object is equally likely to be high, medium or low as specified in the Table 1. In our model, this corresponds to $v_a = 5$, $v_b = 4$, and $\varepsilon_{i\alpha} \in \{-4, 0, 4\}$ each with equal probability. To see the potential for improving over TTC, consider the equilibrium in which $\sigma_1 = a$ and $\sigma_2 = b$. In the events where private values are (M, L) or (L, M) (i.e., $\varepsilon_\sigma = (0, -4)$ or $\varepsilon_\sigma = (-4, 0)$),

[32] A stronger notion would allow the assignment to depend on the realizations of private values directly. Bade (2015) takes this approach and defines efficiency in terms of "outcome functions".

[33] An easy adaptation of part of Theorem 2 in (Bade 2015) suffices to establish ex-ante efficiency, so we omit the formal proof.

	L	M	H
a	1	5	9
b	0	4	8

Table 2: Two objects with three possible personal values.

TTC prescribes that each agent keep his endowment. These yield private values of $(5, 0)$ and $(1, 8)$ respectively. Since these two events occur with equal probability, conditional on one of these two events, agent 1's expected utility is $(5 + 1)/2 = 3$ and agent 2's expected utility is $(0+8)/2 = 4$. If instead we reverse the assignments in these events, the agents receive the common value of the other object. So the expected utilities conditional on one of these two events are 4 and 5 for agents 1 and 2 respectively. If we stick to the recommendations of TTC in all other events, this yields an ex-ante Pareto improvement since these events occur with positive probability.

As Example 1 shows, there may be room for ex-ante Pareto improvement over TTC. If allowed to condition on the realizations of ε, we could directly implement the Pareto improving trades described in the example. In fact, with at least three objects, TTC can even be improved upon by an ordinal rule as required for inefficiency by our definition. Intuitively, the rule infers the magnitude of $\varepsilon_{i\sigma(i)}$ from the position of α in $P_i(\varepsilon_{i\sigma_i})$ relative to the third object. Although coarse, this information allows the rule to prescibe additional exchanges in which agents forego small gains to avoid large losses. As constructing such a rule is somewhat delicate, we relegate a complete example to the Appendix. Of course, neither type of improvement is consistent with *strategy-proofness* at the reporting stage; after investigation, at least one agent prefers to keep his endowment an no longer has an incentive to truthfully reveal his preferences.

Our Proposition 3 identifies an inefficiency qualitatively different from a similar conclusion in Bade (2015) which is driven by learning costs. In particular, Theorem 1 implies that priority rules never dominate TTC rules. For a richer set of learning technologies with varying costs (Bade 2015), under TTC, agents fail to internalize the full benefits of learning and may choose not to investigate. In our setting, this consideration is moot. Consequently, our inefficiency result for TTC strengthens her conclusion in two ways: our result does not rely on the richness of the considered learning technologies and our definition of Pareto dominance is more demanding, requiring comparison with another ordinal rule.

4 Discussion

We extended the object allocation problem to allow learning, proposing a structured model with ex-ante common values and a limited learning technology which allows agents to investigate one object. This structure allows us to compare equilibrium welfare when rules from two leading classes are applied to allocate objects. We show that, while priority rules are Pareto efficient and TTC rules are not, TTC rules are superior according to progressive measures of social welfare. More generally, we have shown how the choice of allocation rule may shape agents' learning choices with implications for equity and welfare. Our analysis suggests that learning incentives deserve expanded scrutiny.

To illustrate the importance of learning incentives, we have made simplifying assumptions to isolate their effects and provide clear descriptions. Loosely, these assumptions divide into four groups. The first group are solely for expositional clarity and made only in the body of the paper, such as equal numbers of agents and objects. These are relaxed in the Appendix where we present general proofs.

The second group are employed in our proofs, but only superficially. For instance, Assumption (2) says that the range of private values is large enough to reverse an agent's ranking between a_1 and a_m with positive probability. This allows us to focus on the single case where learning is most relevant[34] when characterizing best responses under TTC and priority rules. The intuition, proof techniques, and results generalize, for instance, to a setting where only reversals between adjacent objects have positive probability, though our slightly stronger assumption to eases exposition and permits simpler statements of some results.[35]

Third are those assumptions essential to our equilibrium characterizations, including symmetry of the distribution, identical distribution across objects, and unit supply of objects. While our techniques can be adapted to these settings, equilibria will necessarily depend on specific details of the distributions, jointly so in the case where multiple copies of objects may be available. For example, with heteroge-

[34]At the opposite extreme, when no reversal is possible, the setting is equivalent to the standard static model without learning.

[35]In the narrower setting, the equilibrium under TTC is no longer unique as some agents no longer have strictly dominant strategies: The agent endowed with a_1 is indifferent between investigating a_1 and a_2 while the agent endowed with a_n is indifferent between investigating a_n and a_{n-1}.

neous distributions, agents may prefer to investigate the same object. In a limit case where only one object has a meaningful probability of reversal, then all agents will investigate this object regardless of the rule. By allowing multiple copies of objects, we encounter complications which take us even farther afield. With priority rules, agents may prefer to investigate "up" rather than down.[36] With TTC, even defining the rule, which now must be extended to determine with whom one trades among those endowed with a copy of the object, a choice with welfare consequences. Consequently, while our techniques apply in principle to these situations on a case-by-case basis, robust and generalizable results will require a qualitatively different analysis.

In the final group are those assumptions fundamental to our line of inquiry. To expand learning opportunities or incorporate correlation among idiosyncratic values, for example, would require a fundamentally different model. In addition to raising complications analogous to those with multiple copies, expanding investigation opportunities requires first answering new modeling questions: May agents sequence their learning, conditioning on early realizations, or must they commit to a full portfolio of investigations ex-ante? If there are multiple equilibria, how does one select

[36]To elaborate, if there are q units of each object, then the first q agents face the same problem as in the unit supply case. However, the $q+1^{\text{st}}$ agent's optimal strategy now depends on the probability of preference reversal for any one of the first q agents between a_1 and a_2. From the size of q and probability of reversal, this agent may be able to infer that copies of one or both of the better object are available with near certainty and prefer to "risk" an investigation of a_1 rather than a_3. If he does investigate a_3, however, the incentive to investigate higher objects may grow only more pronounced for subsequent agents, possibly even violating even the monotonicity of investigation strategies we observe in our model.

among them? How are rules with multiple equilibria compared? Correlation raises yet deeper questions, rendering none of our rules incentive compatible at the reporting stage.[37] Fundamentally different rules are now appropriate, perhaps ones which allow for structured sequencing of learning or communication among agents. Extensions along these lines deserve attention from future research.

Among other avenues for future research, we may ask about the equilibria of additional rules, particularly rules which are not *strategy-proof*. To the extent that such rules are used in practice, we need to understand how they influence learning decisions. Further extensions might move away from the one-to-one assignment problem. Instead, agents may each receive a fixed number of objects.[38] If we are interested in equal treatment, we may turn to probabilistic assignments in the second stage.[39] The implications of our model for the school choice problem are also important to understand. This would entail adding capacities and priorities to each object. Similarly, in a two-sided matching model, agents on both sides may have opportunities to learn about potential partners. Pursuing this extension may illuminate the mysterious process of dating.

[37] In fact, Che et al. (2015) show that only constant rules are incentive compatible with interdependent values, a robust result that is pervasive across many settings.

[38] See, for example, Pápai (2000b, 2001) and Klaus and Miyagawa (2001).

[39] See Hylland and Zeckhauser (1979) and Bogomolnaia and Moulin (2001).

A Proofs and further examples

A.1 Preliminaries

We start with some notation. For each pair $i, j \in \{1, \ldots, m\}$, let $p_{ij} \equiv 1 - F(|v_i - v_j|) = F(-|v_i - v_j|)$ and $\eta_{ij} \equiv \mathbf{E}_F[x|x > v_j - v_i]$. Then p_{ij} is the probability that learning the private value of either a_i or a_j will reverse their ranking in an agent's preferences. Similarly, η_{ij} is the expectation of an agent's private value for a_i given that it is large enough to keep or raise a_i above a_j in the agent's preference ranking. For each pair $i, j \in A$ with $i < j$, $p_{ij} = p_{ji} > 0$. However, $\eta_{ij} \neq \eta_{ji}$. Instead, $0 < \eta_{ij} < \eta_{ji}$ (see Lemma 1). Finally, for each $i \in N$ and each $P_{-i} \in \mathcal{P}^{N \setminus \{i\}}$, agent i's **option set** under φ is

$$O_i(P_{-i}, \varphi) \equiv \{a \in A : \exists P_i \in \mathcal{P} \text{ such that } \varphi_i(P_i, P_{-i}) = a\}.$$

Taking the preference reports of other agents as given, $O_i(P_{-i}, \varphi)$ represents those objects that agent i may receive by varying his report. Given an event E, agent i's expected utility conditional on E is $U_i(\sigma|E)$.

Next, we present a technical lemma that allows us to compare private values.

Lemma 1. *For each triple $i, j, k \in \{1, \ldots, m\}$ with $i < j < k$,*

1. $p_{ij} = p_{ji}$,

2. $(1 - p_{ij})\eta_{ij} = p_{ij}\eta_{ji}$, *and*

3. $p_{ik}(v_k - v_i + \eta_{ki}) < p_{ij}(v_j - v_i + \eta_{ji})$.

Proof. Let $i, j, k \in A$ with $i < j < k$.

(1) By definition, $p_{ij} = 1 - F(|v_i - v_j|) = 1 - F(|v_j - v_i|) = p_{ji}$.

(2) By symmetry of F, for each $\alpha \in \mathbb{R}_+$,

$$\begin{aligned}\int_{-\alpha}^{\alpha} x\,dF(x) &= \int_{-\alpha}^{0} x\,dF(x) + \int_{0}^{\alpha} x\,dF(x)\\ &= \int_{0}^{\alpha} -x\,dF(x) + \int_{0}^{\alpha} x\,dF(x)\\ &= 0.\end{aligned}$$

In particular,

$$\int_{v_j-v_i}^{\infty} x\,dF(x) = \int_{v_j-v_i}^{v_i-v_j} x\,dF(x) + \int_{v_i-v_j}^{\infty} x\,dF(x) = \int_{v_i-v_j}^{\infty} x\,dF(x).$$

Since $i < j$, $v_j < v_i$ and $p_{ij} = F(-|v_i - v_j|) = F(v_j - v_i) = 1 - F(v_i - v_j)$. By definition of conditional expectations,

$$\begin{aligned}\eta_{ij} = \mathbf{E}_F[x|x > v_j - v_i] &= \frac{\int_{v_j-v_i}^{\infty} x\,dF(x)}{1 - F(v_j - v_i)} = \frac{\int_{v_j-v_i}^{\infty} x\,dF(x)}{1 - p_{ij}} \text{ and}\\ \eta_{ji} = \mathbf{E}_F[x|x > v_i - v_j] &= \frac{\int_{v_i-v_j}^{\infty} x\,dF(x)}{1 - F(v_i - v_j)} = \frac{\int_{v_j-v_i}^{\infty} x\,dF(x)}{p_{ij}}.\end{aligned}$$

Therefore, $(1 - p_{ij})\eta_{ij} = p_{ij}\eta_{ji}$.

(3) Since $i < j < k$, $v_k < v_j < v_i$ and $p_{ij} < p_{ik}$. As computed in (2),

$$p_{ij}\eta_{ji} = \int_{v_i-v_j}^{\infty} x\,dF(x) \text{ and}$$

$$p_{ik}\eta_{ki} = \int_{v_i - v_k}^{\infty} x \, dF(x).$$

Now comparing,

$$\begin{aligned}
p_{ij}\eta_{ji} - p_{ik}\eta_{ki} &= \int_{v_i - v_j}^{\infty} x \, dF(x) - \int_{v_i - v_k}^{\infty} x \, dF(x) \\
&= \int_{v_i - v_j}^{v_i - v_k} x \, dF(x) \\
&> \int_{v_i - v_j}^{v_i - v_k} v_i - v_j \, dF(x).
\end{aligned}$$

Additionally,

$$\begin{aligned}
&p_{ij}(v_j - v_i) - p_{ik}(v_k - v_i) \\
&\qquad = \int_{v_i - v_j}^{\infty} v_j - v_i \, dF(x) - \int_{v_i - v_k}^{\infty} v_k - v_i \, dF(x) \\
&\qquad = \int_{v_i - v_j}^{v_i - v_k} v_j - v_i \, dF(x) + \int_{v_i - v_k}^{\infty} (v_j - v_i) - (v_k - v_i) \, dF(x) \\
&\qquad = \int_{v_i - v_j}^{v_i - v_k} v_j - v_i \, dF(x) + \int_{v_i - v_k}^{\infty} v_j - v_k \, dF(x) \\
&\qquad > \int_{v_i - v_j}^{v_i - v_k} v_j - v_i \, dF(x).
\end{aligned}$$

Combining results,

$$\begin{aligned}
p_{ij}(v_j - v_i + \eta_{ji}) - p_{ik}(v_k - v_i + \eta_{ki}) &= \big(p_{ij}\eta_{ji} - p_{ik}\eta_{ki}\big) + \big(p_{ij}(v_j - v_i) - p_{ik}(v_k - v_i)\big) \\
&> \int_{v_i - v_j}^{v_i - v_k} v_i - v_j \, dF(x) + \int_{v_i - v_j}^{v_i - v_k} v_j - v_i \, dF(x) \\
&= 0.
\end{aligned}$$

Therefore, $p_{ik}(v_k - v_i + \eta_{ki}) < p_{ij}(v_j - v_i + \eta_{ji})$. $\square$

Lemma 1(2) implies that for each pair $i, j \in \{1, \ldots, m\}$ with $i < j$, $0 < \eta_{ij} < \eta_{ji}$ and $v_i < v_j + \eta_{ji}$. Substituting according to definitions, Lemma 1(3) says that for each $h \in N$, $p_{ik}\mathbf{E}[v_{hk}|v_{hk} > v_i] < p_{ij}\mathbf{E}[v_{hj}|v_{hj} > v_i]$. Since $p_{ik} < p_{ij}$, Lemma 1(3) implies that $p_{ik}(v_k + \eta_{ki}) < p_{ij}(v_j + \eta_{ji})$, a form which will be convenient in some computations.

A.2 Hierarchical exchange and unequal numbers of agents and objects

While the case of $3 \leq n = m$ simplifies exposition, our conclusions apply when we allow for $2 \leq m \leq n$. Since TTC rules are only defined for $n = m$, we extend each TTC rule by including all additional objects in agent n's endowment. Formally, each TTC rule is extended to a Hierarchical Exchange rule by generalizing endowments with inheritance hierarchies (Pápai 2000a): Each of the first n objects is the initial endowment of a different agent and all remaining objects are included in agent n's initial endowment. To complete the inheritance hierarchy, we specify that whenever agent n is removed during the top trading cycles algorithm, his untraded endowment is inherited by the remaining agent with the highest index. Similarly, once that agent is removed, his untraded endowment is again inherited by the remaining agent with the highest priority and so on. This specification is for concreteness only; as we will see, the inheritance structure for the last $m - n$ objects plays no role in the analysis.

A.3 Proof of Proposition 1

We first show the result for priority rules in Section A.3.1 and then for TTC rules in Section A.3.2.

A.3.1 Priority rules

Let SD be a priority rule. If $n = m$, then for each $P \in \mathcal{P}^N$, agent n receives the object left over after others' assignments are made, so $SD(P)$ is independent of P_n. Since agent n's investigation strategy has no effect on the allocation, each $\sigma_n \in A$ may be part of an equilibrium. As this is the only difference between the cases $n = m$ and $n < m$, suppose now that $n < m$.

Step 1: Equilibrium strategies for agent 1. Let $\sigma_{-1} \in A^{N \setminus \{1\}}$. First suppose $\sigma_1 = a_1$. If $\varepsilon_{11} > v_2 - v_1$, then agent 1 reports P_0 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of $v_1 + \eta_{12}$. If instead $\varepsilon_{11} < v_2 - v_1$, then agent 1 reports preferences with a_2 at the top and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of v_2. Together,

$$U_1(a_1, \sigma_{-1}) = (1 - p_{12})(v_1 + \eta_{12}) + p_{12} v_2.$$

Next suppose $\sigma_1 = a_k \in A \setminus \{a_1\}$. If $\varepsilon_{1k} < v_1 - v_k$, then agent 1 reports preferences with a_1 on top and receives a_1. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{1k} > v_1 - v_k$, then agent 1 reports preferences with a_k at the top and receives a_k. This occurs with probability p_{1k} and yields a conditional expected utility of $v_k + \eta_{k1}$. Together,

$$U_1(a_k, \sigma_{-1}) = (1 - p_{1k})v_1 + p_{1k}(v_k + \eta_{k1}).$$

By Lemma 1(2), $p_{1k}\eta_{k1} = (1 - p_{1k})\eta_{1k}$. If $k = 2$, then

$$\begin{aligned}
U_1(a_2, \sigma_{-1}) &= (1 - p_{12})v_1 + p_{12}v_2 + p_{12}\eta_{21} \\
&= (1 - p_{12})v_1 + p_{12}v_2 + (1 - p_{12})\eta_{12} \\
&= U_1(a_1, \sigma_{-1}).
\end{aligned}$$

Therefore, agent 1 is indifferent between investigating a_1 and a_2. For $k > 2$,

$$p_{1k}\eta_{k1} = \int_{\alpha \geq v_1 - v_k} \alpha \, dF(\alpha) < \int_{\alpha \geq v_1 - v_2} \alpha \, dF(\alpha) = p_{12}\eta_{21}.$$

Also, $p_{1k} < p_{12}$ and $v_k - v_1 < v_2 - v_1$ so

$$\begin{aligned}
U_1(a_k, \sigma_{-1}) &= (1 - p_{1k})v_1 + p_{1k}(v_k + \eta_{k1}) \\
&= v_1 + p_{1k}(v_k - v_1) + p_{1k}\eta_{k1} \\
&< v_1 + p_{12}(v_2 - v_1) + p_{12}\eta_{21} \\
&= (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21}) \\
&= U_1(a_2, \sigma_{-1}).
\end{aligned}$$

Therefore, for each $k > 2$, investigating a_k is strictly dominated for agent 1.

Step 2: Equilibrium strategy for agent 2. Let $\sigma_{-2} \in A^{N \setminus \{2\}}$ with $\sigma_1 \in \{a_1, a_2\}$.

Given $\sigma_1 \in \{a_1, a_2\}$, agent 1 receives a_1 with probability $1 - p_{12}$ and receives a_2 with probability p_{12} independent of σ_2. By computing expected utilities, we show that $\sigma_2 = a_3$ is a unique best response for agent 2. Let $\sigma_2 = a_k$.

Case 1: $k = 1$. If agent 1 receives a_1, then agent 2 receives a_2 which yields a conditional expected utility of v_2. Suppose instead that agent 1 receives a_2. If $\varepsilon_{21} > v_3 - v_1$, then agent 2 reports preferences with a_1 at the top among $A \backslash \{a_2\}$ and receives a_1. This occurs with probability $1 - p_{13}$ and yields a conditional expected utility of $v_1 + \eta_{13}$. If instead $\varepsilon_{21} < v_3 - v_1$, then agent 2 reports preferences with a_3 at the top among $A \backslash \{a_2\}$ and receives a_3. This occurs with probability p_{13} and yields a conditional expected utility of v_3. Together,

$$U_2(a_1, \sigma_{-2}) = (1 - p_{12})v_2 + p_{12}\big[(1 - p_{13})(v_1 + \eta_{13}) + p_{13}v_3\big].$$

Case 2: $k = 2$. If agent 1 receives a_2, then agent 2 receives a_1 which yields a conditional expected utility of v_1. Suppose instead that agent 1 receives a_1. If $\varepsilon_{22} > v_3 - v_2$, then agent 2 reports preferences with a_2 at the top among $A \backslash \{a_1\}$ and receives a_2. This occurs with probability $1 - p_{23}$ and yields a conditional expected utility of $v_2 + \eta_{23}$. If instead $\varepsilon_{22} < v_3 - v_2$, then agent 2 reports preferences with a_3 at the top among $A \backslash \{a_1\}$ and receives a_3. This occurs with probability p_{23} and yields a conditional expected utility of v_3. Together,

$$U_2(a_2, \sigma_{-2}) = (1 - p_{12})\big[(1 - p_{23})(v_2 + \eta_{23}) + p_{23}v_3\big] + p_{12}v_1.$$

Case 3: $k \geq 3$. First, suppose agent 1 receives a_1. If $\varepsilon_{2k} < v_2 - v_k$, then agent 2 reports preferences with a_1 and a_2 at the top and receives a_2. This occurs with probability $1 - p_{2k}$ and yields a conditional expected utility of v_2. If instead $\varepsilon_{2k} > v_2 - v_k$, then agent 2 reports preferences with a_1 and a_k at the top and receives a_k. This occurs with probability p_{2k} and yields a conditional expected utility of $v_k + \eta_{k2}$. Second, suppose agent 1 receives a_2. If $\varepsilon_{2k} < v_1 - v_k$, then agent 2 reports preferences with a_1 at the top and receives a_1. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{2k} > v_1 - v_k$, then agent 2 reports preferences with a_k at the top and receives a_k. This occurs with probability p_{1k} and yields a conditional expected utility of $v_k + \eta_{k1}$. Together,

$$U_2(a_k, \sigma_{-2}) = (1 - p_{12})\big[(1 - p_{2k})v_2 + p_{2k}(v_k + \eta_{k2})\big] + p_{12}\big[(1 - p_{1k})v_1 + p_{1k}(v_k + \eta_{k1})\big].$$

We now compare these expected utilities. For $k > 3$, by Lemma 1(3), $p_{1k}\eta_{k1} < p_{13}\eta_{31}$ and $p_{2k}\eta_{k2} < p_{23}\eta_{32}$. Also, $p_{1k} < p_{13}$, $p_{2k} < p_{23}$, $v_k - v_1 < v_3 - v_1$, and $v_k - v_2 < v_3 - v_2$. Therefore,

$$(1 - p_{2k})v_2 + p_{2k}(v_k + \eta_{k2}) < (1 - p_{23})v_2 + p_{23}(v_3 + \eta_{32}) \text{ and}$$
$$(1 - p_{1k})v_1 + p_{1k}(v_k + \eta_{k1}) < (1 - p_{13})v_1 + p_{13}(v_3 + \eta_{31})$$

so $U_2(a_k, \sigma_{-2}) < U_2(a_3, \sigma_{-2})$.

Next, by Lemma 1(2), $p_{13}\eta_{31} = (1 - p_{13})\eta_{13}$ and $p_{23}\eta_{32} = (1 - p_{23})\eta_{23}$. Also, by definition, $v_3 + \eta_{32} > v_2$ and $v_3 + \eta_{31} > v_1$. Therefore,

$$
\begin{aligned}
U_2(a_3, \sigma_{-2}) &= (1 - p_{12})\big[(1 - p_{23})v_2 + p_{23}(v_3 + \eta_{32})\big] + p_{12}\big[(1 - p_{13})v_1 + p_{13}(v_3 + \eta_{31})\big] \\
&> (1 - p_{12})v_2 + p_{12}\big[(1 - p_{13})v_1 + p_{13}v_3 + p_{13}\eta_{31}\big] \\
&= (1 - p_{12})v_2 + p_{12}\big[(1 - p_{13})v_1 + p_{13}v_3 + (1 - p_{13})\eta_{13}\big] \\
&= U_2(a_1, \sigma_{-2}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
U_2(a_3, \sigma_{-2}) &= (1 - p_{12})\big[(1 - p_{23})v_2 + p_{23}(v_3 + \eta_{32})\big] + p_{12}\big[(1 - p_{13})v_1 + p_{13}(v_3 + \eta_{31})\big] \\
&> (1 - p_{12})\big[(1 - p_{23})v_2 + p_{23}v_3 + p_{23}\eta_{32}\big] + p_{12}v_1 \\
&> (1 - p_{12})\big[(1 - p_{23})v_2 + p_{23}v_3 + (1 - p_{23})\eta_{23}\big] + p_{12}v_1 \\
&= U_2(a_2, \sigma_{-2}).
\end{aligned}
$$

Altogether, conditional on agent 1 choosing one of his dominant strategies, $\sigma_2 = a_3$ is a unique best response for agent 2.

Step 3: Equilibrium strategy for agent $\boldsymbol{i}$, $\boldsymbol{3 \leq i < m}$. The logic is similar to Step 2. Let $\sigma_{-i} \in A^{N \setminus \{i\}}$ with $\sigma_1 \in \{a_1, a_2\}$ and for each $j \in \{2, \ldots, i-1\}$, $\sigma_j = \sigma_{j+1}$. Then agents $\{1, \ldots, i-1\}$ collectively receive $i - 1$ of the objects $\{a_1, \ldots, a_i\}$. Let $\sigma_i = a_k$. Suppose $a_l \in \{a_1, \ldots, a_i\}$ is the object none of the agents $\{1, \ldots, i-1\}$ receive.

Case 1: $k \leq i$. If $k \neq l$, then agent i receives a_l which yields a conditional expected utility of v_l. By comparison, investigating a_{i+1} yields expected utility $(1 - p_{l(i+1)})v_l +$

$p_{l(i+1)}(v_{i+1} + \eta_{(i+1)l}) > v_l$.

Suppose instead that $k = l$. If $\varepsilon_{il} > v_{i+1} - v_l$, then agent 2 reports preferences with a_l above a_{i+1} and receives a_l. This occurs with probability $1 - p_{l(i+1)}$ and yields a conditional expected utility of $v_l + \eta_{l(i+1)}$. If instead $\varepsilon_{il} < v_{i+1} - v_l$, then agent 2 reports preferences with a_{i+1} above a_l and receives a_{i+1}. This occurs with probability $p_{l(i+1)}$ and yields a conditional expected utility of v_{i+1}. Agent i's expected utility in this case is then $(1 - p_{l(i+1)})(v_l + \eta_{l(i+1)}) + p_{l(i+1)}v_{i+1}$. By comparison, investigating a_{i+1} again yields expected utility $(1 - p_{l(i+1)})v_l + p_{l(i+1)}(v_{i+1} + \eta_{(i+1)l})$. Since $(1 - p_{l(i+1)})\eta_{l(i+1)} = p_{l(i+1)}\eta_{(i+1)l}$, the expected utility is the same under either strategy. Since investigating a_{i+1} yields greater expected utility in the first case and equal expected utility in this case, σ_i is strictly dominated.

Case 2: $k \geq i + 1$. If $\varepsilon_{ik} < v_l - v_k$, then agent 2 reports preferences with a_l above a_k and receives a_l. This occurs with probability $1 - p_{lk}$ and yields a conditional expected utility of v_l. If instead $\varepsilon_{ik} > v_l - v_k$, then agent 2 reports preferences with a_k above a_l and receives a_k. This occurs with probability p_{lk} and yields a conditional expected utility of $v_k + \eta_{kl}$. Therefore, agent i's expected utility conditional on agents $\{1, \ldots, i-1\}$ receiving $\{a_1, \ldots, a_i\} \backslash \{a_l\}$ is $(1 - p_{lk})v_l + p_{lk}(v_k + \eta_{kl})$. By Lemma 1(3), for $k > i + 1$,

$$(1 - p_{lk})v_l + p_{lk}(v_k + \eta_{kl}) < (1 - p_{l(i+1)})v_l + p_{l(i+1)}(v_{i+1} + \eta_{(i+1)l}).$$

Thus, investigating a_k yields strictly lower expected utility than investigating a_{i+1} in each case and the strategy is strictly dominated. Altogether, conditional on agents with higher priority best responding to the strategies of agents with even higher priority, $\sigma_i = a_{i+1}$ is a unique best response for agent i.

A.3.2 TTC rules

Suppose $m \geq 3$. We consider the equilibrium strategies of the agents in order of the common values of their endowments. We argue that each agent has a unique best response to the equilibrium strategies of the preceding agents.

Step 1: Equilibrium strategies for agent 1. We show that $\sigma_1 = a_1$ is a strictly dominant strategy for agent 1 by analyzing his option sets. Let $P_{-1} \in \mathcal{P}^{N\setminus\{1\}}$ and $O_1 \equiv O_1(P_{-1}, TTC)$. By definition of TTC, $a_1 \in O_1$. Let $a_k \in A\setminus\{a_1\}$. To compare $\sigma_1 = a_k$ and $\sigma_1 = a_1$, we consider two cases.

Case 1.1: $a_k \in O_1$. First consider $\sigma_1 = a_k$. If $\varepsilon_{1k} < v_1 - v_k$, then agent 1 reports preferences with a_1 at the top and receives a_1. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{1k} > v_1 - v_k$, then agent 1 reports preferences with a_k at the top and receives a_k. This occurs with probability p_{1k} and yields a conditional expected utility of $v_k + \eta_{k1}$. Then $U_1(a_k, \sigma_{-1}|a_k \in O_1) = (1 - p_{1k})v_1 + p_{1k}(v_k + \eta_{k1})$.

Now consider $\sigma_1 = a_1$. If $\varepsilon_{11} > v_k - v_1$, then agent 1 reports preferences with a_1 above a_k and receives a_1 or a more preferred object. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of at least $v_1 + \eta_{1k}$. If instead $\varepsilon_{11} < v_k - v_1$, then agent 1 reports preferences with a_k above a_1 and receives a_k or a more preferred object. This occurs with probability p_{1k} and yields a conditional expected utility of at least v_k. Agent 1's expected utility in this case is $U_1(a_1, \sigma_{-1}|a_k \in$

$O_1) \geq (1 - p_{1k})(v_1 + \eta_{1k}) + p_{1k}v_k$.

By Lemma 1(2), $p_{1k}\eta_{k1} = (1 - p_{1k})\eta_{1k}$. Therefore, conditional on $a_k \in O_1$, agent 1's expected utility when investigating a_1 is at least as great as his expected utility when investigating a_k.

Case 1.2: $a_k \notin O_1$. First consider $\sigma_1 = a_k$. Then agent 1 reports preferences with a_1 at the top of O_1 and receives a_1. Then $U_1(a_k, \sigma_{-1}|a_k \notin O_1) = v_1$.

Now consider $\sigma_1 = a_1$. Since $a_1 \in O_1$, agent 1's conditional expected utility is at least v_1. To see that it is strictly greater, let $a_l \in A\backslash\{a_1, a_k\}$. For each σ_l, agent l reports preferences with a_1 at the top with positive probability. In this case, $a_l \in O_1$. Also, $\varepsilon_{11} < v_l - v_1$ with probability $p_{1l} > 0$. Then agent 1 reports preferences with a_l above a_1. Since these events are independent, there is positive probability that both $a_l \in O_1$ and $a_l \; P(\varepsilon_{11}) \; a_1$. In this case, agent 1 receives a_l or a more preferred object. Then $U_1(a_1, \sigma_{-1}|a_k \notin O_1) \geq v_l + \eta_{l1} > v_1$.

Altogether, $U_1(a_k, \sigma_{-1}) < U_1(a_1, \sigma_{-1})$. Since this is true for each $k \neq 1$, $\sigma_1 = a_1$ is a strictly dominant strategy for agent 1.

Step 2: Equilibrium strategy for agent 2. We show that $\sigma_2 = a_2$ is a strict best response to $\sigma_1 = a_1$. Let $P_{-2} \in \mathcal{P}^{N\backslash\{2\}}$ with P_1 determined by σ_1 and let $O_2 \equiv O_2(P_{-2}, TTC)$. By definition of TTC, $a_2 \in O_2$. Moreover, $a_1 \in O_2$ with probability p_{12} and $a_1 \notin O_2$ with probability $1 - p_{12}$ independent of agent 2's strategy. Let $a_k \in A\backslash\{a_2\}$.

Case 2.1: $k = 1$. To compare $\sigma_2 = a_k$ and $\sigma_2 = a_2$, we distinguish two subcases.

Subcase 2.1.1: $a_1 \in O_2$. First consider $\sigma_2 = a_1$. If $\varepsilon_{21} > v_2 - v_1$, then agent 2 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability

$1-p_{12}$ and yields a conditional expected utility of $v_1+\eta_{12}$. If instead $\varepsilon_{21}<v_2-v_1$, then agent 2 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of v_2. Then $U_2(a_1,\sigma_{-2}|a_1\in O_2)=(1-p_{12})(v_1+\eta_{12})+p_{12}v_2$.

Now consider $\sigma_2=a_2$. If $\varepsilon_{22}<v_1-v_2$, then agent 2 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1-p_{12}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{22}>v_1-v_2$, then agent 2 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of $v_2+\eta_{21}$. Then $U_2(a_2,\sigma_{-2}|a_1\in O_2)=(1-p_{12})v_1+p_{12}(v_2+\eta_{21})$.

By Lemma 1(2), $p_{12}\eta_{21}=(1-p_{12})\eta_{12}$. Therefore, $U_2(a_1,\sigma_{-2}|a_1\in O_2)=U_2(a_2,\sigma_{-2}|a_1\in O_2)$.

Subcase 2.1.2: $a_1\notin O_2$. First consider $\sigma_2=a_1$. Then agent 2 reports preferences with a_2 at the top of O_2 and receives a_2. Then $U_2(a_1,\sigma_{-2}|a_1\notin O_2)=v_2$.

Now consider $\sigma_2=a_2$. Since $a_2\in O_2$, agent 2's conditional expected utility is at least v_2. To see that it is strictly greater, consider a_3. For each σ_3, agent 3 reports preferences with a_2 at the top of $A\backslash\{a_1\}$ with positive probability. Whenever $a_1\notin O_2$, $a_1\notin O_3$, so $a_3\in O_2$. Also, $\varepsilon_{22}<v_3-v_2$ with probability $p_{23}>0$. In this event, agent 2 reports preferences with a_3 above a_2. Since these events are independent, they occur simultaneously with positive probability. In this joint event, agent 2 receives a_3. Then $U_2(a_2,\sigma_{-2}|a_1\notin O_2)=v_3+\eta_{32}>v_2$.

Altogether, $U_2(a_1,\sigma_{-2})<U_2(a_2,\sigma_{-2})$. Therefore, $\sigma_2=a_1$ is strictly dominated.

Case 2.2: $k>2$. To compare $\sigma_2=a_k$ and $\sigma_2=a_2$, we distinguish four subcases.

Subcase 2.2.1: $a_1\in O_2$ and $a_k\in O_2$. First consider $\sigma_2=a_k$. If $\varepsilon_{2k}<v_1-v_k$,

then agent 2 reports preferences with a_1 at the top of O_2 and receives a_1. This occurs with probability $1 - p_{1k}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{2k} > v_1 - v_k$, then agent 2 reports preferences with a_k at the top of O_2 and receives a_k. This occurs with probability p_{1k} and yields a conditional expected utility of $v_k + \eta_{k1}$. Then $U_2(a_k, \sigma_{-2}|a_1 \in O_2, a_k \in O_2) = (1-p_{1k})v_1 + p_{1k}(v_k + \eta_{k1}) = v_1 + p_{1k}(v_k - v_1 + \eta_{k1})$.

Now consider $\sigma_2 = a_2$. Then agent 2 reports preferences with either a_1 or a_2 at the top of O_2 and receives that object. Thus, as computed in Subcase 2.1.1, $U_2(a_2, \sigma_{-2}|a_1 \in O_2, a_k \in O_2) = (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21}) = v_1 + p_{12}(v_2 - v_1 + \eta_{21})$.

By Lemma 1(3), since $2 < k$, $p_{1k}(v_k - v_1 + \eta_{k1}) < p_{12}(v_2 - v_1 + \eta_{21})$. Therefore, in this subcase agent 2's expected utility when investigating a_2 is higher than his expected utility when investigating a_k.

Subcase 2.2.2: $a_1 \in O_2$ and $a_k \notin O_2$. First consider $\sigma_2 = a_k$. Then agent 2 reports preferences with a_1 at the top of O_2 and receives a_1. This yields a conditional expected utility of v_1. Now consider $\sigma_2 = a_2$. As in Subcase 2.1.1, agent 2's expected utility is again $v_1 + p_{12}(v_2 - v_1 + \eta_{21}) > v_1$. Thus, in this subcase agent 2's expected utility when investigating a_2 is higher than his expected utility when investigating a_k.

Subcase 2.2.3: $a_1 \notin O_2$ and $a_k \in O_2$. First consider $\sigma_2 = a_k$. If $\varepsilon_{2k} < v_2 - v_k$, then agent 2 reports preferences with a_2 above a_k and receives a_2. This occurs with probability $1 - p_{2k}$ and yields a conditional expected utility of v_2. If instead $\varepsilon_{2k} > v_2 - v_k$, then agent 2 reports preferences with a_k above a_2 and receives a_k. This occurs with probability p_{2k} and yields a conditional expected utility of $v_k + \eta_{k2}$. Then $U_2(a_k, \sigma_{-2}|a_1 \notin O_2, a_k \in O_2) = (1 - p_{2k})v_2 + p_{2k}(v_k + \eta_{2k})$.

Now consider $\sigma_2 = a_2$. If $\varepsilon_{22} > v_k - v_2$, then agent 2 reports preferences with a_2

above a_k and receives a_2 or a more preferred object. This occurs with probability $1-p_{2k}$ and yields a conditional expected utility of at least $v_2+\eta_{2k}$. If instead $\varepsilon_{22} < v_k - v_2$, then agent 2 reports preferences with a_k above a_2 and receives a_k or a more preferred object. This occurs with probability p_{2k} and yields a conditional expected utility of at least v_k. Then $U_2(a_2, \sigma_{-2}|a_1 \notin O_2, a_k \in O_2) \geq (1-p_{2k})(v_2+\eta_{2k})+p_{2k}v_k$.

By Lemma 1(2), $p_{2k}\eta_{k2} = (1-p_{2k})\eta_{2k}$. Therefore, $U_2(a_k, \sigma_{-2}|a_1 \notin O_2, a_k \in O_2) \leq U_2(a_2, \sigma_{-2}|a_1 \notin O_2, a_k \in O_2)$.

Subcase 2.2.4: $a_1 \notin O_2$ and $a_k \notin O_2$. First consider $\sigma_2 = a_k$. Then agent 2 reports preferences with a_2 at the top of O_2 and receives a_2. This yields a conditional expected utility of v_2. Now consider $\sigma_2 = a_2$. Since $a_2 \in O_2$, agent 2's conditional expected utility is at least v_2. Thus, $U_2(a_k, \sigma_{-2}|a_1 \notin O_2, a_k \notin O_2) \leq U_2(a_2, \sigma_{-2}|a_1 \notin O_2, a_k \notin O_2)$.

Now $a_1 \in O_2$ with positive probability, so at least one of Subcases 2.2.1 and 2.2.2 occurs with positive probability. Therefore, $U_2(a_k, \sigma_{-2}) < U_2(a_2, \sigma_{-2})$. Combining results, $\sigma_2 = a_2$ is a unique best response for agent 2.

Step 3: Equilibrium strategy for agent i, $i \geq 3$. We show that $\sigma_i = a_i$ is a best response to $(\sigma_1, \ldots, \sigma_{i-1}) = (a_1, \ldots, a_{i-1})$. Let $P_{-i} \in \mathcal{P}^{N\setminus\{i\}}$ where for each $j \in \{1, \ldots, i-1\}$, P_j is determined by σ_j and let $O_i \equiv O_i(P_{-i}, TTC)$. By definition of TTC, $a_i \in O_i$. Moreover, for each $l \in \{1, \ldots, i-1\}$, there is positive probability that agent l reports preferences such that $a_i \; P_l \; a_l$ and for each $j \in \{1, \ldots, i-1\}\setminus\{l\}$, $a_j \; P_j \; a_l$. In this case, $a_l \in O_i$. Therefore, for each $a_l \in \{a_1, \ldots, a_{i-1}\}$, $a_l \in O_i$ with positive probability independent of agent i's strategy.

Let $a_k \in A\setminus\{a_i\}$, let $v_h \equiv \max\{v_l : a_l \in O_i\setminus\{a_k\}\}$, and let a_h be the object with common value v_h. Since $a_i \in O_i$, $h \leq i$.

Case 3.1: $k < i$. We claim that ($\star$) for each pair $j, l \in \{1, \ldots, i-1\}$ with $j \neq l$, $\{a_j, a_l\} \not\subseteq O_i$. To see this, suppose by way of contradiction that there is such a pair. Then $a_i \; P_j \; a_j$ and $a_i \; P_l \; a_l$. Also, $v_i < v_j$ and $v_i < v_l$. Since P_j and P_l are determined by σ_j and σ_l, we have $a_l \; P_j \; a_i$ and $a_j \; P_l \; a_i$. But then agents j and l prefer trading with each other to trading with agent i, so $a_j \notin O_i$ and $a_l \notin O_i$. To compare $\sigma_i = a_k$ and $\sigma_i = a_i$, we distinguish two subcases.

Subcase 3.1.1: $a_k \in O_i$. Then by ($\star$), $a_h = a_i$ and agent i receives either a_k or a_i. First consider $\sigma_i = a_k$. If $\varepsilon_{ik} > v_k - v_i$, then agent i reports preferences with a_k at the top of O_i and receives a_k. This occurs with probability $1 - p_{ik}$ and yields a conditional expected utility of $v_k + \eta_{ki}$. If instead $\varepsilon_{ik} < v_k - v_i$, then agent i reports preferences with a_i at the top of O_i and receives a_i. This occurs with probability p_{ik} and yields a conditional expected utility of v_h. Then $U_i(a_k, \sigma_{-i} | a_k \in O_i) = (1 - p_{ik})(v_k + \eta_{ki}) + p_{ik} v_i$.

Now consider $\sigma_i = a_i$. If $\varepsilon_{ii} < v_k - v_i$, then agent i reports preferences with a_k at the top of O_i and receives a_k. This occurs with probability $1 - p_{ik}$ and yields a conditional expected utility of v_k. If instead $\varepsilon_{ii} > v_k - v_i$, then agent i reports preferences with a_i at the top of O_i and receives a_i. This occurs with probability p_{ik} and yields a conditional expected utility of $v_i + \eta_{ik}$. Then $U_i(a_i, \sigma_{-i} | a_k \in O_i) = (1 - p_{ik}) v_k + p_{ik}(v_i + \eta_{ik})$.

By Lemma 1(2), $p_{ik}\eta_{ik} = (1 - p_{ik})\eta_{ki}$. Therefore, $U_i(a_k, \sigma_{-i} | a_k \in O_i) = U_i(a_i, \sigma_{-i} | a_k \in O_i)$.

Subcase 3.1.2: $a_k \notin O_i$. First consider $\sigma_i = a_k$. Then agent i reports preferences which rank O_i according to their common values and receives a_h. Then for each

$j \in \{1, \ldots, i\}, U_i(a_k, \sigma_{-i} | a_k \notin O_i, h = j) = v_h$.

Now consider $\sigma_i = a_i$. For each $j \in \{1, \ldots, i-1\}$ there is positive probability that $h = j$, so we consider each of these events separately. If $\varepsilon_{ii} > v_h - v_i$, then i reports preferences with a_i at the top of O_i and receives a_i. This occurs with probability p_{ih} and yields a conditional expected utility of $v_i + \eta_{ih}$. If instead $\varepsilon_{ii} < v_h - v_i$, then i reports preferences with a_h at the top of O_i and receives a_h. This occurs with probability $1 - p_{ih}$ and yields a conditional expected utility of v_h. Then $U_i(a_i, \sigma_{-i} | a_k \notin O_i, h = j) = (1 - p_{ih})v_h + p_{ih}(v_i + \eta_{ih})$. By Lemma 1(2), $p_{ih}\eta_{ih} = (1 - p_{ih})\eta_{hi}$. Therefore $U_i(a_i, \sigma_{-i} | a_k \notin O_i, h = j) > U_i(a_k, \sigma_{-i} | a_k \notin O_i, h = j) = v_h$. If $h = i$, then $U_i(a_i, \sigma_{-i} | a_k \notin O_i, h = i) \geq v_i = U_i(a_k, \sigma_{-i} | a_k \notin O_i, h = j)$.

Altogether, $U_i(a_k, \sigma_{-i}) < U_i(a_i, \sigma_{-i})$. Therefore, for $k < i$, $\sigma_i = a_k$ is not a best response to $(\sigma_1, \ldots, \sigma_{i-1})$.

Case 3.2: $k > i$. To compare $\sigma_i = a_k$ and $\sigma_i = a_i$, we distinguish four subcases.

Subcase 3.2.1: $h < i$ and $a_k \in O_i$. First consider $\sigma_i = a_k$. If $\varepsilon_{ik} < v_h - v_k$, then agent 2 reports preferences with a_h at the top of O_i and receives a_h. This occurs with probability $1 - p_{hk}$ and yields a conditional expected utility of v_h. If instead $\varepsilon_{ik} > v_h - v_k$, then agent i reports preferences with a_k at the top of O_i and receives a_k. This occurs with probability p_{hk} and yields a conditional expected utility of $v_k + \eta_{kh}$. Then $U_i(a_k, \sigma_{-i} | h < i, a_k \in O_i) = (1 - p_{hk})v_h + p_{hk}(v_k + \eta_{kh}) = v_h + p_{hk}(v_k - v_h + \eta_{kh})$.

Now consider $\sigma_i = a_i$. Then agent i reports preferences with either a_h or a_i at the top of O_i and receives that object. Thus, as computed in Subcase 3.1.1, $U_i(a_i, \sigma_{-i} | h < i, a_k \in O_i) = (1 - p_{hi})v_h + p_{hi}(v_i + \eta_{ih}) = v_h + p_{hi}(v_i - v_h + \eta_{ih})$.

By Lemma 1(3), since $h < i < k$, $p_{hk}(v_k - v_h + \eta_{kh}) < p_{hi}(v_i - v_h + \eta_{ih})$. Therefore,

$U_i(a_k, \sigma_{-i}|h < i, a_k \in O_i) < U_i(a_i, \sigma_{-i}|h < i, a_k \in O_i)$.

Subcase 3.2.2: $h < i$ and $a_k \notin O_i$. An argument identical to that in Subcase 3.1.2 shows that $U_i(a_k, \sigma_{-i}|h < i, a_k \notin O_i) < U_i(a_i, \sigma_{-i}|h < i, a_k \notin O_i)$.

Subcase 3.2.3: $h = i$ and $a_k \in O_i$. First consider $\sigma_i = a_k$. If $\varepsilon_{ik} < v_i - v_k$, then agent i reports preferences with a_i above a_k and receives a_i. This occurs with probability $1 - p_{ik}$ and yields a conditional expected utility of v_i. If instead $\varepsilon_{ik} > v_i - v_k$, then agent i reports preferences with a_k above a_i and receives a_k. This occurs with probability p_{ik} and yields a conditional expected utility of $v_k + \eta_{ki}$. Then $U_i(a_k, \sigma_{-i}|h = i, a_k \in O_i) = (1 - p_{ik})v_i + p_{ik}(v_k + \eta_{ki})$.

Now consider $\sigma_i = a_i$. If $\varepsilon_{ii} > v_k - v_i$, then agent i reports preferences with a_i above a_k and receives a_i or a more preferred object. This occurs with probability $1 - p_{ik}$ and yields a conditional expected utility of at least $v_i + \eta_{ik}$. If instead $\varepsilon_{ii} < v_k - v_i$, then agent i reports preferences with a_k above a_i and receives a_k or a more preferred object. This occurs with probability p_{ik} and yields a conditional expected utility of at least v_k. Then $U_i(a_i, \sigma_{-i}|h = i, a_k \in O_i) \geq (1 - p_{ik})(v_i + \eta_{ik}) + p_{ik}v_k$.

By Lemma 1(2), $p_{ik}\eta_{ki} = (1 - p_{ik})\eta_{ik}$. Therefore, $U_i(a_k, \sigma_{-i}|h = i, a_k \in O_i) \leq U_i(a_i, \sigma_{-i}|h = i, a_k \in O_i)$.

Subcase 3.2.4: $h = i$ and $a_k \notin O_i$. First consider $\sigma_i = a_k$. Then agent i reports preferences with a_i at the top of O_i and receives a_i. Then $U_i(a_k, \sigma_{-i}|h = i, a_k \notin O_i) = v_i$. Now consider $\sigma_i = a_i$. Since $a_i \in O_i$, $U_i(a_i, \sigma_{-i}|h = i, a_k \notin O_i) \geq v_i = U_i(a_k, \sigma_{-i}|h = i, a_k \notin O_i)$.

Now $a_h \neq a_i$ with positive probability, so at least one of Subcases 3.2.1 and 3.2.2 occurs with positive probability. Therefore, $U_i(a_k, \sigma_{-i}) < U_i(a_i, \sigma_{-i})$. Thus, for $k > i$,

$\sigma_i = a_k$ is not a best response to $(\sigma_1, \ldots, \sigma_{i-1})$. Instead, $\sigma_i = a_i$ is the unique best response.

Summarizing, if $m \geq 3$ and $\sigma \in A^N$ is an equilibrium profile, then for each $i \in N$, $\sigma_i = a_i$.

We conclude by analyzing the special case $n = m = 2$. Let $\sigma \in A^N$.

Agent 1: First suppose $a_2 \in O_1$. Under $\sigma_1 = a_1$, if $\varepsilon_{11} > v_2 - v_1$, then agent 1 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of $v_1 + \eta_{12}$. If instead $\varepsilon_{11} < v_2 - v_1$, then agent 1 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of v_2. Then $U_1(a_1, \sigma_2 | a_2 \in O_1) = (1 - p_{12})(v_1 + \eta_{12}) + p_{12} v_2$.

Under $\sigma_1 = a_2$, if $\varepsilon_{12} < v_1 - v_2$, then agent 1 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{12} > v_1 - v_2$, then agent 1 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of $v_2 + \eta_{21}$. Then $U_1(a_2, \sigma_2 | a_2 \in O_1) = (1 - p_{12}) v_1 + p_{12}(v_2 + \eta_{21})$. By Lemma 1(2), $p_{12}\eta_{21} = (1 - p_{12})\eta_{12}$. Therefore, $U_1(a_1, \sigma_2 | a_2 \in O_1) = U_1(a_2, \sigma_2 | a_2 \in O_1)$.

Now suppose $a_2 \notin O_1$. Then agent 1 receives a_1. As this is independent of σ_1, $U_1(a_1, \sigma_2 | a_2 \notin O_1) = U_1(a_2, \sigma_2 | a_2 \notin O_1) = v_1$. Altogether, $U_1(a_1, \sigma_2) = U_1(a_2, \sigma_2)$.

Agent 2: First suppose $a_1 \in O_2$. Under $\sigma_2 = a_1$, if $\varepsilon_{21} > v_2 - v_1$, then agent 2 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of $v_1 + \eta_{12}$. If instead $\varepsilon_{21} < v_2 - v_1$, then agent 2 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of v_2. Then $U_2(a_1, \sigma_1 | a_1 \in O_2) = (1 - p_{12})(v_1 + \eta_{12}) + p_{12} v_2$.

Under $\sigma_2 = a_2$, if $\varepsilon_{22} < v_1 - v_2$, then agent 2 reports preferences with a_1 above a_2 and receives a_1. This occurs with probability $1 - p_{12}$ and yields a conditional expected utility of v_1. If instead $\varepsilon_{22} > v_1 - v_2$, then agent 2 reports preferences with a_2 above a_1 and receives a_2. This occurs with probability p_{12} and yields a conditional expected utility of $v_2 + \eta_{21}$. Then $U_2(a_2, \sigma_1 | a_1 \in O_2) = (1 - p_{12}) v_1 + p_{12}(v_2 + \eta_{21})$. By Lemma 1(2), $p_{12}\eta_{21} = (1 - p_{12})\eta_{12}$. Therefore, $U_2(a_1, \sigma_1 | a_1 \in O_2) = U_2(a_2, \sigma_1 | a_1 \in O_2)$.

Now suppose $a_1 \notin O_2$. Then agent 2 receives a_2. As this is independent of σ_2, $U_2(a_1, \sigma_1 | a_1 \notin O_2) = U_2(a_2, \sigma_1 | a_1 \notin O_2) = v_2$. Altogether, $U_2(a_1, \sigma_1) = U_2(a_2, \sigma_1)$. Combining results, if $n = m = 2$, then each profile $\sigma \in A^N$ constitutes an equilibrium.

A.4 Proofs of Proposition 2 and Theorem 1

We first provide explicit formulas for the equilibrium welfare of each agent under SD and TTC. To do so, we introduce some additional notation. For each $k \in N$, let $N^k \equiv \{1, \ldots, k-1\}$. Then under SD, N^k is the set agents with higher priority than agent k. Similarly, under TTC, N^k is the set of agents whose endowments have common values higher than the common value of agent k's endowment. For each pair $l, k \in \{1, \ldots, m\}$ with $l \leq k$, let

$$P(l,k) \equiv \begin{cases} 1 \text{ if } l = k \\ p_{l(l+1)} \cdot p_{l(l+2)} \cdots p_{lk} \quad \text{if } l < k \end{cases}$$

and recursively define

$$Q(l,k) \equiv \begin{cases} P(l,k) & \text{if } l = 1 \\ 1 - \sum_{h=1}^{k-1} Q(h,k) & \text{if } 1 < l = k \\ Q(l,l) \cdot P(l,k) & \text{if } 1 < l < k \end{cases} \cdot$$

Lemma 2 below allows us to interpret these products in terms of agents' options sets: under either SD or TTC, $Q(l,k)$ represents the probability that a_l is the object with the highest common value in agent k's option set.

Lemma 2. *For each pair $l, k \in \{1, \ldots, n\}$ with $l \leq k$, $Q(l,k)$ is the probability that (i) a_l has the highest common value in agent k's option set under SD and (ii) a_l has the highest common value in agent k's option set under TTC.*

Proof. Let $l, k \in \{1, \ldots, n\}$ with $l < k$. After verifying the probabilities for $l < k$, the formula for $l = k$ follows by subtraction. Let $\sigma^{\prec}, \sigma^{\mu} \in A^N$ be equilibria under SD and TTC respectively and let $P^{\prec}, P^{\mu} \in \mathcal{P}^N$ be preferences determined by $\sigma^{\prec}$ and σ^{μ}. That is, $P^{\prec}$ and P^{μ} are random variables that depend on the realizations of $\varepsilon_{\sigma^{\prec}}$ and $\varepsilon_{\sigma^{\mu}}$ respectively.

Without loss of generality, suppose that for each $i \in N$, $\sigma_i^{\prec} = a_{i+1}$ and $\sigma_i^{\mu} = a_i$

where $a_{m+1} = a_1$ if necessary.

(i) Priority rule. According to $\sigma^{\prec}$, agents in N^k receive at least $k-1$ of the objects $\{a_1, \ldots, a_k\}$. Thus, agent k's option set contains exactly one of these objects and the cases are mutually exclusive. Now suppose $a_l \in O_k(P^{\prec}_{-k}, SD)$. Then no agent in N^k receives a_l. We verify the probabilities by induction on l.

Case 1.1: $l = 1$. Then for each $i \in N^k$, agent i receives a_{i+1}. Therefore, $a_{i+1}\ P^{\prec}_i\ a_1$ which occurs with probability $p_{1(i+1)}$. Since these events are independent, the probability that $a_1 \in O_k(P^{\prec}_{-k}, SD)$ is $p_{12} \cdot p_{13} \cdots p_{1k} = Q(1, k)$.

Case 1.2: $l = 2$. Then agent 1 receives a_1 and for each $i \in N^k \backslash \{1\}$, agent i receives a_{i+1}. Therefore, $a_1\ P^{\prec}_1\ a_2$ which occurs with probability $1-p_{12} = 1-Q(1, 2) = Q(2, 2)$. Also, for each $i \in N^k \backslash \{1\}$, $a_{i+1}\ P^{\prec}_i\ a_2$ which occurs with probability $p_{2(i+1)}$. Since these events are independent, the probability that $a_2 \in O_k(P^{\prec}_{-k}, SD)$ is

$$(1 - p_{12})p_{23} \cdot p_{24} \cdots p_{2k} = Q(2, 2)P(2, k) = Q(2, k).$$

Case 1.3: $l \geq 3$. Suppose the claim is true for each $i < l$. Then the agents in N^k collectively receive $\{a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_k\}$. Moreover, since each agent i always receives an object among $\{a_1, \ldots, a_{i+1}\}$, this implies that the agents N^l collectively receive $\{a_1, \ldots, a_{l-1}\}$. Therefore, $\{a_1, \ldots, a_{l-1}\} \cap O_l(P^{\prec}_{-l}, SD) = \emptyset$. By hypothesis, for each $i < l$, $Q(i, l)$ is the probability that $a_i \in O_l(P^{\prec}_{-l}, SD)$, so $1 - Q(i, l)$ is the probability $a_i \notin O_l(P^{\prec}_{-l}, SD)$. By mutual exclusivity, $1 - \sum_{i=1}^{l-1} Q(i, l)$ is the probability that $\{a_1, \ldots, a_{l-1}\} \cap O_l(P^{\prec}_{-l}, SD) = \emptyset$. Next, for each $i \in N^k \backslash (N^l \cup \{l\})$, agent i receives a_{i+1}. Therefore, $a_{i+1}\ P^{\prec}_i\ a_l$ which occurs with probability $p_{l(i+1)}$.

Since these events are independent, the probability that $a_l \in O_k(P^{\prec}_{-k}, SD)$ is

$$\left(1 - \sum_{i=1}^{l-1} Q(i,l)\right) p_{l(l+1)} \cdot p_{l(l+2)} \cdots p_{lk} = Q(l,l) \cdot P(l,k) = Q(l,k).$$

Finally, since agent k's option set under SD always includes exactly one of $\{a_1, \ldots, a_k\}$, the probability that $a_k \in O_k(P^{\prec}_{-k}, SD)$ is $1 - \sum_{i=1}^{k-1} Q(i,k) = Q(k,k)$.

(ii) Top trading cycles rule. According to σ^μ, each agent i reports preferences that agree with P_0 on $A \backslash \{\mu_i\}$. Thus, for each pair $i, j \in N$ with $i < j$, $O_j(P^\mu_{-j}, TTC) \subseteq O_i(P^\mu_{-i}, TTC)$. Moreover, if agent i receives a_j, then agent j receives a_i and for each $h \in N$ with $i < h < j$, agent h receives $\mu_h = a_h$. In particular, the events $a_i \in O_j(P^\mu_{-j}, TTC)$ and $a_h \in O_j(P^\mu_{-j}, TTC)$ are mutually exclusive. Now suppose $a_l \in O_k(P^\mu_{-k}, TTC)$. Then no agent in N^k receives a_l. We verify the probabilities by induction on l.

Case 2.1: $l = 1$. Then $a_k \; P^\mu_1 \; a_1$. This occurs with probability p_{1k}. Furthermore, $\{a_1, \ldots, a_{k-1}\} \cap O_1(P^\mu_{-1}, TTC) = \{a_1\}$. For each $i \in N^k \backslash \{1\}$, $a_i \notin O_1(P^\mu_{-1}, TTC)$ implies $a_i \; P^\mu_i \; a_1$ which occurs with probability p_{1i}. Since these events are independent, the probability that $a_1 \in O_k(P^\mu_{-k}, TTC)$ is $p_{1k} \cdot p_{12} \cdot p_{13} \cdots p_{1(k-1)} = Q(1,k)$.

Case 2.2: $l = 2$. Then $a_k \; P^\mu_2 \; a_2$. This occurs with probability p_{2k}. Furthermore, $\{a_1, \ldots, a_{k-1}\} \cap O_2(P^\mu_{-2}, TTC) = \{a_2\}$. For each $i \in N^k \backslash \{2\}$, $a_i \notin O_2(P^\mu_{-2}, TTC)$ implies $a_i \; P^\mu_i \; a_2$. This occurs for $i = 1$ with probability $1 - p_{12} = 1 - Q(1,2)$ and for $i > 2$ with probability p_{2i}. Since these events are independent, the probability that $a_2 \in O_k(P^\mu_{-k}, TTC)$ is $p_{2k}(1 - p_{12}) \cdot p_{23} \cdots p_{2(k-1)} = Q(2,k)$.

Case 2.3: $l \geq 3$. Suppose the claim is true for each $i < l$. Then $a_k \; P^\mu_l \; a_l$. This occurs with probability p_{lk}. Furthermore, $\{a_1, \dots, a_{k-1}\} \cap O_l(P^\mu_{-l}, TTC) = \{a_l\}$. By hypothesis, for each $i < l$, $Q(i,l)$ is the probability that $a_i \in O_l(P^\mu_{-l}, TTC)$, so $1 - Q(i,l)$ is the probability $a_i \notin O_l(P^\mu_{-l}, TTC)$. By mutual exclusivity, $1 - \sum_{i=1}^{l-1} Q(i,l)$ is the probability that $\{a_1, \dots, a_{l-1}\} \cap O_l(P^\mu_{-l}, TTC) = \emptyset$. Next, for each $i \in N^k \backslash (N^l \cup \{l\})$, $a_i \notin O_l(P^\mu_{-l}, TTC)$ implies $a_i \ P^\mu_i \ a_l$ which occurs with probability p_{il}. Since these events are independent, the probability that $a_l \in O_k(P^\mu_{-k}, TTC)$ is

$$p_{lk}\left(1 - \sum_{i=1}^{l-1} Q(i,l)\right) p_{l(l+1)} \cdot p_{l(l+2)} \cdots p_{l(k-1)} = Q(l,l) \cdot P(l,k) = Q(l,k).$$

Finally, since agent k's option set under TTC always includes a_k, the probability that a_k is the object in $O_k(P^\mu_{-k}, TTC)$ with the highest common value is then $O_k(P^\mu_{-k}, TTC)$ is $1 - \sum_{i=1}^{k-1} Q(i,k) = Q(k,k)$. □

We now provide formulas for the each agent's equilibrium utility. For ease of notation, we adopt the conventions that $a_{m+1} \equiv a_1$ and for each $k \in N$, $p_{k(m+1)} \equiv 0$ and $\eta_{(m+1)k} \equiv 0$.

Lemma 3. *Let $\sigma \in A^N$ be an equilibrium under SD. Then for each $k \in N$,*

$$U_k(\sigma, SD) = \sum_{l=1}^{k} Q(l,k) \left[(1 - p_{l(k+1)})v_l + p_{l(k+1)}(v_{k+1} + \eta_{(k+1)l})\right].$$

Proof. Let $\sigma \in A^N$ be an equilibrium under SD, $P \in \mathcal{P}^N$ be preferences determined by σ, and $k \in N$. Without loss of generality, suppose $\sigma_k = a_{k+1}$. We interpret the expressions in the formula as expected utilities conditional on the realization of agent k's option set.

According to SD, exactly one of $\{a_1, \ldots, a_k\}$ is in $O_k(P_{-k}, SD)$ so these events are mutually exclusive and exhaustive. By Lemma 2, for each $l \in \{1, \ldots, k\}$, $a_l \in O_k(P_{-k}, SD)$ with probability $Q(l,k)$. In this case, agent k reports preferences with either a_l or a_{k+1} at the top of $O_k(P_{-k}, SD)$ and receives that object. If $\varepsilon_{k(k+1)} < v_l - v_{k+1}$, then $a_l \ P_k \ a_{k+1}$ and agent k receives a_l. This occurs with probability $1 - p_{l(k+1)}$ and yields a conditional expected utility of v_l. If instead $\varepsilon_{k(k+1)} > v_l - v_{k+1}$, then $a_{k+1} \ P_k \ a_l$ and agent k receives a_{k+1}. This occurs with probability $p_{l(k+1)}$ and yields a conditional expected utility of $v_{k+1} + \eta_{(k+1)l}$. Thus

$$U_k\big(\sigma, SD \big| a_l \in O_k(P_{-k}, SD)\big) = (1 - p_{l(k+1)})v_l + p_{l(k+1)}(v_{k+1} + \eta_{(k+1)l}).$$

The stated formula follows by taking an expectation over the realization of the option set. $\square$

Lemma 4. *Let $\sigma \in A^N$ be an equilibrium under TTC. Then for each $k \in N$,*

$$\begin{aligned} U_k(\sigma, TTC) = \sum_{l=1}^{k-1} & Q(l,k) \left[(1 - p_{lk})v_l + p_{lk}(v_k + \eta_{kl})\right] \\ & + \sum_{l=k+1}^{n} \big(Q(k, l-1) - Q(k,l)\big) \left[(1 - p_{kl})(v_k + \eta_{kl}) + p_{kl} v_l\right] \\ & + Q(k,n) \left[(1 - p_{k(n+1)})(v_k + \eta_{k(n+1)}) + p_{k(n+1)} v_{n+1}\right]. \end{aligned}$$

Proof. Let $\sigma \in A^N$ be an equilibrium under TTC, $P \in \mathcal{P}^N$ be preferences determined by σ, and $k \in N$. Without loss of generality, suppose that $\sigma_k = a_k$. We interpret the expressions in the formula as expected utilities conditional on the realization of agent k's option set. Suppose $O_k(P_{-k}, TTC)\backslash\{a_k\} \neq \emptyset$ and let $a_l \in O_k(P_{-k}, TTC)\backslash\{a_k\}$ be the object with the highest common value.

Case 1: $\boldsymbol{l < k}$. Then agent k reports preferences with either a_l or a_k at the top of $O_k(P_{-k}, TTC)$ and receives that object. If $\varepsilon_{kk} < v_l - v_k$, then $a_l\ P_k\ a_k$ and agent k receives a_l. This occurs with probability $1 - p_{lk}$ and yields a conditional expected utility of v_l. If instead $\varepsilon_{kk} > v_l - v_k$, then $a_k\ P_k\ a_l$ and agent k receives a_k. This occurs with probability p_{lk} and yields a conditional expected utility of $v_k + \eta_{kl}$. Then

$$U_k\big(\sigma, TTC \big| a_l \in O_k(P_{-k}, TTC)\big) = (1 - p_{lk})v_l + p_{lk}(v_k + \eta_{kl}).$$

By Lemma 2, this occurs with probability $Q(l, k)$. Taking expectations yields the first summation in the stated formula.

Case 2: $\boldsymbol{k < l \leq n}$. Then agent k reports preferences with either a_k or a_l at the top of $O_k(P_{-k}, TTC)$ and receives that object. If $\varepsilon_{kk} > v_l - v_k$, then $a_k\ P_k\ a_l$ and agent k receives a_k. This occurs with probability $1 - p_{kl}$ and yields a conditional expected utility of $v_k + \eta_{kl}$. If instead $\varepsilon_{kk} < v_l - v_k$, then $a_l\ P_k\ a_k$ and agent k receives a_l. This occurs with probability p_{kl} and yields a conditional expected utility of v_l. Then

$$U_k\big(\sigma, TTC \big| O_k(P_{-k}, TTC) \cap \{a_1, \ldots, a_l\} = \{a_k, a_l\}\big) = (1 - p_{kl})v_l + p_{kl}(v_k + \eta_{kl}).$$

We now compute the probability that $O_k(P_{-k}, TTC) \cap \{a_1, \ldots, a_l\} = \{a_k, a_l\}$. This requires that $\{a_1, \ldots, a_{k-1}\} \cap O_k(P_{-k}, TTC) = \emptyset$. By Lemma 2, this occurs with probability $Q(k, k)$ and this depends only on the preferences of agents in N^k.

Now consider $i \in N$ with $k < i$. Then also $\{a_1, \dots, a_{k-1}\} \cap O_i(P_{-i}, TTC) = \emptyset$. Since each such agent i reports preferences that agree with P_0 on $A \backslash \{\mu_i\}$, agent i reports preferences with either a_k or a_i at the top of $A \backslash \{a_1, \dots, a_{k-1}\}$. If $\varepsilon_{ii} < v_k - v_i$, then $a_k \ P_i \ a_i$ and $a_i \in O_k(P_{-k}, TTC)$. This occurs with probability $1 - p_{ki}$. If instead $\varepsilon_{ii} > v_k - v_i$, then $a_i \ P_i \ a_k$ and $a_i \notin O_k(P_{-k}, TTC)$. This occurs with probability p_{ki}. Conditional on the preferences of agents in N^k, for each pair $i, j \in N$ with $k < i < j$, the events $a_i \in O_k(P_{-k}, TTC)$ and $a_j \in O_k(P_{-k}, TTC)$ are independent. Combining results,

$$\begin{aligned} &Pr\big(O_k(P_{-k}, TTC) \cap \{a_1, \dots, a_l\} = \{a_k, a_l\}\big) \\ &\qquad = Q(k, k) \cdot p_{k(k+1)} \cdot p_{k(k+2)} \cdots p_{k(l-1)} \cdot (1 - p_{kl}) \\ &\qquad = Q(k, l-1) - Q(k, l). \end{aligned}$$

Taking expectations yields the second summation in the stated formula.

Case 3: $\boldsymbol{n < l}$. Then $l = n + 1$. Agent k reports preferences with one of a_k and a_{n+1} at the top of $O_k(P_{-k}, TTC)$ and receives that object. If $\varepsilon_{kk} > v_{n+1} - v_k$, then $a_k \ P_k \ a_{n+1}$ and agent k receives a_k. This occurs with probability $1 - p_{k(n+1)}$ and yields a conditional expected utility of $v_k + \eta_{k(n+1)}$. If instead $\varepsilon_{kk} < v_{n+1} - v_k$, then $a_{n+1} \ P_k \ a_k$ and agent k receives a_l. This occurs with probability $p_{k(n+1)}$ and yields a conditional expected utility of v_{n+1}. Then

$$\begin{aligned} &U_k\big(\sigma, TTC \big| O_k(P_{-k}, TTC) \cap \{a_1, \dots, a_l\} = \{a_k, a_{n+1}\}\big) \\ &\qquad = (1 - p_{k(n+1)})(v_k + \eta_{k(n+1)}) + p_{k(n+1)} v_{n+1}. \end{aligned}$$

We now compute $Pr(O_k(P_{-k}, TTC) \cap \{a_1, \ldots, a_l\} = \{a_k, a_l\})$. If $n < m$, then Case 3 occurs with the remaining probability after considering the events of Cases 1 and 2,

$$\begin{aligned} 1 - \sum_{h=1}^{k-1} Q(h,k) - \sum_{l=k+1}^{n} \big[Q(k,l-1) - Q(k,l)\big] &= Q(k,k) - \big[Q(k,k) - Q(k,n)\big] \\ &= Q(k,n). \end{aligned}$$

This yields final term in the stated formula.

If $n = m$, then Case 3 does not occur. Instead, still with probability $Q(k,n)$, $O_k(P_{-k}, TTC) \backslash \{a_k\} = \emptyset$. Then $O_k(P_{-k}, TTC) = \{a_k\}$ and agent k receives a_k regardless of his preferences. This yields a conditional expected utility of v_k. Given our conventions $\eta_{k(m+1)} = 0$ and $p_{k(m+1)} = 0$, the expression in Case 3 simplifies to v_k and the stated formula applies. $\square$

For reference, Corollaries 2 and 3 simplify the utility formulas from Lemmas 3 and 4 for the special case of $m = n = 2$.

Corollary 2. *Let $\sigma^{\prec} \in A^N$ be an equilibrium under SD. If $n = m = 2$, then*

$$\begin{aligned} U_1(\sigma^{\prec}, SD) &= (1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21}) \ \textit{and} \\ U_2(\sigma^{\prec}, SD) &= (1 - p_{12})v_2 + p_{12}v_1. \end{aligned}$$

Corollary 3. *Let $\sigma^{\mu} \in A^N$ be an equilibrium under TTC. If $n = m = 2$, then*

$$U_1(\sigma^{\mu}, TTC) = (1 - p_{12})\big[(1 - p_{12})(v_1 + \eta_{12}) + p_{12}v_2\big] + p_{12}v_1 \ \textit{and}$$

$$U_2(\sigma^\mu, TTC) = p_{12}\big[p_{12}(v_2 + \eta_{21}) + (1 - p_{12})v_1\big] + (1 - p_{12})v_2.$$

A.4.1 Proof of Proposition 2

We argue by inspecting the formulas derived in Lemmas 3 and 4. Let $n, m \in \mathbb{N}$ with $n \leq m$ and let $\sigma^{\prec}, \sigma^\mu \in A^N$ be equilibria under SD and TTC respectively. We proceed by induction on the number of agents and objects. Let $U^{n,m}(\varphi)$ be the sum of the agents' ex-ante equilibrium utilities under φ. Let $U_0(k) \equiv \sum_{i=1}^k v_i$. So $U_0(n)$ is the utilitarian welfare without learning.

Step 1: $\boldsymbol{n = m = 2}$. By Corollary 2,

$$\begin{aligned}
U^{2,2}(SD) &= U_1^{2,2}(\sigma^{\prec}, SD) + U_2^{2,2}(\sigma^{\prec}, SD) \\
&= [(1 - p_{12})v_1 + p_{12}(v_2 + \eta_{21})] + [(1 - p_{12})v_2 + p_{12}v_1] \\
&= U_0(n) + p_{12}\eta_{21}.
\end{aligned}$$

By Lemma 1(2), $(1 - p_{12})\eta_{12} = p_{12}\eta_{21}$. Now by Corollary 3,

$$\begin{aligned}
U^{2,2}(TTC) &= U_1^{2,2}(\sigma^\mu, TTC) + U_2^{2,2}(\sigma^\mu, TTC) \\
&= (1 - p_{12})\big[(1 - p_{12})(v_1 + \eta_{12}) + p_{12}v_2\big] + p_{12}v_1 \\
&\qquad + p_{12}\big[p_{12}(v_2 + \eta_{21}) + (1 - p_{12})v_1\big] + (1 - p_{12})v_2 \\
&= U_0(n) + (1 - p_{12})(1 - p_{12})\eta_{12} + p_{12}p_{12}\eta_{21} \\
&= U_0(n) + (1 - p_{12})p_{12}\eta_{21} + p_{12}p_{12}\eta_{21} \\
&= U_0(n) + p_{12}\eta_{21}.
\end{aligned}$$

Therefore, $U^{2,2}(SD) = U^{2,2}(TTC)$.

Step 2: Increasing the number of objects. Suppose $U^{n,n}(SD) = U^{n,n}(TTC)$. Under either rule, objects $\{a_{n+2}, a_{n+3}, \ldots, a_m\}$ are never allocated and so all agents are indifferent to their presence. Thus, it suffices to consider $m = n+1$. Since our hypothesis applies to all problems with n agents and n objects, we may suppose the new object has the lowest common value among all objects, namely v_{n+1}.

Under SD, only agent n ever receives a_{n+1}, and the difference in utilitarian welfare between $U^{n,n}(SD)$ and $U^{n,n+1}(SD)$ is the difference in agent n's ex-ante expected utility. Then by Lemma 3,

$$\begin{aligned}
&U^{n,n+1}(SD) - U^{n,n}(SD) \\
&\qquad = U_n^{n,n+1}(\sigma^{\prec}, SD) - U_n^{n,n}(\sigma^{\prec}, SD) \\
&\qquad = \sum_{l=1}^{n} Q(l,n)\big[(1 - p_{l(n+1)})v_l + p_{l(n+1)}(v_{n+1} + \eta_{(n+1)l})\big] - \sum_{l=1}^{n} Q(l,n)v_l \\
&\qquad = \sum_{l=1}^{n} Q(l,n)p_{l(n+1)}(v_{n+1} + \eta_{(n+1)l} - v_l).
\end{aligned}$$

Under TTC, the arrival of the new object increases each agent's ex-ante expected utility. For each $k \in N$, the difference is the last term in the expression for U_k in Lemma 4:

$$\begin{aligned}
&U_k^{n,n+1}(\sigma^{\mu}, TTC) - U_k^{n,n}(\sigma^{\mu}, TTC) \\
&\qquad = Q(k,n)\big[(1 - p_{k(n+1)})(v_k + \eta_{k(n+1)}) + p_{k(n+1)}v_{n+1}\big] - Q(k,n)v_k \\
&\qquad = Q(k,n)\big[(1 - p_{k(n+1)})\eta_{k(n+1)} + p_{k(n+1)}(v_{n+1} - v_k)\big].
\end{aligned}$$

By Lemma 1(2), $(1 - p_{k(n+1)})\eta_{k(n+1)l} = p_{k(n+1)}\eta_{(n+1)k}$. Substituting and summing over agents,

$$
\begin{aligned}
U^{n,n+1}(TTC) - U^{n,n}(TTC) &= \sum_{k=1}^{n} Q(k,n)\big[(1 - p_{k(n+1)})\eta_{k(n+1)} + p_{k(n+1)}(v_{n+1} - v_k)\big] \\
&= \sum_{k=1}^{n} Q(k,n)\big[p_{k(n+1)}\eta_{(n+1)k} + p_{k(n+1)}(v_{n+1} - v_k)\big] \\
&= \sum_{k=1}^{n} Q(k,n) p_{k(n+1)}(v_{n+1} + \eta_{(n+1)k} - v_k).
\end{aligned}
$$

Then $U^{n,n+1}(SD) - U^{n,n}(SD) = U^{n,n+1}(TTC) - U^{n,n}(TTC)$. By hypothesis, $U^{n,n}(SD) = U^{n,n}(TTC)$, so $U^{n,n+1}(SD) = U^{n,n+1}(TTC)$ as well.

Step 3: Increasing the number of agents. Suppose $U^{n-1,n}(SD) = U^{n-1,n}(TTC)$. Since our hypothesis applies to all problems with $n-1$ agents and n objects, we may assume that the new agent has the lowest priority under SD and is endowed with a_n under TTC.

Under SD, the arrival of agent n has no effect on the allocations or ex-ante expected utilities of the original agents. Therefore, the difference in utilitarian welfare is the ex-ante expected utility of agent n. Also, agent n has no meaningful investigation decision. This is reflected by our conventions $p_{l(m+1)} = 0$ and $\eta_{(m+1)l} = 0$. Then by Lemma 3,

$$
\begin{aligned}
U^{n,n}(SD) - U^{n-1,n}(SD) &= U_n^{n,n}(\sigma^{\prec}, SD) \\
&= \sum_{l=1}^{n} Q(l,n)\big[(1 - p_{l(n+1)})v_l + p_{l(n+1)}(v_{n+1} + \eta_{(n+1)l})\big]
\end{aligned}
$$

$$= \sum_{l=1}^{n} Q(l,n)v_l.$$

Under TTC, the arrival of the new agent decreases each original agent's ex-ante expected utility. The difference arises because a_n may now be unavailable. In terms of Lemma 4, for each $k \in N$, the comparison between a_k and a_n moves from the final term into the second summation. The difference is

$$\begin{aligned}
&U_k^{n,n}(\sigma^\mu,TTC) - U_k^{n-1,n}(\sigma^\mu,TTC) \\
&\qquad = \big(Q(k,n-1) - Q(k,n)\big)\big[(1-p_{kn})(v_k+\eta_{kn}) + p_{kn}v_n\big] + Q(k,n)v_k \\
&\qquad\qquad - Q(k,n-1)\left[(1-p_{kn})(v_k+\eta_{kn}) + p_{kn}v_n\right] \\
&\qquad = -Q(k,n)\big[(1-p_{kn})(v_k+\eta_{kn}) + p_{kn}v_n\big] + Q(k,n)v_k \\
&\qquad = -Q(k,n)\big[(1-p_{kn})\eta_{kn} + p_{kn}(v_n - v_k)\big].
\end{aligned}$$

By Lemma 1(2), $(1-p_{kn})\eta_{kn} = p_{kn}\eta_{nk}$. Substituting and summing over the original agents,

$$\begin{aligned}
\sum_{k=1}^{n-1} U_k^{n,n}(\sigma^\mu,TTC) - U_k^{n-1,n}(\sigma^\mu,TTC) &= \sum_{k=1}^{n-1} -Q(k,n)\big[(1-p_{kn})\eta_{kn} + p_{kn}(v_n-v_k)\big] \\
&= \sum_{k=1}^{n-1} -Q(k,n)p_{kn}(v_n + \eta_{nk} - v_k).
\end{aligned}$$

Now agent n's ex-ante expected utility is

$$U_n^{n,n}(\sigma^\mu,TTC) = \sum_{l=1}^{n-1} Q(l,n)\big[(1-p_{ln})v_l + p_{ln}(v_n+\eta_{nl})\big] + Q(n,n)v_n.$$

Combining results,

$$
\begin{aligned}
&U^{n,n}(TTC) - U^{n-1,n}(TTC) \\
&\qquad = U^{n,n}_n(\sigma^\mu, TTC) + \sum_{k=1}^{n-1} U^{n,n}_k(\sigma^\mu, TTC) - U^{n-1,n}_k(\sigma^\mu, TTC) \\
&\qquad = Q(n,n)v_n + \sum_{l=1}^{n-1} Q(l,n)\big[(1-p_{ln})v_l + p_{ln}(v_n + \eta_{nl})\big] \\
&\qquad\qquad\qquad - \sum_{k=1}^{n-1} Q(k,n)p_{kn}(v_n + \eta_{nk} - v_k) \\
&\qquad = Q(n,n)v_n + \sum_{l=1}^{n-1} Q(l,n)\big[(1-p_{ln})v_l + p_{ln}(v_n + \eta_{nl}) - p_{ln}(v_n + \eta_{nl} - v_l)\big] \\
&\qquad = Q(n,n)v_n + \sum_{l=1}^{n-1} Q(l,n)v_l \\
&\qquad = \sum_{l=1}^{n} Q(l,n)v_l.
\end{aligned}
$$

Therefore, $U^{n,n}(SD) = U^{n,n}(TTC)$.

A.4.2 Proof of Theorem 1

Let $m \in \mathbb{N}$. We show that for each $n \in \mathbb{N}$ with $n \leq m$, the equilibrium utility profile under TTC Lorenz dominates the equilibrium utility profile under SD when there are n agents. For each $n \in \mathbb{N}$ and each $k \in \mathbb{N}$ with $k \leq n$, let $U^n_k(SD)$ be the equilibrium utility under SD of the agent with kth highest priority when there are n agents. Similarly, let $U^n_k(TTC)$ be the equilibrium utility under TTC of the agent whose endowment has the kth highest priority when there are n agents. Explicit formulas for theses expressions are computed in Lemmas 3 and 4.

By Proposition 2, each rule yields the same total utility. Moreover, the proof of Proposition 2 shows that (i) the utility of a given agent under SD is independent of the number of agents with lower priority; and (ii) the utility of a given agent under TTC is decreasing in the number of agents with lower priority. That is, for each $n, k \in \mathbb{N}$ with $k \leq m$,

$$\begin{aligned}
&\sum_{k=1}^{n} U_k^n(SD) = \sum_{k=1}^{n} U_k^n(TTC), \\
&U_k^k(SD) = U_k^{k+1}(SD) = \cdots = U_k^n(SD), \text{ and} \\
&U_k^k(TTC) > U_k^{k+1}(TTC) > \cdots > U_k^n(TTC).
\end{aligned}$$

First, $U_1^1(SD) = U_1^1(TTC)$. Now comparing for $n = 2$, $U_1^2(SD) = U_1^1(SD) = U_1^1(TTC) > U_1^2(TTC)$. Also, $U_1^2(SD) + U_2^2(SD) = U_1^2(TTC) + U_2^2(TTC)$, so $U_2^2(SD) = U_1^2(TTC) + U_2^2(TTC) - U_1^2(SD) < U_2^2(TTC)$. Thus, $U^2(TTC)$ Lorenz dominates $U^2(SD)$.

Now in general, for each pair $n, l \in \mathbb{N}$ with $l \leq n$,

$$\sum_{k=1}^{l} U_k^n(SD) = \sum_{k=1}^{l} U_k^l(SD) = \sum_{k=1}^{l} U_k^l(TTC) < \sum_{k=1}^{l} U_k^n(TTC).$$

Comparing the utilities of the less well off agents,

$$\begin{aligned}
\sum_{k=l}^{n} U_k^n(SD) &= \sum_{k=1}^{n} U_k^n(SD) - \sum_{k=1}^{l} U_k^n(SD) \\
&= \sum_{k=1}^{n} U_k^n(TTC) - \sum_{k=1}^{l} U_k^n(SD)
\end{aligned}$$

$$< \sum_{k=1}^{n} U_k^n(TTC) - \sum_{k=1}^{l} U_k^n(TTC)$$
$$= \sum_{k=l}^{n} U_k^n(TTC).$$

That is, $U^n(TTC)$ Lorenz dominates $U^n(SD)$.

A.5 Proof of Proposition 3

We omit the straightforward argument that priority rules are efficient.[40] There are two cases to consider in showing that TTC rules are not efficient when $m \geq 3$.

Case 1: $2 = n$ and $m = 3$. We show by example that TTC is not *ex-ante efficient*. Let $(v_1, v_2, v_3) \equiv (50, 49, 30)$ and $F \sim Unif[-30, 30]$. Then $p_{12} = \frac{29}{60}$, $p_{13} = \frac{10}{60}$, and $p_{23} = \frac{11}{60}$. Let $\sigma \equiv (a_1, a_2)$, which is the unique equilibrium under TTC. We construct a rule that yields an ex-ante Pareto improvement with σ. Let $\bar{P}, \bar{P}' \in \mathcal{P}^N$ be such that

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}'_1$	$\bar{P}'_2$
a_1	a_1	a_2	a_2
a_2	a_3	a_3	a_1
a_3	a_2	a_1	a_3

Now define $\varphi\colon \mathcal{P}^N \to X$ by

$$\varphi(P) \equiv \begin{cases} (a_2, a_1) & \text{if } P = \bar{P} \text{ or } P = \bar{P}' \\ TTC(P) & \text{otherwise} \end{cases}.$$

[40] See Bade (2015) for a proof that can be translated to our setting.

To compare ex-ante expected utilities, let $\varepsilon_\sigma \in \mathbb{R}^N$. If $P(\varepsilon_\sigma) \notin \{\bar{P}, \bar{P}'\}$, then (σ, φ) and (σ, TTC) coincide, so suppose $P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}$. The allocations are $\varphi(\bar{P}) = \varphi(\bar{P}') = (a_2, a_1)$, $TTC(\bar{P}) = (a_1, a_3)$, and $TTC(\bar{P}') = (a_3, a_2)$. Under (σ, φ), agent 1's conditional expected utility is $v_2 = 49$ and agent 2's conditional expected utility is $v_1 = 50$.

We now compute the conditional expected utilities under (σ, TTC). First, $Pr(P_1(\varepsilon_\sigma) = \bar{P}_1) = 1 - p_{12} = \frac{31}{60}$, $Pr(P_2(\varepsilon_\sigma) = \bar{P}_2) = p_{23} = \frac{11}{60}$, $Pr(P_1(\varepsilon_\sigma) = \bar{P}_1') = p_{13} = \frac{10}{60}$, and $Pr(P_2(\varepsilon_\sigma) = \bar{P}_2') = p_{12} = \frac{29}{60}$. Therefore, by independence,

$$
\begin{aligned}
Pr\big(P(\varepsilon_\sigma) = \bar{P}\big) &= \tfrac{31}{60} \cdot \tfrac{11}{60} = \tfrac{341}{3600}, \\
Pr\big(P(\varepsilon_\sigma) = \bar{P}'\big) &= \tfrac{10}{60} \cdot \tfrac{29}{60} = \tfrac{290}{3600}, \\
Pr\big(P(\varepsilon_\sigma) = \bar{P} \big| P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}\big) &= \tfrac{341}{631}, \text{ and} \\
Pr\big(P(\varepsilon_\sigma) = \bar{P}' \big| P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}\big) &= \tfrac{290}{631}.
\end{aligned}
$$

Next, computing expectations,

$$
\begin{aligned}
\mathbf{E}[v_1 + \varepsilon_{11} | P(\varepsilon_\sigma) = \bar{P}] &= 50 + \tfrac{1}{2}(-1 + 30) = 64.5 \text{ and} \\
\mathbf{E}[v_2 + \varepsilon_{22} | P(\varepsilon_\sigma) = \bar{P}'] &= 49 + \tfrac{1}{2}(1 + 30) = 64.5.
\end{aligned}
$$

Combining results,

$$
\begin{aligned}
U_1\big(\sigma, TTC \big| P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}\big) &= \tfrac{341}{631} \cdot 64.5 + \tfrac{290}{631} \cdot 30 = \tfrac{61389}{1262} \approx 48.64 < 49 \text{ and} \\
U_2\big(\sigma, TTC \big| P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}\big) &= \tfrac{341}{631} \cdot 30 + \tfrac{290}{631} \cdot 64.5 = \tfrac{57870}{1262} \approx 45.86 < 50.
\end{aligned}
$$

Therefore, each agent's ex-ante expected utility is higher under φ than under TTC.

Case 2: $3 < m$. We show how to embed the example in Case 1 as a subproblem. First, for $2 = n < m$, only the objects with the three highest common values are relevant and so the computations carry over unchanged.

Now suppose $2 < n \leq m$. Since no agent receives an object $a_{n+2}, \ldots, a_m$ in equilibrium, suppose without loss of generality that $m \in \{n, n+1\}$. Again let $F \sim Unif[-30, 30]$ and let $v \in \mathbb{R}^m$ be such that $(v_1, v_2, v_m) \equiv (50, 49, 30)$ and $49 > v_4 > \cdots > 30 = v_m$. Let $\bar{P}, \bar{P}' \in \mathcal{P}^N$ be such that for each $i \in N \backslash \{1, 2\}$,

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_i$	$\bar{P}'_1$	$\bar{P}'_2$	$\bar{P}'_i$
a_1	a_1	a_i	a_2	a_2	a_i
a_2	a_3	a_1	a_3	a_1	a_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{m-1}	a_m	a_{m-1}	a_m	a_{m-1}	a_{m-1}
a_m	a_2	a_m	a_1	a_m	a_m

Then $\bar{P}$ and $\bar{P}'$ each occur with positive probability under σ^μ. Moreover, for each $i \in N \backslash \{1, 2\}$ and each $\varepsilon_\sigma \in \mathbb{R}^N$ such that $P(\varepsilon_\sigma) \in \{\bar{P}, \bar{P}'\}$, $TTC_i(P(\varepsilon_\sigma)) = a_i = \mu_i$. Therefore, we may extend the definition of φ in Case 1 so that for each $i \in N \backslash \{1, 2\}$ and each $\varepsilon_\sigma \in \mathbb{R}^N$, $\varphi_i(P(\varepsilon_\sigma)) = TTC_i(P(\varepsilon_\sigma))$. Then agent i's expected utility is the same under φ and under TTC. If $m = n + 1$, then the conditional expected utilities for agents 1 and 2 are the same as in Case 1. If instead $m = n$, then the conditional expected utilities for agents 1 and 2 are the lower than in Case 1 because each agent receives his endowment. In either case, φ represents an ex-ante Pareto improvement over TTC.

While we have shown TTC may be Pareto-dominated, Theorem 1 implies that SD does not dominate it.

A.6 Further examples

A.6.1 Randomizing before versus after learning

Randomizing over priority orders is a typical way to (re-)introduce fairness into the assignment problem. When the randomization is conducted *before* agents make learning decisions, the procedure simply redistributes ex-ante expected utility with uniform randomization equalizing these values across agents. The sum of these utilities, the Utilitarian social welfare, is unaffected by the randomization. However, if the randomization instead occurs *after* agents investigate, it mutes learning benefits, reducing Utilitarian social welfare. In fact, the loss may be large enough that an asymmetric fixed-endowment TTC rule yields a profile of ex-ante utilities which *Pareto dominates* that achieved by randomizing over priority orders. To illustrate, we revisit the motivating example.

Example 2. Suppose that there are three schools with utility information as specified in Table 1. To keep computations simple, we restrict the environment to two students.

Fixed priorities and endowments. Label the students so that student 1 has higher priority or is endowed with the Ivy school and student 2 has lower priority or is endowed with the state school. Under SD and TTC, the investigation strategies and ex-ante utilities of the two students are the same as computed previously:

	SD
Student	Expected utility
1	$\frac{1}{2}\cdot 14+\frac{1}{2}\cdot 7=10.5$
2	$\frac{1}{2}\cdot 10+\frac{1}{4}\cdot 9+\frac{1}{4}\cdot 7=9.0$

	TTC
Student	Expected utility
1	$\frac{1}{2}\cdot 14+\frac{1}{4}\cdot 7+\frac{1}{4}\cdot 5=10.0$
2	$\frac{1}{2}\cdot 12+\frac{1}{4}\cdot 9+\frac{1}{4}\cdot 5=9.5$

Randomization over priority orders. Suppose that the designer randomizes uniformly over priority orders *after* students investigate. There are six possible combinations of (pure) investigation strategies, each with four outcomes of investigation according to whether the realized values are high or low. Students' ex-ante utilities in each case are:

Strategy	HH	HL	LH	LL	utility
Ivy vs. Ivy	$\frac{1}{2}\cdot 14+\frac{1}{2}\cdot 7$	14	7	$\frac{1}{2}\cdot 7+\frac{1}{2}\cdot 5$	9.375
State vs. Ivy	12	$\frac{1}{2}\cdot 12+\frac{1}{2}\cdot 9$	$\frac{1}{2}\cdot 9+\frac{1}{2}\cdot 5$	9	9.675
Tech vs. Ivy	10	$\frac{1}{2}\cdot 9+\frac{1}{2}\cdot 7$	10	9	9.25
Ivy vs. State	14	$\frac{1}{2}\cdot 14+\frac{1}{2}\cdot 7$	$\frac{1}{2}\cdot 7+\frac{1}{2}\cdot 5$	7	9.375
State vs. State	$\frac{1}{2}\cdot 12+\frac{1}{2}\cdot 9$	12	9	$\frac{1}{2}\cdot 9+\frac{1}{2}\cdot 5$	9.625
Tech vs. State	10	10	9	$\frac{1}{2}\cdot 9+\frac{1}{2}\cdot 7$	9.25
Ivy vs. Tech	14	$\frac{1}{2}\cdot 14+\frac{1}{2}\cdot 7$	7	7	9.625
State vs. Tech	12	12	9	$\frac{1}{2}\cdot 9+\frac{1}{2}\cdot 5$	10.0
Tech vs . Tech	$\frac{1}{2}\cdot 10+\frac{1}{2}\cdot 9$	10	9	$\frac{1}{2}\cdot 9+\frac{1}{2}\cdot 7$	9.125

Investigating the State school is a strictly dominant strategy for each student and State vs. State in the unique equilibrium.

Equilibrium welfare comparison. The students equilibrium ex-ante utilities under SD, TTC, and randomization are:

Rule	Student 1	Student 2	Utilitarian social welfare
SD	10.5	9	19.5
TTC	10	9.5	19.5
Random priority	9.675	9.675	19.25

Randomization achieves the lowest Utilitarian social welfare, illustrating the efficiency cost of randomization after investigation decisions are set.

Potential for Pareto inferior equilibrium. Although the ex-ante utility profiles in the previous example are not Pareto-comparable, it is possible for TTC to Pareto dominate randomization. Modify the utilities in the example so that:

School	Good fit	Bad fit	Expectation
Ivy	8	2	5
State	7	1	4
Technical	6	0	3

In the equilibrium under randomization, both students again investigate the State school. At equilibrium under randomization, the ex-ante utility of each student is:

$$\tfrac{1}{4}\left[\tfrac{1}{2}(7+5)+7+5+\tfrac{1}{2}(5+3)\right]=5.5.$$

By contrast, under TTC, their ex-ante utilities are:

$$\begin{aligned}
&\text{Student 1:} \quad \tfrac{1}{2}\cdot 8+\tfrac{1}{4}\cdot 4+\tfrac{1}{4}\cdot 3=5.75\\
&\text{Student 2:} \quad \tfrac{1}{2}\cdot 7+\tfrac{1}{4}\cdot 5+\tfrac{1}{4}\cdot 3=5.5
\end{aligned}$$

The outcome under TTC ex-ante Pareto-dominates the outcome under randomization.

A.6.2 Welfare under Hierarchical Exchange rules

The priority and TTC rules we study are examples of Hierarchical Exchange rules (Pápai 2000a), all of which are themselves members of an even larger family of *group* *strategy-proof* exchange rules (Pycia and Ünver 2016). Intuitively, priority and TTC rules lie at opposite extremes within either family, priority rules maximally concentrating endowments and TTC rules distributing them most widely. Despite their differences, these rules achieve the same Utilitarian social welfare (Proposition 2). Surprisingly, this equivalence does not extend even to the family of Hierarchical exchange rules with other rules falling short of this benchmark. To illustrate, we consider a *conditional* priority rule, a narrow modification of a priority rule which allows assignments of high priority agents to determine the order of those who remain. The intuition is similar to randomization: Conditional ordering increases the uncertainly agents face about their option sets, thereby dampening their benefits from investigation.

Example 3. Suppose that there are three agents and four objects with $v = (16, 14, 12, 8)$ and $\varepsilon_{ia} \in \{-10, 10\}$ with equal probability. Let $\hat{SD}$ be a conditional priority rule in which agent 1 chooses first and is followed by agent 2 if assigned a_1 and by agent 3 otherwise.

Fixed priority. Under SD, agent 1 optimally investigates either a_1 or a_2, agent 2 optimally investigates a_3, and agent 3 optimally investigates a_4. Each agent's ex-ante equilibrium utility is:

	Expected utility
Agent 1	$\frac{1}{2} \cdot 26 + \frac{1}{2} \cdot 24 = \frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 24 = 20$
Agent 2	$\frac{1}{2} \cdot 22 + \frac{1}{4} \cdot 16 + \frac{1}{4} \cdot 14 = 18.5$
Agent 3	$\frac{1}{2} \cdot 18 + \frac{1}{8} \cdot 16 + \frac{1}{8} \cdot 14 + \frac{1}{4} \cdot 12 = 15.75$

Conditional priority. Agent 1 faces the same problem as under a priority rule.

He investigates either a_1 or a_2, is assigned each object with equal probability, and obtains an ex-ante utility of 20.

Consider agent 2 problem. He faces one of four possible options sets: If agent 1 is assigned a_1, then his option set is $\{a_2, a_3, a_4\}$; if agent 1 is assigned a_2, then his option set is $\{a_3, a_4\}$, $\{a_1, a_4\}$, or $\{a_1, a_3\}$, according to whether agent 3 receives a_1, a_3, or a_4. If agent 3 investigates a_1, then $\{a_3, a_4\}$ and $\{a_1, a_4\}$ are equally likely; if agent 3 investigates a_2, then $\{a_3, a_4\}$ is certain; if agent 3 investigates a_3, then $\{a_3, a_4\}$ and $\{a_1, a_4\}$ are equally likely; and if agent 3 investigates a_4, then $\{a_1, a_3\}$ and $\{a_3, a_4\}$ are equally likely. Based on these observations, we compute agent 2's ex-ante utility for each combination of investigation strategies by agent 2 and agent 3 as:

$\sigma_2 \setminus \sigma_3$	a_1	a_2	a_3	a_4
a_1	15.5	16	16	16
a_2	14	15	15	15.5
a_3	17	16.75	16.75	17.75
a_4	16	16.25	16.25	15.75

Comparing, it is a strictly dominant strategy for agent 2 to investigate a_3.

Given agent 2's investigation strategy, agent 3 faces one of three possible options sets: If agent 1 is assigned a_1, then his option is $\{a_3, a_4\}$ or $\{a_1, a_4\}$ with equal probability; if agent 1 is assigned a_2, then his option set is $\{a_1, a_3, a_4\}$ with certainty. Based on these observations, we compute agent 3's ex-ante utility for each investigation strategy as:

a_1	a_2	a_3	a_4
14.25	16	16.75	16

Agent 3 optimally investigates a_3 and so there is a unique equilibrium in which both agents investigate a_3.

Equilibrium welfare comparison. The agents' equilibrium ex-ante utilities under SD and $\hat{SD}$ are:

Rule	Agent 1	Agent 2	Agent 3	Utilitarian social welfare
SD	20	18.5	15.75	54.25
$\hat{SD}$	20	16.75	16.75	53.5

Utilitarian social welfare is higher under SD than under $\hat{SD}$.

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