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# On the exponential form of the displacement operator for different systems

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## Abstract

The family of displacement operators  $D(x, p)$ , a central concept in the theory of coherent states of a quantum mechanical harmonic oscillator, has been successfully generalized to systems of quantized, cyclic or finite position coordinates. However, out of the plethora of mutually equivalent expressions for the displacement operators valid in the continuous case, only few are directly applicable in the other systems of interest. The aim of this paper is to strengthen the analogy between the different cases by identifying the root cause of the issues accompanying the straightforward generalization of certain important expressions and, more importantly, offering alternative ones of general validity. Ultimately we arrive at an algorithm allowing one to express any displacement operator as an exponential of a pure imaginary multiple of a generalized ‘quadrature’ observable that is not obtained by a linear combination of position and momentum observables but rather by a shear transform of one of them in the system’s phase space.

Keywords: displacement operator, quantum rotator, discrete quantum systems, orbital angular momentum, Wigner function

## 1. Introduction

The displacement operators represent a powerful tool in quantum optics as well as other parts of quantum theory. Among other uses, they can be employed in generating coherent states from the vacuum, defining the characteristic function of a quantum state, converting expressions for measurement probabilities involving coherent states to vacuum expectation values, reconstruction of a state from its  $P$ -representation, provides an integral resolution of unity, and more [1].

Concepts analogous to the continuous displacement have been proposed in other systems where position and momentum representations are naturally defined and linked via the Fourier transform; namely, systems with quantized position, momentum, or both, or discrete analogues thereof. In many works the displacement operators (or closely related phase-space point operators) have been defined independently and

presented in several different mathematical forms fit for their respective purposes [2–7]. To our knowledge, no general framework encompassing all the cases and covering the potential of the displacement operators in a global manner has yet been presented. The main obstacle on the way to a full formal analogy is the lack of the Heisenberg algebra corresponding to infinitesimal displacements in discrete systems. The aim of this paper is to establish such a framework by introducing a number of mathematical expressions for the displacement operator in each of the cases and comparing them with the reference continuous-variable system side by side, avoiding the reliance on the Lie algebraic methods where they are inaccessible.

The main motivation for strengthening the analogy between the different systems in question and for broadening the number of available equivalent formulae for the displacement operator is the ease of use in various situations. Consider, for example, the calculation of the transformation of the ground state of a harmonic oscillator to a coherent state by displacement when the result is expected in the  $x$ -,  $p$ -, or Fock representation. Yet other forms are practical for calculations of operator transforms. We believe that by providing a



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number of formulae bearing a strong analogy between the different physical systems, not only can we make the framework more robust and more widely applicable but also more transparent for future extensions and analogies.

The work is structured as follows. In section 2, main results familiar from the theory of continuous variable systems are listed for subsequent reference and briefly discussed with respect to their generality. The connection of displacement operators to characteristic functions and quasiprobability distributions is also mentioned. In section 3, the formalism is generalized to rotational systems, or to closely related systems with a discretized position coordinate. An important class of shear transforms is introduced and shown to transform between the displacement operators and to allow a definition of generic displacement as an exponential of a quadrature observable. In section 4, the formalism is further extended to finite-dimensional systems for which both position and momentum observables are discretized, and similar results are drawn. In section 5, the theory described so far is demonstrated on a simple two-dimensional system, a qubit, and illustrated geometrically in the Bloch sphere representation, and in section 6, a similar example is carried out on a three-dimensional system. The two examples display different facets of the theory presented in the main part of the paper and are particularly powerful in a side-by-side comparison. Finally, we summarize our results.

## 2. Displacement in continuous variable systems—an overview

Originating in the study of a quantum mechanical harmonic oscillator, the Heisenberg unitary group of continuous variable displacement operators and their complex unit multiples plays a key role in the theory of coherent states, semiclassical methods and quasiprobability distributions. In quantum optics, perhaps the most influential field in which it has become thoroughly familiar, the commonly used definition of a single-mode displacement by a complex offset  $\alpha$  in the Gauss plane is

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \quad (1)$$

where  $a$  and  $a^\dagger$  denote annihilation and creation operators of the optical mode in question, respectively [1]. Such an operator, acting on the vacuum state of a single field mode, produces the coherent state  $|\alpha\rangle$ . More generally, for any state  $\rho$  with well-defined moments of a quadrature operator

$$x_\lambda = (e^{-i\lambda} a + e^{i\lambda} a^\dagger) / \sqrt{2}, \quad \lambda \in \mathbb{R}, \quad (2)$$

it shifts the expectation value of  $x_\lambda$  by a constant of

$$\begin{aligned} & \langle D(\alpha)^\dagger x_\lambda D(\alpha) \rangle_\rho - \langle x_\lambda \rangle_\rho \\ &= (e^{-i\lambda} \alpha + e^{i\lambda} \alpha^*) / \sqrt{2}, \end{aligned} \quad (3)$$

keeping all central moments intact. This is particularly apparent if Wigner functions of a state before and after a

displacement are compared; the action of  $D(\alpha)$  is a pure geometrical shift of the quasiprobability distribution  $W(\alpha)$  in its independent variable [1].

In many situations, it is mathematically more convenient to work with another, ‘ordered exponential’ form, that is, one in which any products of creation and annihilation operators act in one specified order. From the formal Taylor expansion of (1), it can be noticed that the coefficients by any product of  $a$  and  $a^\dagger$  constituting it are the same as in the expansion of the product of separate exponentials of  $\alpha a^\dagger$  and  $-\alpha^* a$  and the latter differs from (1) only by a real factor:

$$\begin{aligned} D(\alpha) &= e^{-|\alpha|^2/2} \exp(\alpha a^\dagger) \exp(-\alpha^* a) \\ &= e^{|\alpha|^2/2} \exp(-\alpha^* a) \exp(\alpha a^\dagger). \end{aligned} \quad (4)$$

Note, however, that this form decomposes a unitary operator into a product of two unbounded operators and thus will not be valid for all states of the Hilbert space.

In mechanical systems, two of the quadratures,  $X = x_0$  and  $P = x_{\pi/2}$ , have the distinct physical meaning of the quantum particle’s position and momentum observables (in dimensionless units), respectively. By denoting

$$\alpha = (x + ip) / \sqrt{2}, \quad x, p \in \mathbb{R}, \quad (5)$$

position is displaced by  $x$  and momentum by  $p$  at the action of  $D(\alpha)$ . One may then find it more useful to rewrite the displacement operator in terms of the variables  $x$  and  $p$  and the generators  $X$  and  $P$  instead of  $\alpha$  and the creation/annihilation operators:

$$D(x, p) = \exp(ipX - ixP). \quad (6)$$

Displacement operators compose according to the formula

$$D(\alpha)D(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2} D(\alpha + \beta), \quad (7)$$

or

$$\begin{aligned} D(x, p)D(x', p') &= e^{i(px' - xp')/2} D(x + x', p + p'); \end{aligned} \quad (8)$$

in other words, the action of the composition differs from that of a single displacement by the Euclidean sum of the two displacement vectors only by a global phase factor. This leads to a decomposition a generic displacement into displacements along the Cartesian axes

$$\begin{aligned} D(x, p) &= e^{ixp/2} D(x, 0)D(0, p) \\ &= e^{-ixp/2} D(0, p)D(x, 0). \end{aligned} \quad (9)$$

Axis displacements have a simple action in the  $x$ - and  $p$ -representations of the state. For example, in the  $x$ -representation

$$\begin{aligned} [D(x, 0)\psi](\chi) &= \psi(\chi - x), \\ [D(0, p)\psi](\chi) &= e^{ip\chi}\psi(\chi). \end{aligned} \quad (10)$$

Equation (9) then allows to find the action of a generic  $D(x, p)$  in the  $x$ -representation promptly:

$$D(x, p) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \\ : \psi(x) \mapsto e^{ip(x-x/2)}\psi(x-x). \quad (11)$$

The expectation value of the displacement operator as a function of  $x$  and  $p$  is called the characteristic function  $\tilde{W}(x, p)$  of the state in question [1]. The Wigner function  $W(x, p)$  is related to  $\tilde{W}(x, p)$  by two-dimensional Fourier transform:

$$W(x, p) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(p x' - x p')} \tilde{W}(x', p') dx' dp', \quad (12)$$

and can also be thought of as the expectation value of another, phase-space point operator

$$\Delta(x, p) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(p x' - x p')} D(x', p') dx' dp'. \quad (13)$$

Phase-space point operators can also be related to displacement operators through the parity operator  $\Pi$ :  $\psi(x) \mapsto \psi(-x)$  in several ways:

$$\pi\Delta(x, p) = D(x, p)\Pi D(x, p)^\dagger \\ = D(2x, 2p)\Pi = \Pi D(2x, 2p)^\dagger. \quad (14)$$

Note that  $\Pi$  itself satisfies  $\Pi = \pi\Delta(0, 0)$ . The  $x$ -representation of  $\Delta(x, p)$  is

$$\Delta(x, p) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \\ : \psi(x) \mapsto \pi^{-1} e^{2ip(x-x)}\psi(2x-x). \quad (15)$$

### 3. Infinite one-dimensional lattice / bead on a wire

The first physical system featuring a generalization of displacement operators we will turn our attention to is a quantum particle constrained in motion to a discrete infinite one-dimensional lattice. This model has drawn attention recently, mainly in connection to the study of quantum random walks [8, 9] and the theory of Laguerre–Gaussian optical modes and their superpositions [10] (with a fixed radial degree of freedom). The discrete nature of position coordinate leads to a cyclic momentum coordinate [9]. The phase space of a quantum particle moving on an integer lattice can be identified with the Cartesian product  $\mathbb{Z} \times S^1$ .

The above model is closely related to that of a bead on a wire, that is, a point quantum particle constrained in motion to a circle and possibly subject to potential forces. Here, the position is of an angular nature and therefore cyclic, while the conjugate variable is represented by angular momentum, which has an unbounded discrete spectrum of equidistant eigenvalues. In other words, the roles of position and momentum are simply swapped in comparison to the phase space of a particle on an infinite lattice.

Note that wave functions in the momentum representation in the former system or the angle representation in the latter are naturally defined as functions on a circle. This can be mapped to a finite interval by choosing a particular

parametrization. In any case, however, there will be an inevitable coordinate singularity at some point of the circle where the respective angle variable jumps between the end-points of the interval. This implies that a momentum observable for an infinite lattice or an angle observable for a bead on a wire can be defined as a multiplicative operator in the corresponding representation but will carry a signature of the parametrization chosen [11, 12].

The phase space formalism in a bead on a wire has been pioneered by Rigas *et al* [2], and in an infinite lattice by Hinajeros *et al* [3] Notably, most of the theory had been earlier described by Lukš and Peřinová [4] as a side result of their work on phase space methods for a quantum mechanical harmonic oscillator when temporarily allowing excitation numbers with a negative sign in addition to the standard nonnegative ones. A remarkable related topic is the number-phase Wigner function studied intensively by Vaccaro [13–15] as a polar version of the standard Wigner function of a harmonic oscillator or single-mode Fock space which also allows a use in rotational systems without substantial changes [4]. In the following, we will adapt the notion of a bead on a wire but all the results can easily be reformulated in the case of the infinite lattice if needed.

Let us consider a particle constrained in motion to the unit circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}. \quad (16)$$

The state space of the quantum system is  $\mathcal{H} = L^2(S^1)$ . By fixing a reference angle  $\theta_0$ , we can describe pure states  $\Psi(z) \in \mathcal{H}$  by functions

$$\psi(\phi) : \mathcal{I} \rightarrow \mathbb{C} : \phi \mapsto \Psi(e^{i\phi}), \\ \mathcal{I} = [\theta_0, \theta_0 + 2\pi). \quad (17)$$

Here,  $\psi(\phi) \in L^2(\mathcal{I})$  plays the role of a position (or angle) representation of the state  $\Psi$ . Employing the isomorphism between  $L^2(S^1)$  and  $L^2(\mathcal{I})$  represented by the mapping  $\Psi(z) \mapsto \psi(\phi)$ , we can easily carry operators between the two Lebesgue spaces; in the following, we will take the liberty of defining operators on the former space by their action on the latter.

The Hermitean (that is, bounded and self-adjoint) multiplicative operator

$$\Phi : L^2(\mathcal{I}) \rightarrow L^2(\mathcal{I}) : \psi(\phi) \mapsto \phi\psi(\phi) \quad (18)$$

represents the position (or angle) observable in the system and the differential operator

$$L : \psi(\phi) \mapsto -i\hbar\psi'(\phi), \quad (19)$$

which is self-adjoint on the subspace of absolutely continuous functions restricted by a periodic boundary condition and with their derivative  $L^2$ -integrable over  $\mathcal{I}$ , the angular momentum. Note that both are properly defined observables with  $\sigma(\Phi) = \overline{\mathcal{I}}$  and  $\sigma(L) = \hbar\mathbb{Z}$ , yet their commutator is only sparsely defined in  $L^2(\mathcal{I})$ . This fact has far-reaching consequences to operations involving both observables at once, as well as the angle-angular momentum uncertainty relation [11].

Importantly, the choice of the reference angle  $\theta_0$ , directly linked to the definition of  $\Phi$  observable (18) through  $\mathcal{I}$ , does not affect exponentials of the form  $\exp(ik\Phi)$  with  $k \in \mathbb{Z}$  and thus not the behaviour of the phase-space point operator

$$\Delta(\phi, l) = (2\pi)^{-1} \exp(2il\phi) \exp(-2i\phi L/\hbar) \times \Pi \exp(-2il\Phi), \quad \phi \in \mathbb{R}, l \in \frac{1}{2}\mathbb{Z} \quad (20)$$

as introduced (in the conjugate infinite lattice system) by Hinajeros *et al* [3] and earlier independently reached in the momentum representation within an isomorphic system by Lukš and Peřinová [4]. In the angle representation, the action

$$\Delta(\phi, l) : L^2(\mathcal{I}) \rightarrow L^2(\mathcal{I}) : \psi(\chi) \mapsto (2\pi)^{-1} e^{2il(\chi-\phi)} \psi(2\phi \ominus \chi) \quad (21)$$

can be seen as a direct analogue to (15). In the last formula,  $\ominus$  represents subtraction followed by a realignment into  $\mathcal{I}$  by an addition of an integer multiple of  $2\pi$  where necessary.

The above operator can take half-integer values of the  $l$  parameter, reflecting the geometrical argument that an integer lattice is mapped onto itself under reflection about any half-integer as well as any integer. The displacement operator for a bead on a wire can be obtained hence via two-dimensional Fourier transform on a ‘doubled’ phase space  $(\frac{1}{2}\mathbb{Z}) \times S^1$

$$D(\phi, l) = \sum_{l' \in \frac{1}{2}\mathbb{Z}} \int_{\mathcal{I}} e^{il\phi' - i\phi l'} \Delta(\phi', l') d\phi', \quad (22)$$

or via an analogue of (14)

$$D(\phi, l) = 2\pi \Delta(\phi/2, l/2) \Pi. \quad (23)$$

The result in both cases agrees and, in the  $\phi$ -representation, bears a close resemblance to (11)

$$D(\phi, l) : L^2(\mathcal{I}) \rightarrow L^2(\mathcal{I}) : \psi(\chi) \mapsto e^{il(\chi-\phi/2)} \psi(\chi \ominus \phi), \quad \phi \in \mathbb{R}, l \in \mathbb{Z}. \quad (24)$$

(Note that an earlier work on rotational systems by Rigas *et al* [2] analyses more possible forms of  $D(\phi, l)$  satisfying a prescribed set of axioms but arrives at this form ultimately.)

As a result of the doubled range of the momentum coordinate in (20),  $D(\phi, l)$  is not  $2\pi$ -periodic in  $\phi$ , rather, it satisfies

$$D(\phi + 2\pi, l) = (-1)^l D(\phi, l). \quad (25)$$

In other words, the fundamental domain of both operators is doubled with respect to  $\mathbb{Z} \times S^1$ , via the introduction of half-integer values of the  $l$  argument in (20) and via the  $4\pi$  periodicity in  $\phi$  of (24), causing that for example, in order to reconstruct  $\Delta(\phi, l)$  from  $D(\phi, l)$  via Fourier transform, one needs to take an integration range of length  $4\pi$ . This is one of the reasons which prevented Rigas *et al*, in their work [2], from making their phase-space point operator  $\hat{w}(l, \phi)$ , obtained by integrating over  $(-\pi, \pi)$  only, identified with the parity operator in the special case  $(l, \phi) = (0, 0)$ .

In cases  $\phi = 0$  or  $l = 0$ , the displacement simplifies to an exponential of one of the observables

$$D(0, l) = \exp(i l \Phi), \quad D(\phi, 0) = \exp(-i \phi L / \hbar). \quad (26)$$

An important difference between the two above equations is that while operators of the latter class form a continuous one-parametric group generated by the observable  $L$ , the former only form a discrete group as  $l$  is limited to integer values by the structure of the phase space. Therefore, the observable  $\Phi$  does not serve as a generator for the displacements in a Lie algebraic sense. The exponentials are still well-defined in terms of spectral theory.

In a generic case, a direct analogue of (6) no longer works, as a study of the eigenfunctions of  $i l \Phi - i \phi L / \hbar$  reveals. It is still possible, however, to construct a generic  $D(\phi, l)$  by a composition of the axis displacements with a phase correction, akin to (9):

$$D(\phi, l) = e^{i l \phi / 2} D(\phi, 0) D(0, l) = e^{-i l \phi / 2} D(0, l) D(\phi, 0). \quad (27)$$

Clearly, the phase factor is an artifact of the ordering of the two noncommuting operations involved. A symmetrization, which removes the need for any outer phase factor, is possible by splitting the angle displacement in two:

$$D(\phi, l) = D(\phi/2, 0) D(0, l) D(\phi/2, 0). \quad (28)$$

Note that a similar trick can be applied back in the  $(x, p)$  case, where either coordinate displacement may be halved:

$$D(x, p) = D(x/2, 0) D(0, p) D(x/2, 0) = D(0, p/2) D(x, 0) D(0, p/2). \quad (29)$$

A formula analogous to the rightmost expression involving  $D(0, l/2)$  is also valid for rotational systems but only applicable for even values of  $l$ .

The ultimate goal of this paper is, however, to present a single exponential formula for the displacement operator in each system. To this end, let us define a one-parametric unitary group of ‘twist’ operators  $T(\alpha)$  [2, 5, 16]:

$$T(\alpha) = \exp(i \alpha L^2 / \hbar^2), \quad \alpha \in \mathbb{R}. \quad (30)$$

Their name stems from the action they perform on the Wigner function [3] of the system. An important property of the twist operators is their commutation relation with the displacement operator:

$$D(\phi, l) T(\alpha) = T(\alpha) D(\phi + 2\alpha l, l). \quad (31)$$

This can easily be seen in the momentum representation (not discussed in this paper) or equivalently by comparing the action of both sides on the eigenfunctions of  $L$

$$L \psi_l(\phi) = \hbar l \psi_l(\phi) \Leftrightarrow \psi_l(\phi) \propto e^{i l \phi}, \quad l \in \mathbb{Z}. \quad (32)$$

Equation (31) shows that for a nonzero fixed  $l$  (for  $l = 0$ , (26) can be used), any displacement  $D(\phi, l)$  can be



constructed from  $D(0, l)$  by a unitary transform:

$$D(\phi, l) = T(\alpha)^\dagger D(0, l) T(\alpha), \quad \alpha = \phi/(2l). \quad (33)$$

Noting that the middle term of the right-hand side is the exponential of  $i\phi$ , the unitary transform surrounding it can be transferred into the exponent, resulting in the last line rewritten in the form of a single exponential:

$$D(\phi, l) = \exp \left[ T \left( -\frac{\phi}{2l} \right) i\phi T \left( \frac{\phi}{2l} \right) \right]. \quad (34)$$

The term in the exponent, except for the prefactor  $i\phi$ , is a ‘sheared’ angle observable, one obtained by evolving  $\Phi$  under a Hamiltonian proportional to  $L^2$ , which generates (30). This replaces the difference  $pX - xP$  in (6) and presents a generic quadrature operator for a bead on a wire. The spectrum of this operator is the same as that of  $\Phi$ , as opposed to a linear combination of  $\Phi$  and  $L$ , and the domain is the full Hilbert space  $\mathcal{H}$ . Therefore the use of this operator naturally evades any issues related to unboundedness or to the pathological properties of the commutator  $[\Phi, L]$ .

One more notable property of the sheared angle observable is that it gives physical meaning to the probability distributions obtained by marginalizing the angle-angular momentum Wigner function [3] along helices: let  $\alpha \in \mathbb{R}$  and  $A \in \mathcal{B}(\mathcal{I})$ , then

$$\int_A d\phi \sum_{l \in \frac{1}{2}\mathbb{Z}} W(\phi + \alpha l, l) = \text{Tr} \left( \rho \int_A dE(\phi) \right), \quad (35)$$

where  $E(\phi)$  is the spectral decomposition of  $T(-\alpha/2)\Phi T(\alpha/2)$ .

The equation (34) is a new contribution to the theory of phase-space methods for a bead on a wire. An analogous formula can also be obtained in continuous  $(x, p)$  systems by back-tracking the idea of using unitary shear transforms (that is, free space propagation or its Fourier transform, arising as a thin-lens or ‘chirp’ transform in paraxial wave optics [17]):

$$\begin{aligned} D(x, p) &= \exp \left[ \exp(-i\alpha P^2) i p X \exp(i\alpha P^2) \right], \\ &= \exp \left[ -\exp(i\beta X^2) i x P \exp(-i\beta X^2) \right], \\ \alpha &= x/(2p), \quad \beta = p/(2x) \end{aligned} \quad (36)$$

(that the operator in the exponent is equal to  $ipX - ixP$  in both cases is a result of the Baker–Hausdorff lemma [18] in the special case of early termination of the expansion at the terms  $[P^2, X] = -iP$  or  $[X^2, P] = iX$ , respectively, which make all the higher-order commutators zero). In  $(\phi, l)$  systems, a corresponding formula employing an exponential of  $\Phi^2$  is not available due to the incompatible nature of the two phase space coordinates:  $\exp(i\beta\Phi^2)$  does not induce any simple geometrical transform of the phase space  $S^1 \times \mathbb{Z}$ . As a side note, however, the following formula surprisingly

recovers  $D(\phi, l)$  for nonzero  $\phi$ :

$$\begin{aligned} D(\phi, l) &= \exp(i\beta\Phi_\phi^2) \exp(-i\phi L/\hbar) \exp(-i\beta\Phi^2), \\ \Phi_\phi &: L^2(\mathcal{I}) \rightarrow L^2(\mathcal{I}): \psi(\chi) \mapsto (\chi \ominus \phi + \phi)\psi(\phi). \end{aligned} \quad (37)$$

Here,  $\beta = l/(2\phi)$  and  $\Phi_\phi$  is a new angle observable, effectively replacing the reference angle  $\theta_0$  hidden in  $\mathcal{I}$  in the definition (18) by  $\theta_0 + \phi$ . However, as the unitary operators on both sides of  $\exp(-i\phi L/\hbar)$  differ, this form, albeit interesting, cannot be cast into a single exponential.

#### 4. Finite-dimensional systems

In a system with Hilbert space  $\mathcal{H} = \mathbb{C}^N$ ,  $N \in \mathbb{N}$ , position observable can be defined using the operator

$$A : \mathbb{C}^N \rightarrow \mathbb{C}^N : \{\alpha_k\}_{k=0}^{N-1} \mapsto \{k\alpha_k\}_{k=0}^{N-1}, \quad (38)$$

with the position eigenstates being the elements of the standard basis  $\{|a\rangle | 0 \leq a < N\}$ . In this basis,  $A$  is described by a diagonal matrix  $(A_{kl})$  with elements from 0 to  $N - 1$  on the diagonal; both descriptions will be used interchangeably. Momentum is then naturally defined via its eigenstates

$$|p\rangle\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i p k/N} |k\rangle, \quad (39)$$

where  $p$  is an integer defined up to a modular class mod  $N$ . A suitable momentum observable can be obtained using a two-dimensional discrete Fourier transform of  $A$ :

$$B_{kl} = \frac{1}{N} \sum_{m,n=0}^{N-1} e^{2\pi i(km-nl)/N} A_{mn}. \quad (40)$$

The eigenvalues of  $B$  are then also integers from  $b = 0$  to  $N - 1$  with eigenvectors  $|b\rangle\rangle$  as defined above.

Note that there are significant differences between  $A, B$  and position and momentum observables in continuous variable systems. In particular, the definition of  $B$  is nowhere close to the differential operator (19). Most notably, however, no two finite-dimensional operators can commute to a non-zero multiple of the identity. In fact, the commutator  $[A, B]$  would be described by a matrix all the diagonal elements of which are zero in the standard basis, as opposed to being the only nonzero elements as in the case of  $[X, P] = i\hbar 1$ .

In the following, we shall assume that both the position and momentum coordinates are defined up to an integer multiple of  $N$ , working with the eigenvalues of the two observables as the representatives of their respective modular classes. An expression involving  $a$  or  $b$  will only be considered well-defined if it is invariant under an addition of  $N$  to either of the variables. The phase space is  $\Omega = (\mathbb{Z}/N\mathbb{Z})^{\times 2}$ .

In the pioneering work on phase-space formalism in finite-dimensional quantum systems, Wooters [6] specified the Wigner function by a set of axioms based on the direct generalization of inverse Radon transform—reconstruction from independent marginals. The concept of straight and parallel lines along which the marginalization is performed,

along with the requirement of statistical sufficiency, lead to the restriction of the Hilbert space dimension to a prime or a prime power. Later works lifted this restriction by redefining the Wigner function using a generalization of phase-space point operators [7]

$$\Delta(a, b) = \frac{1}{N} \sum_{\alpha=0}^{N-1} e^{4\pi i b(a-\alpha)/N} |2a - \alpha\rangle \langle \alpha|, \quad a, b \in \mathbb{Z}/2 \quad (41)$$

closely linked to displacement operators via

$$D(a, b) = \frac{1}{2} \sum_{\alpha, \beta=0}^{2N-1} e^{\pi i (ab - a\beta)/N} \Delta(\alpha/2, \beta/2) \quad (42)$$

or equivalently [19]

$$D(a, b) = N\Delta(a/2, b/2)\Pi = \sum_{\alpha=0}^{N-1} e^{2\pi i b(\alpha+a/2)/N} |\alpha + a\rangle \langle \alpha|, \quad a, b \in \mathbb{Z}, \quad (43)$$

where  $\Pi = N\Delta(0, 0)$  is the position inversion operator. Displacement operators of this form, in turn, lead to the definition of coherent states [19] and finite-dimensional analogues of Glauber–Sudarshan  $P$  and Husimi  $Q$  functions [20–22].

Fundamental properties of  $D(a, b)$  comparable to those listed in the cases of  $D(x, p)$  and  $D(\phi, l)$  in the preceding sections include

$$D(a, 0) = \exp(-2\pi i a B/N) = D(1, 0)^a, \quad D(0, b) = \exp(2\pi i b A/N) = D(0, 1)^b, \quad (44)$$

and

$$D(a, b) = e^{\pi i ab/N} D(a, 0) D(0, b) = e^{-\pi i ab/N} D(0, b) D(a, 0), \quad (45)$$

among others. The operators  $D(1, 0)$  and  $D(0, 1)$  correspond precisely to the  $(A, B)$  or  $(U, V)$  operators studied by Weyl and Schwinger [23, 24], who derive by algebraic means that, in the current notation

$$D(0, b) D(a, 0) = e^{2\pi i ab/N} D(a, 0) D(0, b).$$

The operator  $D(a, b)$  thus represents a phase-symmetrized product of the two axis displacements. Moreover, in analogy to (25),  $D(a, b)$  satisfy

$$D(a + N, b) = (-1)^b D(a, b), \quad D(a, b + N) = (-1)^a D(a, b) \quad (46)$$

and are  $2N$ -periodic in both their arguments. If one of the arguments is fixed at an even value (particularly, at zero), and only then, the minimal period in the other argument reduces to  $N$ .

In the pursuit for a single exponential form for  $D(a, b)$ , one may follow the protocol laid out in the previous section

and define two *discrete* unitary groups of shear operators

$$T_a(q) = \exp\left(2\pi i q A^2/N\right), \quad T_b(q) = \exp\left(2\pi i q B^2/N\right). \quad (47)$$

When the operators  $A$  and  $B$  are formally replaced by their eigenvalues, the condition of modular class consistency restricts  $q$  to the integers if  $N$  is odd and to integer or half-integer values if  $N$  is even.

Analogously to (31) in the case of continuous shear transforms in the last section, it holds that

$$T_b(q)^\dagger D(a, b) T_b(q) = D(a + 2qb, b) \quad (48)$$

as well as

$$T_a(q)^\dagger D(a, b) T_a(q) = D(a, b - 2qa). \quad (49)$$

Hence we can see that the unital superoperators generated by the two shear operators  $T_a(q)$  and  $T_b(q)$  act on the set of displacement operators via special linear transforms of their arguments with integer coefficients. It might not be possible to obtain  $D(a, b)$  by a shear transform of  $D(0, b)$  (or  $D(a, 0)$ ) for generic  $a$  and  $b$ , as in (34), obvious counterexamples being cases where  $b$  (or  $a$ , respectively) shares a nontrivial common divisor with  $N$ . However, one might apply multiple shear transforms to generate more general  $SL(2, \mathbb{Z})$  transforms on the displacement offset.

Considering (48) and (49) together with the property (46), we find for odd  $N$  that

$$T_b((N+1)/2)^\dagger D(a, b) T_b((N+1)/2) = D(a + b + Nb, b) = (-1)^b D(a + b, b), \quad T_a((N-1)/2)^\dagger D(a, b) T_a((N-1)/2) = D(a, b + a - Na) = (-1)^a D(a, a + b). \quad (50)$$

Defining

$$D'(a, b) = (-1)^{ab} D(a, b), \quad (51)$$

(50) can be rewritten as

$$T_b((N+1)/2)^\dagger \times D'(a, b) T_b((N+1)/2) = D'(a + b, b), \quad T_a((N-1)/2)^\dagger \times D'(a, b) T_a((N-1)/2) = D'(a, a + b), \quad (52)$$

which means that the fundamental generators of  $SL(2, \mathbb{Z})$  are reachable on the set of sign-corrected displacements and thus, by composition, any element thereof.

The case of even  $N$  is even simpler as half-integer values of  $q$  are permitted, thus one can use

$$T_b(1/2)^\dagger D(a, b) T_b(1/2) = D(a + b, b), \quad T_a(-1/2)^\dagger D(a, b) T_a(-1/2) = T_a(1/2) D(a, b) T_a(1/2)^\dagger = D(a, b + a). \quad (53)$$

In fact, half-integer values of  $q$  are the only way to obtain  $D(a + b, b)$  or  $D(a, b + a)$  from  $D(a, b)$  in the case of even dimension. Let us mention at this point an intriguing resemblance to the result that phase-space point operators

need to be defined in half-integer values of  $a$  and  $b$  in even-dimensional systems in order to generate the operator space [25] and similarly the Wigner function needs to be defined on a phase space doubled in both coordinates,  $\Omega' = ((\frac{1}{2}\mathbb{Z})/N\mathbb{Z})^{\times 2}$ , for the state to be reconstructible thence, while integer coordinates are sufficient in odd dimensions.

Depending on the parity of  $N$ , the equations (50) or (53) induce a representation of  $SL(2, \mathbb{Z})$  on the operator space by shear transforms. Indeed, for all  $X \in \mathcal{B}(\mathbb{C}^N)$

$$\begin{aligned} T_a(q)^\dagger T_b(q)^\dagger T_a(q)^\dagger X T_a(q) T_b(q) T_a(q) \\ = T_b(q)^\dagger T_a(q)^\dagger T_b(q)^\dagger X T_b(q) T_a(q) T_b(q), \end{aligned} \quad (54)$$

$$[T_a(q) T_b(q) T_a(q)]^{\dagger 4} X [T_a(q) T_b(q) T_a(q)]^{\dagger 4} = X, \quad (55)$$

where  $q = 1/2$  for  $N$  even and  $q = (N + 1)/2$  for  $N$  odd. For  $N$  fixed and any  $M \in SL(2, \mathbb{Z})$ , let  $\mathcal{T}_N(M)$  denote the superoperator assigned to  $M$  in the respective representation.

Given any  $a, b \in \mathbb{Z}$  with  $d = \text{gcd}(a, b)$ , there exists a  $M \in SL(2, \mathbb{Z})$  such that

$$(a, b)^\top \equiv M(0, d)^\top \pmod{N}. \quad (56)$$

Then, in the case of even  $N$

$$\mathcal{T}_N(M)[D(0, d)] = D(a, b) \quad (57)$$

and if  $N$  is odd

$$\begin{aligned} \mathcal{T}_N(M)[D(0, d)] &= \mathcal{T}_N(M)[D'(0, d)] \\ &= D'(a, b) = (-1)^{ab} D(a, b). \end{aligned} \quad (58)$$

Equations (57) and (58) can be written uniformly as

$$\mathcal{T}_N(M)[D(0, d)] = (-1)^{abN} D(a, b). \quad (59)$$

From this form one directly obtains

$$D(a, b) = (-1)^{abN} \exp\left(\frac{2\pi i d}{N} \mathcal{T}_N(M)[A]\right). \quad (60)$$

This is the required single exponential form of the displacement operator in finite-dimensional systems.

### 5. Example: a qubit

Let us illustrate the above theory with the simplest nontrivial example, a two-dimensional quantum mechanical system ( $\mathcal{H} = \mathbb{C}^2$ ), or a qubit. Following the above framework we define a position observable as the diagonal matrix (38)

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(I - \sigma_z) \quad (61)$$

and a momentum observable using two-dimensional discrete Fourier transform

$$\begin{aligned} B &= F_2 A F_2^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}(I - \sigma_x), \end{aligned} \quad (62)$$

where  $\sigma_x, \sigma_y, \sigma_z$  are the familiar Pauli matrices. The unit displacements along axes are defined by

$$\begin{aligned} D(1, 0) &= \exp(-i\pi B) = \sigma_x, \\ D(0, 1) &= \exp(i\pi A) = \sigma_z. \end{aligned} \quad (63)$$

A simultaneous displacement  $D(1, 1)$  can be reached using the composition formula (45)

$$\begin{aligned} D(1, 1) &= e^{i\pi/2} D(1, 0) D(0, 1) \\ &= e^{-i\pi/2} D(0, 1) D(1, 0) = \sigma_y, \end{aligned} \quad (64)$$

or using either of the fundamental shear operations

$$\begin{aligned} T_a(1/2) &= \exp(i\pi A^2/2) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \\ T_b(1/2) &= \exp(i\pi B^2/2) = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}; \end{aligned} \quad (65)$$

$$\begin{aligned} D(1, 1) &= T_a(1/2) D(1, 0) T_a(1/2)^\dagger \\ &= T_b(1/2)^\dagger D(0, 1) T_b(1/2). \end{aligned} \quad (66)$$

Observing that

$$\begin{aligned} T_a(1/2) B T_a(1/2)^\dagger &= T_b(1/2)^\dagger A T_b(1/2) \\ &= \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{2}(I - \sigma_y) =: C, \end{aligned} \quad (67)$$

one can identify  $D(1, 1)$  with  $\exp(\pm i\pi C)$ .

The odd-valued displacements manifest the sign change of (46), for example,  $D(3, 1) = -D(1, 1)$ . This is consistent with (45)

$$\begin{aligned} D(3, 1) &= e^{3i\pi/2} D(1, 0) D(0, 1) \\ &= e^{-3i\pi/2} D(0, 1) D(1, 0) = -\sigma_y, \end{aligned} \quad (68)$$

as well as with (59). To illustrate the latter, take

$$\begin{aligned} M &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3: \quad \mathcal{T}(M)[X] \\ &= [T_b(1/2)^\dagger]^3 X [T_b(1/2)]^3 = T_b(3/2)^\dagger X T_b(3/2), \end{aligned} \quad (69)$$

whence

$$\begin{aligned} D(3, 1) &= T_b(3/2)^\dagger D(0, 1) T_b(3/2) \\ &= \exp(i\pi T_b(3/2)^\dagger A T_b(3/2)). \end{aligned} \quad (70)$$

As  $T_b(3/2) = T_b(1/2)^*$  and  $A^* = A$ , the right hand side equals  $\exp(i\pi C^*)$  and thus  $D(1, 1)^*$ , or  $-\sigma_y$ .

Finally, consider the effects of the shear and displacement operators on the Bloch sphere representation of a qubit state. The group generated by the set of shear superoperators is isomorphic to the chiral octahedral symmetry group and can be visualized as the symmetry group of the Bloch sphere preserving the set of Cartesian axes as a set of three lines. This is also the group of all linear canonical transforms of the doubled phase space  $\mathbb{Z}_4^{\otimes 2}$ . On the other hand, the superoperators corresponding to the three independent displacement operators act on a quantum state as rotations about the three axes by  $\pi$  and form, together with the identity, the Klein four-group  $\mathbb{Z}_2^{\otimes 2}$ . Note that this group is necessarily Abelian



by construction, as the total displacement only adds as a vector in  $\mathbb{Z}_2^2$  under composition and the global phase is irrelevant in the superoperator representation. Finer transforms acting on the state such as smooth  $SO(3)$  rotations can be considered in different formalisms [1] but their behaviour differs from that of the displacement operators described within this work.

## 6. Example: a qutrit

It is very instructive to complement the steps illustrated in the above example by an analogous example done with a three-dimensional system, often called a qutrit. Not only there are subtle but important differences in the above theory between even- and odd-dimensional systems (for example, in the restrictions on the shear transform parameters), also in two dimensions some simplifications take place that do not allow to fully appreciate all details of the theory, resulting from many of the key matrices being idempotent, real-valued or Hermitean where they would not be were the dimension higher.

Let us start our second example by defining the position observable

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (71)$$

and the momentum observable as given by (40)

$$\begin{aligned} B &= F_3 A F_3^\dagger = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{8\pi i/3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &\times \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{-4\pi i/3} \\ 1 & e^{-4\pi i/3} & e^{-8\pi i/3} \end{pmatrix} \\ &= \begin{pmatrix} 1 & e^{5\pi i/6/\sqrt{3}} & e^{-5\pi i/6/\sqrt{3}} \\ e^{-5\pi i/6/\sqrt{3}} & 1 & e^{5\pi i/6/\sqrt{3}} \\ e^{5\pi i/6/\sqrt{3}} & e^{-5\pi i/6/\sqrt{3}} & 1 \end{pmatrix}. \end{aligned} \quad (72)$$

The fundamental displacements are

$$\begin{aligned} D(1, 0) &= \exp(-2i\pi B/3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ D(0, 1) &= \exp(2i\pi A/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}. \end{aligned} \quad (73)$$

In contrast to the previous example, the argument of the shear transforms (47) needs to be integer so only even changes in the arguments of the displacements are possible at the action given by (48), (49). Thus  $D(1, 1)$  (or any displacement with both offsets odd) is not reachable from any axis displacement directly. However, we might reach  $D(1, 4)$  or other instances which differ from  $D(1, 1)$  only in sign. This should not be surprising as the unitary transforms (48), (49) preserve the spectrum of the displacement being transformed. All

nontrivial axis displacements in three dimensions have spectra of the same form

$$\begin{aligned} \sigma(D(d, 0)) &= \sigma(D(0, d)) \\ &= \left\{ 1, e^{2\pi i/3}, e^{4\pi i/3} \right\} \quad (\forall d \in \mathbb{Z}: d \perp 3) \end{aligned} \quad (74)$$

and so does  $D(1, 4)$ , which may be obtained, for example, as  $T_a(2)D(1, 0)T_a(2)^\dagger$ . But by (46)

$$\begin{aligned} D(1, 1) &= -D(1, 4) \Rightarrow \sigma(D(1, 1)) \\ &= -\sigma(D(1, 4)) = \left\{ -1, -e^{2\pi i/3}, -e^{4\pi i/3} \right\}. \end{aligned} \quad (75)$$

As a side note we would like to point out that in even-dimensional systems the spectrum of each displacement operator is symmetric with respect to sign inversion so this argument does not preclude reachability of doubly odd displacements from axis displacements by unitary transforms. Indeed, as we saw in the qubit example, the two-dimensional case of  $D(1, 1)$  could be reached by shear transforms of  $D(1, 0)$  or  $D(0, 1)$  without resorting to (46).

Other ways of obtaining  $D(1, 1)$  include

$$\begin{aligned} D(1, 1) &= e^{\pi i/3} D(1, 0) D(0, 1) \\ &= e^{-\pi i/3} D(0, 1) D(1, 0) \\ &= -T_a(1)^\dagger D(1, 0) T_a(1) \\ &= -T_b(1) D(0, 1) T_b(1)^\dagger \\ &= -\exp\left(2\pi i T_b(1) A T_b(1)^\dagger / 3\right) \\ &= -\exp\left(-2\pi i T_a(1)^\dagger B T_a(1) / 3\right). \end{aligned} \quad (76)$$

All of these forms agree in the result

$$D(1, 1) = \begin{pmatrix} 0 & 0 & -e^{2\pi i/3} \\ -e^{-2\pi i/3} & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \quad (77)$$

In particular, the last two lines of (76) show two different exponential forms of the latter operator. Evaluating the unit shear operators using the theory of quadratic Gauss sums

$$\begin{aligned} T_a(1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix}, \\ T_b(1) &= \frac{1}{\sqrt{3}} \begin{pmatrix} e^{\pi i/2} & e^{-\pi i/6} & e^{-\pi i/6} \\ e^{-\pi i/6} & e^{\pi i/2} & e^{-\pi i/6} \\ e^{-\pi i/6} & e^{-\pi i/6} & e^{\pi i/2} \end{pmatrix}, \end{aligned} \quad (78)$$

we note that the two exponents are not equal:

$$\begin{aligned} C_1 &= T_b(1) A T_b(1)^\dagger \\ &= \begin{pmatrix} 1 & e^{-\pi i/6/\sqrt{3}} & e^{-\pi i/2/\sqrt{3}} \\ e^{\pi i/6/\sqrt{3}} & 1 & e^{-5\pi i/6/\sqrt{3}} \\ e^{\pi i/2/\sqrt{3}} & e^{-5\pi i/6/\sqrt{3}} & 1 \end{pmatrix}, \end{aligned}$$

$$C_2 = -T_a(1)^\dagger B T_a(1) = -\begin{pmatrix} 1 & e^{-\pi i/2/\sqrt{3}} & e^{-\pi i/6/\sqrt{3}} \\ e^{\pi i/2/\sqrt{3}} & 1 & e^{5\pi i/6/\sqrt{3}} \\ e^{\pi i/6/\sqrt{3}} & e^{-5\pi i/6/\sqrt{3}} & 1 \end{pmatrix}; \quad (79)$$

however, they commute and differ by a matrix whose spectrum is  $\{0, 3\}$  so they both lead to the same result

$$\exp(2\pi i C_1/3) = \exp(2\pi i C_2/3) = -D(1, 1). \quad (80)$$

Alternatively an  $i\pi$ -multiple of the identity matrix could be added to the exponent if the external minus sign is not desired.

## 7. Conclusions

We have presented a number of alternative expressions used for the displacement operator in the continuous variable systems and discussed the possibilities of their generalization to infinite lattice, angle—angular momentum systems, and finite dimensional systems. We have found a single exponential form using a sheared angle observable as a generator in angle—angular momentum systems, and found analogues for other classes of quantum systems. The shear observable only reduces to a linear combination of position and momentum observables in the case of two continuous variables, therefore it provides a new look at constructing generalized quadratures in all three systems in question. As shear phase space transforms can be equally well applied even for systems where the spectra of position and momentum are not isomorphic, we argue that they are more universal than general linear transforms and more general than phase space rotations.

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