Local and global finite branching of solutions of ordinary differential equations

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Dedicated to Ilpo Laine on his 70th birthday

Abstract. We consider ordinary differential equations such that the only movable singularities of solutions that can be reached by analytic continuation along finite length curves are either poles or algebraic branch points. We review results in the literature about such equations. These results generalise some known proofs that the Painlevé equations possess the Painlevé property. Although locally the singularity structure of such solutions is simple, the global structure is often very complicated. We consider a class of second-order equations and classify the admissible solutions that are globally quadratic over the field of meromorphic functions.

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1. INTRODUCTION

Cauchy's theorem guarantees the existence of a unique analytic local solution to any regular initial value problem for an ordinary differential equation (ODE). For a given ODE it is natural to ask what kind of singularities can develop after the analytic continuation of such a local solution. For linear ODEs, singularities in solutions can only occur at singularities of the coefficients (when the coefficient of the highest derivative has been set to 1). Such singularities are called *fixed*. In contrast, solutions of nonlinear equations can also

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be singular at values of the independent variable for which all coefficient functions in the equation are regular. Such singularities are called *movable* as their locations vary as we move from one solution to another by varying the initial conditions. For example, the general solution of

$$w' = \frac{w}{2}\left(z^2w^2 - \frac{2}{z} + \frac{1}{z^2w^2}\right)$$

is $w(z) = z^{-1}\sqrt{\tan(z-c)}$, where c is an arbitrary constant. The singularity at $z = 0$ is fixed while all other singularities (which are located at $z = c + (n\pi/2)$, $n \in \mathbb{Z}$) are movable square-root branch points. This is a particular kind of algebraic singularity, which means that in a neighbourhood of such a singularity at $z = z_0$, there is a rational number $r > 0$ such that the solution can be represented as the sum of a Laurent series in $(z - z_0)^r$ with finite principal part. This behaviour is typical for first-order ODEs as shown by Painlevé's theorem (see, e.g., Ince [10] or Hille [5]).

Theorem 1.1. (Painlevé) *All movable singularities of all solutions of an equation of the form $y' = R(z, y)$, where R is rational in y with coefficients that are analytic in z on some common open set, are either poles or algebraic branch points.*

The situation is much more complicated for higher-order equations. Indeed it is simple to construct equations with solutions having movable essential singularities, logarithmic branch points and even movable natural barriers. Section 2 of this paper will be a review in which we study several classes of second-order ODEs of the form

$$y'' = E(z, y)(y')^2 + F(z, y)y' + G(z, y), \tag{1.1}$$

such that the only movable singularities of any solution that can be reached by analytic continuation along finite length curves are either poles or algebraic branch points. For the equations considered it is a straightforward matter to verify when they possess enough formal series solutions of the desired form. The difficulty arises in showing that such series represent the only kinds of movable singularities that can be reached. The simplest class of equations considered (other than those that can be solved by quadrature) includes the Painlevé equations. The proofs of all of the theorems described in Section 2 generalise the proofs that the Painlevé equations possess the Painlevé property (that all solutions are single-valued about all movable singularities) in the spirit of Painlevé [17], Hukuhara [16], Hinkkanen and Laine [6], and Shimomura [20]. This property is closely related to the *weak Painlevé property* [18] and has been called the *quasi-Painlevé property* by Shimomura [21–23]. We will also describe a recent result on algebraic singularities of certain Hamiltonian systems [12].

The Painlevé property is important as it appears to imply that an equation is integrable (in some sense solvable). In particular, the only first-order rational equations of the form $y' = R(z, y)$ with the Painlevé property are Riccati equations,

$$y' = a(z)y^2 + b(z)y + c(z),$$

which can be solved in terms of solutions of a second-order linear ODE. Also, each of the Painlevé equations can be written as the compatibility condition for a pair of linear problems with spectral parameters (iso-monodromy problems) from which many remarkable properties follow. However, the Painlevé property (but not integrability) is easily destroyed by making an algebraic change of variables.

Although functions with only algebraic singularities are very simple objects locally, they can be very complicated globally and generally require a complicated infinitely-sheeted Riemann surface. So although the Painlevé property of say the first Painlevé equation can be destroyed by an algebraic change of variables, the resulting equation can be distinguished from the generic case by the fact that its solutions are globally, not just locally, finitely branched.

Let F be the set of fixed singularities of some ODE and let M be the set of meromorphic functions over $\mathbb{C} \setminus F$. Let us say that the equation has the *algebro-Painlevé property* if all solutions are algebraic over M. It is natural to speculate that ODEs with this property are integrable. In Section 3 of this paper we consider a related problem. Following the standard conventions of Nevanlinna theory, for any meromorphic function f, we denote any quantity that is $o(T(r, f))$ as $r \to \infty$ outside of some possible exceptional set of finite linear measure by $S(r, f)$. We will prove the following theorem.

Theorem 1.2. *Let y be a solution of the equation*

$$y'' = \frac{3}{4}y^5 + \sum_{k=0}^{4} a_k(z)y^k, \tag{1.2}$$

such that y also satisfies

$$y(z)^2 + s_1(z)y(z) + s_2(z) = 0, \tag{1.3}$$

s_1, s_2, $a_0, \ldots, a_4$ being meromorphic functions such that for some $j \in \{1,2\}$, $T(r, a_k) = S(r, s_j)$ for all $k \in \{0, \ldots, 4\}$. Suppose that equation (1.3) is irreducible over the meromorphic functions. Then s_1 is proportional to a_4, and s_2 reduces either to the solution of a Riccati equation with coefficients that are rational expressions in $a_0, \ldots, a_4$ and their derivatives, or to the equation

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4(az+b)w^2 + 2((az+b)^2 - c)w, \tag{1.4}$$

which, in case of $a \neq 0$ is equivalent to a special case of the fourth Painlevé equation and in case of $a = 0$ can be solved in terms of elliptic functions.

2. MOVABLE ALGEBRAIC SINGULARITIES

Painlevé, Gambier and Fuchs studied equations in the complex domain of the form

$$y'' = F(z, y, y'), \tag{2.1}$$

where $F(z, p, q)$ is rational in p and q with coefficients that are analytic in some common domain. They showed that any equation of the form (2.1) can be mapped by a transformation of the form

$$z \mapsto \Phi(z), \qquad y \mapsto \frac{\alpha(z)y + \beta(z)}{\gamma(z)y + \delta(z)}$$

to one of fifty canonical equations. Among these equations were the six known today as the Painlevé equations P_I–P_{VI}, the first three of which are

$$\begin{aligned}
y'' &= 6y^2 + z, \\
y'' &= 2y^3 + zy + \alpha, \\
y'' &= \frac{(y')^2}{y} - \frac{y'}{z} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},
\end{aligned}$$

where α, β, γ and δ are arbitrary constants. The solutions of each of the fifty canonical differential equations could be solved in terms of linear differential equations, classically known functions such as elliptic functions, and the solutions of the six Painlevé equations.

Painlevé [17] presented a proof that the first Painlevé equation does indeed have the Painlevé property (which for this equation is equivalent to showing that all solutions are meromorphic), however the proof contained gaps, which were only properly fixed in the published literature in 1999 by Hinkkanen and Laine [6]. A correct proof had also been circulating around the University of Tokyo since the 1960s by Hukuhara, which was eventually published (in the original Esperanto!) by Okamoto and Takano in [16]. Hinkkanen and Laine subsequently published a series of papers in which they proved the Painlevé property for all of the Painlevé equations [7–9] using similar methods. Shimomura [20] also provided proofs that the Painlevé equations possess the Painlevé property. All of these proofs have in common the fact that they work directly with the nonlinear equations, and by showing that certain quantities must be bounded they are able to construct different regular initial value problems that correspond to the possible singularities of a solution. Broadly speaking, these are the same tools that will be described below for analysing algebraic singularities. Most other approaches to proving the Painlevé property use the related linear (iso-monodromy) problems, so these approaches are essentially using the integrability of the equations, which will not generalise to the class of equations that we consider.

In 1953, Smith [24] proved the following.

Theorem 2.1. *Let f and g be polynomials of degree n and m respectively, where $n > m$, and let P be analytic at some point z_0. Then there is an infinite family of solutions of*

$$y'' + f(y)y' + g(y) = P(z), \tag{2.2}$$

which have an algebraic critical point at z_0. In a neighbourhood of z_0 these solutions can be expressed in the form

$$y(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^{(j-1)/n}, \tag{2.3}$$

where $a_0 \neq 0$. Furthermore, let Γ be a contour of finite length which lies in the z-plane and has z_0 as an end point. If $y(z)$ is a solution of equation (2.2) which can be continued analytically along Γ as far *as z_0 but not over it, then the singularity of $y(z)$ at z_0 must be an algebraic critical point of the type described in (2.3).*

In the same paper, Smith also showed that the only singularities in the finite plane that can be reached by continuation along infinite length curves are accumulation points of such algebraic branch points. In particular, he was able to demonstrate this phenomenon in solutions of the equation

$$y'' + 4y^3y' + y = 0,$$

by using the fact that the general solution of this equation can be given implicitly in terms of Bessel functions.

In a series of papers [21–23], Shimomura considered the equations

$$y'' = \frac{2(2k+1)}{(2k-1)^2}y^{2k} + z, \qquad k \in \mathbb{N} \tag{2.4}$$

$$y'' = \frac{k+1}{k^2}y^{2k+1} + zy + \alpha, \qquad k \in \mathbb{N} \setminus \{2\}, \tag{2.5}$$

which he referred to as P_{I}-type and P_{II}-type respectively. Shimomura's main results concerning these equations can be summarised as follows.

Theorem 2.2. *Any singularity of a solution of equation (2.4) or (2.5) that can be reached by analytic continuation along a finite length curve is algebraic.*

Before moving on to discuss more general equations, we will first study some obstructions to the existence of algebraic singularities. To this end, consider the ODE

$$y'' = \sum_{n=0}^{N-2} a_n(z)y(z)^n + \frac{2(N+1)}{(N-1)^2}y(z)^N, \tag{2.6}$$

where $N \geq 2$ is an integer and the $a_0, \ldots, a_{N-2}$ are analytic in a neighbourhood of some point $z = z_0$. The coefficient of y^N has been normalised for convenience. We wish to find a formal series solution to equation (2.6) that is a Laurent series in some fractional power of $z - z_0$. We begin by looking for leading-order behaviour of the form $y(z) = c_0(z-z_0)^{-p} + o\left((z-z_0)^{-p}\right)$ as $z \to z_0$. We find that $c_0^{N-1} = 1$ and $p = 2/(N-1)$. Superficially there appear to be $N-1$ different leading-order behaviours. However, if N is even then

$(z-z_0)^{-2/(N-1)}$ is a branched function with $N-1$ branches. If we fix any choice of c_0 such that $c_0^{N-1}=1$, then the other $N-2$ values of c_0 simply correspond to the other sheets of the Riemann surface that can be reached by the analytic continuation of $c_0(z-z_0)^{-2/(N-1)}$ around $z=z_0$. So there is only one leading-order behaviour from this point of view. However, when N is odd then $(z-z_0)^{-2/(N-1)}$ has only $(N-1)/2$ branches whereas there are still $N-1$ choices for c_0. So we see that when N is odd there are effectively two leading-order behaviours.

Having obtained the leading-order behaviour, we now look for a formal Laurent series expansion of the form

$$y(z)=\sum_{n=0}^{\infty} c_n(z-z_0)^{(n-2)/(N-1)}. \tag{2.7}$$

If N is odd it is in fact sufficient to have a Laurent series in $(z-z_0)^{2/(N-1)}$, meaning that $c_k=0$ for k odd. Substituting the expansion (2.7) in equation (2.6) leads to a recurrence relation of the form

$$(r+N-1)(r-2N-2)c_r=P_r(c_0,\dots,c_{r-1}), \qquad r\geq 1, \tag{2.8}$$

where P_r is a polynomial in its arguments. Having determined c_0, equation (2.8) allows us to determine $c_1,\dots,c_{2N+1}$. However, on substituting $r=2N+2$ in (2.8) we obtain the condition

$$P_{2N+2}(c_0,\dots,c_{2N+1})=0. \tag{2.9}$$

If this condition is satisfied then c_{2N+2} can be chosen arbitrarily and then the recurrence relation (2.8) uniquely determines all other c_j for all $j>2N+2$. If (2.9) is not true then there is no formal series solution of the form (2.7).

Recall that if N is even then there is effectively only one leading-order behaviour and so there is only one obstruction of the form (2.9), which in this case turns out to be $a''_{N-2}(z_0)=0$. When N is odd, we get two conditions. They are equivalent to $a''_{N-2}(z_0)=0$ and one other condition on the coefficients.

In [1], Filipuk and Halburd proved the following.

Theorem 2.3. *For $N\geq 2$, suppose that there is a domain $\Omega\subset\mathbb{C}$ such that $a_0,\dots,a_N$ are analytic and that $a_N(z_0)\neq 0$ on Ω. Suppose* *further that for each $z_0 \in \Omega$ and for each c_0 such that*

$$c_0^{N-1} = \frac{2}{a_N(z_0)} \frac{N+1}{(N-1)^2}, \tag{2.10}$$

the equation

$$y''(z) = \sum_{n=0}^{N} a_n(z) y(z)^n, \qquad n \in \mathbb{N}, \quad n \geq 2. \tag{2.11}$$

admits a formal series solution of the form

$$y(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^{\frac{j-2}{N-1}}. \tag{2.12}$$

Then

(i) *For each c_0 satisfying* (2.10) *and for each $\beta \in \mathbb{C}$, there is a unique formal series solution of the form* (2.12) *such that $c_{2(N+1)} = \beta$.*

(ii) *Given c_0 and $c_{2(N+1)}$ as above, the series* (2.12) *converges in a neighbourhood of z_0.*

(iii) *Now let y be a solution of equation* (2.11) *that can be continued analytically along a curve Γ up to but not including the endpoint z_0, where the coefficients a_j are analytic on $\Gamma \cup \{z_0\}$ and a_N is nowhere zero on $\Gamma \cup \{z_0\}$. If Γ is of finite length, then y has a convergent series expansion about z_0 of the form* (2.12).

(iv) *If y cannot be represented by a series expansion about z_0 of the form* (2.12) *then Γ is of infinite length and z_0 is an accumulation point of such algebraic singularities.*

The following theorem [2] is a generalisation of Smith's Theorem 2.1.

Theorem 2.4. *Let Γ be a finite length curve with z_0 as one of its endpoints and let*

$$F(z, y) = \sum_{j=0}^{n} f_j(z) y^j, \qquad G(z, y) = \sum_{k=0}^{n+1} g_k(z) y^k,$$

where n is a positive integer, $f_0,\dots,f_n; g_0,\dots,g_{n+1}$ are analytic on $\Gamma\cup\{z_0\}$ and f_n is nowhere zero there. Suppose that y is a solution of the equation

$$y'' = F(z,y)y' + G(z,y),$$

that is analytic on Γ but cannot be analytically continued to $\Gamma\cup\{z_0\}$. If, in a neighbourhood of z_0, either

$$f'_{n-1}f_n - f_{n-1}f'_n + (n+1)f_{n-1}g_{n+1} - nf_ng_n = 0, \qquad (n>1)$$

or

$$\begin{aligned} f_0f_1(2g_2-f_1') + (2g_2-f_1')^2 - f_1^2g_1 + f_0'f_1^2 + f_1(2g_2'-f_1'') \\ - f_1'(2g_2-f_1') = 0, \qquad (n=1) \end{aligned}$$

then y has a series expansion of the form

$$y(z) = \sum_{j=0}^{\infty} c_j(z-z_0)^{(j-1)/n},$$

where $c_0^n = -(n+1)/(nf_n(z_0))$, that converges in a neighbourhood of $z=z_0$.

The papers [3, 11] concern equations of the form (1.1) where E, F and G have the form

$$E(z,y) = \sum_{\mu=1}^{n} \frac{k_\mu}{y-a_\mu(z)}, \qquad F(z,y) = \frac{f(z,y)}{\prod_{\mu=1}^{n}(y-a_\mu(z))^{l_\mu}},$$
$$G(z,y) = \frac{g(z,y)}{\prod_{\mu=1}^{n}(y-a_\mu(z))^{m_\mu}},$$

in a neighbourhood of a point $z=z_0\in\mathbb{C}$, where $f(z,y)$ and $g(z,y)$ are polynomials in y with coefficients that are analytic in a neighbourhood of $z=z_0$. Empty sums and products are taken to be zero and one respectively. All of the functions $a_\mu(z)$ are analytic in a neighbourhood of $z=z_0$ and $a_\mu(z_0)=a_\nu(z_0)$ only if $\mu=\nu$. Finally we let

$$f_\mu(z) = \frac{f(z,a_\mu(z))}{\prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(a_\mu(z)-a_\nu(z))^{l_\nu}}, \qquad g_\mu(z) = \frac{g(z,a_\mu(z))}{\prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(a_\mu(z)-a_\nu(z))^{m_\nu}}.$$

The following theorem was proved in [3].

Theorem 2.5. *Consider equation* (1.1) *with* E, F *and* G *as described above. Let the degree of* $f(z,y)$ *as a polynomial in* y *be at most* n *and define*

$$k_0 := 2 - \sum_{\mu=1}^{n} k_\mu, \quad m_0 := deg_y g(z,y) - \sum_{\mu=1}^{n} m_\mu - 2 \quad \textit{and}$$
$$g_0(z) := \lim_{y\to\infty} y^{-(m_0+2)} G(z,y).$$

We make the following assumptions.

(a) *For all* $\mu \in \{0,\ldots,n\}$, $g_\mu(z_0) \neq 0$.

(b) *For all* $\mu \in \{0,\ldots,n\}$, $2k_\mu$ *and* m_μ *are integers such that* $2k_\mu + m_\mu > 1$ *and* $m_\mu \geq 1$.

(c) *For all* $\mu \in \{1,\ldots,n\}$, $f_\mu + 2k_\mu a'_\mu \equiv 0$. *Furthermore, if* $m_\mu = 1$, *then* $g_\mu \neq k_\mu a'^2_\mu$.

We also assume the existence of the following formal series solutions.

(i) *For all* $\mu \in \{1,\ldots,n\}$, *if* m_μ *is even then there is a neighborhood* Ω *of* z_0 *such that for all* $\hat{z} \in \Omega$ *there is a formal series solution of the form*

$$a_\mu(\hat{z}) + \sum_{j=0}^{\infty} \alpha_j (z-\hat{z})^{(j+2)/(m_\mu+1)}, \tag{2.13}$$

where $\alpha_0 \neq 0$.

(ii) *For all* $\mu \in \{1,\ldots,n\}$, *if* m_μ *is odd then there is a neighborhood* Ω *of* z_0 *such that for all* $\hat{z} \in \Omega$ *there are two formal series solutions of the form*

$$\begin{aligned} a_\mu(\hat{z}) + \sum_{j=0}^{\infty} \alpha_j (z-\hat{z})^{2(j+1)/(m_\mu+1)}, \\ a_\mu(\hat{z}) + \sum_{j=0}^{\infty} \beta_j (z-\hat{z})^{2(j+1)/(m_\mu+1)}, \end{aligned} \tag{2.14}$$

where $\alpha_0\beta_0 \neq 0$ *and* $\alpha_0^{(m_\mu+1)/2} = -\beta_0^{(m_\mu+1)/2}$.

(iii) *If m_0 is even then there is a neighborhood Ω of z_0 such that for all $\hat{z} \in \Omega$ there is a formal series solution of the form*

$$\sum_{j=0}^{\infty} \alpha_j (z - \hat{z})^{(j-2)/(m_0+1)}, \tag{2.15}$$

where $\alpha_0 \neq 0$.

(iv) *If m_0 is odd then there is a neighborhood Ω of z_0 such that for all $\hat{z} \in \Omega$ there are two formal series solutions of the form*

$$\sum_{j=0}^{\infty} \alpha_j (z - \hat{z})^{2(j-1)/(m_0+1)}, \quad \sum_{j=0}^{\infty} \beta_j (z - \hat{z})^{2(j-1)/(m_0+1)}, \tag{2.16}$$

where $\alpha_0 \beta_0 \neq 0$ and $\alpha_0^{(m_0+1)/2} = -\beta_0^{(m_0+1)/2}$.

Let Γ be a finite-length curve with endpoint $z_0 \in \mathbb{C}$. Suppose that y is analytic on $\Gamma \setminus \{z_0\}$, where it solves equation (1.1). If y cannot be analytically continued to $\Gamma \cup \{z_0\}$ then in a neighbourhood of $z = z_0$, $y(z)$ is the sum of a series of one of the forms (2.13–2.16) with $\hat{z} = z_0$.

In [11], Kecker proved the following, which is a generalisation of Theorem 2.4.

Theorem 2.6. *Consider equation (1.1) with E, F and G as described before Theorem 2.5, where k_μ, l_μ, and m_μ are integers for all $\mu \in \{1, \ldots, n\}$. Suppose that $deg_y f(z, y) > \sum_{\mu=1}^{n} l_\mu$ and that*

$$\sum_{\mu=1}^{n} (l_\mu - m_\mu) - 1 \leq deg_y f - deg_y g.$$

For all $\mu \in \{1, \ldots, n\}$ such that $a'_\mu \equiv 0$, we have $l_\mu > m_\mu \geq 0$. Otherwise we have $l_\mu = m_\mu > 0$ and $g_\mu + a'_\mu f_\mu \equiv 0$. If additionally $l_\mu = m_\mu = 1$ we require that

$$k_\mu a'_\mu(z_0) + f_\mu(z_0) \neq 0.$$

For all $\mu \in \{1, \ldots, n\}$, we assume the existence of a formal series *solution for all $\hat{z}$ in some neighbourhood of z_0 of the form*

$$y(z) = a_\mu(\hat{z}) + \sum_{j=1}^{\infty} c_j (z-\hat{z})^{k/l_\mu}, \qquad c_1 \neq 0. \tag{2.17}$$

Furthermore, we assume the existence of a formal series solution of the form

$$y(z) = \sum_{j=0}^{\infty} c_j (z-\hat{z})^{(j-1)/l_0}, \qquad c_0 \neq 0, \tag{2.18}$$

where $l_0 = deg_y f - \sum_{\mu=1}^{n} l_\mu$. Then in a neighbourhood of any singularity z_0 that can be reached by the analytic continuation of a solution y of equation (1.1)*, y has a convergent series expansion of the form (2.17) or (2.18) with $\hat{z} = z_0$.*

Finally, consider the system of equations in [12],

$$\begin{aligned} y_1' =&(N+1)\alpha_{0,N+1}(z)y_2^N + \sum_{(i,j)\in I} j\alpha_{ij}(z)y_1^i y_2^{j-1} \\ y_2' =&-(M+1)\alpha_{M+1,0}(z)y_1^M - \sum_{(i,j)\in I} i\alpha_{ij}(z)y_1^{i-1}y_2^j, \end{aligned} \tag{2.19}$$

where the set of indices I is defined by

$$I = \{(i,j) \in \mathbb{N}^2 : i(N+1) + j(M+1) < (N+1)(M+1)\},$$

and $\alpha_{ij}(z)$, $(i,j) \in I \cup \{(M+1,0),(0,N+1)\}$, are analytic functions in some common domain $\Omega \subset \mathbb{C}$. This is a Hamiltonian system with Hamiltonian

$$H(z,y_1,y_2) = \alpha_{M+1,0}(z)y_1^{M+1} + \alpha_{0,N+1}(z)y_2^{N+1} + \sum_{(i,j)\in I} \alpha_{ij}(z)y_1^i y_2^j.$$

We define the set

$$\Phi = \{z \in \Omega | \alpha_{M+1,0}(z) = 0\} \cup \{z \in \Omega | \alpha_{0,N+1}(z) = 0\}.$$

Theorem 2.7. *Suppose that at every point $\hat{z} \in \Omega\backslash\Phi$ the Hamiltonian system (2.19) admits formal series solutions of the form*

$$y_1(z) = \sum_{k=-N-1}^{\infty} c_{1,k}(z-\hat{z})^{\frac{k}{MN-1}}, \quad y_2(z) = \sum_{k=-M-1}^{\infty} c_{2,k}(z-\hat{z})^{\frac{k}{MN-1}},$$

about any point $\hat{z} \in \Omega \setminus \Phi$*, for every pair of values* $(c_{1,-N-1}, c_{2,-M-1})$ *satisfying*

$$c_{1,-N-1}^{MN-1} = -\left(\alpha_{0,N+1}(\hat{z})\alpha_{M+1,0}(\hat{z})^N (MN-1)^{N+1}\right)^{-1},$$
$$c_{2,-M-1} = (MN-1)\alpha_{M+1,0}(\hat{z})c_{1,-N-1}^M.$$

Let $\Gamma \subset \Omega$ *be a finite length curve with endpoint* $z_0 \in \Omega \setminus \Phi$ *such that a solution* (y_1, y_2) *can be analytically continued along* Γ *up to, but not including* z_0*. Then the solution can be represented by series expansions*

$$y_1(z) = \sum_{k=-\frac{N+1}{d}}^{\infty} C_{1,k}(z-z_0)^{\frac{kd}{MN-1}},$$
$$y_2(z) = \sum_{k=-\frac{M+1}{d}}^{\infty} C_{2,k}(z-z_0)^{\frac{kd}{MN-1}},$$

where $d = \gcd\{M+1, N+1\}$*, convergent in some branched, punctured, neighbourhood of* z_0*.*

3. DIFFERENTIAL EQUATIONS WITH ALGEBROID SOLUTIONS

Having discussed the singularity structure of certain classes of second-order differential equations with locally finitely branched singularities in the first part of this article, we now focus on the global singularity structure of these equations. In particlular we are interested in finding equations which allow for globally finitely branched solutions. This leads to the notion of algebroid functions, i.e. functions that are algebraic over the field of meromorphic functions.

3.1. Properties of algebroid functions

An n-valued algebroid function f satisfies an irreducible algebraic equation

$$f^n + s_1(z)f^{n-1} + \cdots + s_{n-1}(z)f + s_n(z) = 0, \tag{3.1}$$

where $s_1, \ldots, s_n$ are meromorphic functions. If all functions $s_1, \ldots, s_n$ are rational then f is called algebraic. If at least one of the functions $s_1, \ldots, s_n$ is non-rational then f is called transcendental algebroid.

Over every point $z_0 \in \mathbb{C}$ an algebroid function takes on at most $k \leq n$ values $a_1, \ldots, a_k \in \mathbb{C} \cup \{\infty\}$ and allows for series expansions

$$f(z) = a_i + \sum_{j=\tau_i}^{\infty} c_j (z - z_0)^{\frac{j}{\lambda_i}}$$

or, in case $a_i = \infty$,

$$f(z) = \sum_{j=-\tau_i}^{\infty} c_j (z - z_0)^{\frac{j}{\lambda_i}}. \tag{3.2}$$

Here it is assumed that the number λ_i in each series expansion has no common factor with all the indices j where $c_j \neq 0$. We then have $\lambda_1 + \cdots + \lambda_k = n$.

3.2. First-order equations

For first-order equations, Malmquist [13] proved the following theorem in 1913.

Theorem 3.1. *Let $P(z,y)$ and $Q(z,y)$ be polynomials in y with rational coefficients. Suppose the rational first-order differential equation*

$$y' = \frac{P(z,y)}{Q(z,y)}, \tag{3.3}$$

has a transcendental algebroid solution. Then it can be reduced, by a rational transformation $w = R(z,y)$, to a Riccati equation

$$w' = a(z)w^2 + b(z)w + c(z),$$

with $a(z)$, $b(z)$ and $c(z)$ rational.

Usually, Malmquist's theorem is quoted as the following: If equation (3.3) has a transcendental meromorphic solution, then it is a Riccati equation. But in fact, Malmquist proved the more general Theorem 3.1. Malmquist's original proof involved asymptotic methods. Yosida [28] gave a much shorter proof of Malmquist theorem using Nevanlinna Theory, but only for the case of a meromorphic solution. Nevanlinna Theory also allows one to generalise Theorem 3.1 to the notion of admissible solutions as explained below.

3.3. Tools from Nevanlinna Theory

We denote the Nevanlinna functions by their usual symbols, the integrated counting function $N(r,f)$, the proximity function $m(r,f)$ and the Nevanlinna characteristic $T(r,f)$. Nevanlinna Theory of meromorphic functions was generalised by Selberg [19] and Ullrich [26] to algebroid functions. Most of the notation and some standard theorems carry over to the algebroid case with some modifications. Let f be an n-valued algebroid function. We denote $n(r,f) = \sum_{|z_0|\le r} \tau$, where the sum is over the numbers τ of all points z_0 where f has an expansion of the form (3.2). The algebroid Nevanlinna functions are then defined as follows:

$$\begin{aligned} N(r,f) &= \frac{1}{n}\int_0^r \frac{n(t,f)-n(0,f)}{t}\mathrm{d}t + \frac{1}{n}n(0,f)\ln(r), \\ m(r,f) &= \frac{1}{2\pi n}\sum_{\nu=1}^{n}\int_0^{2\pi} \ln^+ \left|f_\nu\left(r\mathrm{e}^{i\phi}\right)\right|\mathrm{d}\phi, \\ T(r,f) &= m(r,f) + N(r,f), \end{aligned}$$

where $f_1,\dots,f_n$ are the n branches of f. In the single-valued (meromorphic) case these functions reduce to the usual Nevanlinna functions. However one needs to be slightly careful when applying the Nevanlinna functions to a composition of algebroid functions, e.g. sums and products. In [14, 15] Mokhon'ko proves some theorems for the application of the Nevanlinna functions to rational expressions of algebroid functions which are analogous to the meromorphic case, e.g.

$$m\left(r,\sum_{i=1}^n f_i\right) \le \sum_{i=1}^n m(r,f_i) + O(1), \quad m\left(r,\prod_{i=1}^n f_i\right) \le \sum_{i=1}^n m(r,f_i).$$

Many results from the meromorphic case have an analogue in the algebroid case. We only quote here the important Lemma on the Logarithmic Derivative [27]:

Lemma 3.2. *Suppose f is a transcendental algebroid function. Then we have*

$$m\left(r,\frac{f'}{f}\right) = o(\log T(r,f)),$$

as $r\to\infty$, possibly outside an exceptional set of finite linear measure.

Any function that behaves like $o(T(r,f))$ as $r \to \infty$, possibly outside an exceptional set of finite measure, is denoted by $S(r,f)$. In particular, Lemma 3.2 states that $m\left(r, f'/f\right) = S(r,f)$. As we seek to apply Nevanlinna theory to the solutions of differential equations we give the following definition. Suppose f satisfies an algebraic differential equation

$$F(z, f, f', \ldots, f^{(k)}) = 0, \tag{3.4}$$

where F is polynomial in f and its derivatives with meromorphic coefficients $\{a_\lambda, \lambda \in I\}$. An algebroid solution of (3.4), satisfying the algebraic equation (3.1), is called *admissible* if, for some $j \in \{1, \ldots, n\}$, $T(r, a_\lambda) = S(r, s_j)\ \forall\ \lambda \in I$. For the notion of admissible solutions, Malmquist's Theorem 3.1 will generalise to the following form: Let $P(z,y)$ and $Q(z,y)$ be polynomial in y with meromorphic coefficients and suppose that the first-order equation (3.3) has an admissible algebroid solution. Then it can be reduced, by a rational transformation $w = R(z,y)$, to a Riccati equation in w.

3.4. Algebroid solutions of second-order equations

We now consider equations in the class

$$y'' = \frac{2(N+1)}{(N-1)^2} y^N + \sum_{k=0}^{N-1} a_k(z) y^k, \tag{3.5}$$

the normalisation factor being chosen for convenience. Suppose that (3.5) has an admissible algebroid solution y. Then, rearranging (3.5) and using Lemma 3.2, one obtains

$$\begin{aligned}
Nm(r,y) &= m(r, y^N) \\
&= m(r, y'' - a_{N-1}y^{N-1} - \cdots - a_1 y - a_0) + O(1) \\
&\leq m(r,y) + m\left(r, \frac{y''}{y} - a_{N-1}y^{N-2} - \cdots - a_1\right) \\
&\qquad + m(r, a_0) + O(1) \\
&\leq 2m(r,y) + m(r,a_0) + m(r,a_1) \\
&\qquad + m(r, a_{N-1}y^{N-3} - \cdots - a_2) + S(r,y) \\
&\leq \cdots \leq (N-1)m(r,y) + \sum_{j=0}^{N-1} m(r, a_j) + S(r,y),
\end{aligned}$$

and therefore, since y is assumed to be admissible,

$$m(r,y) = S(r,y). \tag{3.6}$$

This shows that $N(r,y) = T(r,y) + S(r,y)$. In particular, this means that at least one of the functions $s_1, \ldots, s_n$ has a number of poles of order $O(T(r,y))$.

3.5. Simplest case: 2-valued algebroid solutions

We now consider an equation that has solutions with branched singularities of the form

$$y(z) = \sum_{k=-1}^{\infty} c_k (z - z_0)^{\frac{k}{2}}.$$

A candidate of such an equation in the class (3.5) is equation (1.2) of Theorem 1.2,

$$y'' = \frac{3}{4} y^5 + a_4(z) y^4 + a_3(z) y^3 + a_2(z) y^2 + a_1(z) y + a_0(z). \tag{3.7}$$

If we are seeking globally 2-valued algebroid solutions, y also satisfies a quadratic equation (1.3) where s_1 and s_2 are meromorphic functions. They are also the elementary symmetric functions of the two branches y_1, y_2 of y, i.e.

$$s_1 = -(y_1 + y_2), \quad s_2 = y_1 y_2.$$

It follows from (3.6) that also $m(r, s_1) = S(r,y)$ and $m(r, s_2) = S(r,y)$.

At any singularity z_0 of y, where $a_k(z)$, $k \in \{0, \ldots, 4\}$ are analytic, we have $y_1, y_2 \sim (z - z_0)^{-1/2}$. Therefore, since s_1 is single-valued, it has no pole at these points z_0 and hence we have $T(r, s_1) = S(r,y)$. On the other hand, since y is an admissible solution, s_2 must have a number of poles of order $T(r,y)$. Differentiating (1.3) once yields

$$2yy' + s_1' y + s_1 y' + s_2' = 0 \quad \Longrightarrow \quad y' = -\frac{s_1' y + s_2'}{2y + s_1}. \tag{3.8}$$

We differentiate again and insert y' from (3.8) and y'' from (3.7). Multiplying by the common denominator $(2y + s_1)^2$ one obtains an equation polynomial in y, s_1 and s_2 and their first and second derivatives. One can use (1.3) repeatedly to reduce the order in y, and in a finite number of steps one obtains an equation

$$F_1(s_1, s_1', s_1'', s_2, s_2', s_2'')y + F_0(s_1, s_1', s_1'', s_2, s_2', s_2'') = 0.$$

Since (1.3) was assumed to be irreducible, y does not satisfy a linear equation of this kind, i.e. we have in fact shown that $F_1 \equiv F_0 \equiv 0$. For F_1 we have

$$0 = F_1 = \left(4s_2 - s_1^2\right)\Big[s_1'' - s_1^5 + a_4 s_1^4 - a_3 s_1^3 + a_2 s_1^2 - a_1 s_1 + 2a_0 \\ + s_2(2a_2 + 3a_3 s_1 - 4a_4 s_1^2 + 5s_1^3) + s_2^2(2a_4 - 5s_1)\Big],$$

and, since $4s_2 - s_1^2$ is the discriminant of the irreducible quadratic equation (1.3), the expression in the brackets must vanish identically, which yields an equation of the form

$$s_1'' + p(s_1) = s_2 q(s_1) + s_2^2(2a_4 - 5s_1),$$

where p and q are polynomial in s_1. However, the left hand side of this equation is of order $S(r, y)$ whereas the right hand side involves s_2. This is only possible if both sides vanish identically, giving the conditions

$$s_1 = \frac{2}{5}a_4, \quad q(s_1) = 0, \quad s_1'' + p(s_1) = 0. \tag{3.9}$$

By a linear transformation in y we could have set $a_4 = 0$ (and therefore $s_1 = 0$) from the start, which we will assume to be done in the following. The other conditions in (3.9) then become $a_2 = 0$ and $a_0 = 0$. The equation $F_0 = 0$ now yields an equation satisfied by s_2:

$$s_2'' = \frac{(s_2')^2}{2s_2} + \frac{3}{2}s_2^3 - 2a_3(z)s_2^2 + 2a_1(z)s_2. \tag{3.10}$$

We will now examine this equation further which must have an admissible meromorphic solution. At any pole z_0 of s_2, where $a_3(z)$ and $a_1(z)$ are analytic,

$$s_2 \sim \alpha(z - z_0)^p, \quad p \in \mathbb{Z},$$

one easily finds that $p = -1$ and $\alpha = \pm 1$. Inserting the full Laurent series

$$\frac{\alpha}{z-z_0} + \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

into (3.10) one can determine the coefficients c_k, $k = 0, 1, 2, \ldots$ recursively and finds the expansion

$$\frac{\alpha}{z-z_0} + \frac{1}{2}a_3(z_0) + \left(\frac{\alpha}{4}a_3(z_0)^2 + \frac{2}{3}a_3'(z_0) - \frac{2\alpha}{3}a_1(z_0)\right)(z-z_0) \\ + h(z-z_0)^2 + \cdots, \tag{3.11}$$

where the coefficient h cannot be determined by the recursion, which breaks down for $k = 2$. Instead one finds the resonance condition

$$\alpha a_3''(z_0) + a_3(z_0)a_3'(z_0) - 2a_1'(z_0) = 0. \tag{3.12}$$

From equation (3.10) one obtains, using Lemma 3.2,

$$\begin{aligned} 2m(r, s_2) &= m(r, s_2^2) \\ &\leq m\left(r, \frac{s_2''}{s_2}\right) + 2m\left(r, \frac{s_2'}{s_2}\right) + m(r, s_2) + m(r, 2a_3) \\ &\qquad + m(r, 2a_1) + O(1), \\ \implies \quad m(r, s_2) &= S(r, s_2). \end{aligned}$$

It follows that we must have $N(r, s_2) = O(T(r, s_2))$. However, it is not certain whether both cases of the leading order behaviour $\alpha = \pm 1$ occur with frequency of order $T(r, s_2)$. We denote the integrated counting function of the number of poles of s_2 with leading order behaviour $\alpha/(z - z_0)$ by $N_\alpha(r, s_2)$. Essentially we consider two different cases. First suppose that both leading order behaviours at the poles of s_2 occur with frequency of order $N_{\pm 1}(r, s_2) = O(T(r, s_2))$. We then consider the functions

$$\alpha a_3''(z) + a_3(z)a_3'(z) - 2a_1'(z), \quad \alpha = \pm 1.$$

By (3.12) each of these functions has zeros with frequency of order $T(r, s_2)$. But therefore, since s_2 is admissible, they must both vanish identically and one obtains the two conditions

$$a_3'' \equiv 0, \quad (a_3^2 - 4a_1)' \equiv 0,$$

and letting $a_3(z) = -2(az + b)$ and $a_1(z) = (az + b)^2 - c$, equation

(3.10) becomes equation (1.4). In case of $a \neq 0$, equation (3.10) reduces, by a linear transformation in z, to the equation

$$s_2'' = \frac{(s_2')^2}{2s_2} + \frac{3}{2}s_2^3 + 4zs_2^2 + 2(z^2 - c)s_2,$$

which is a special case of the fourth Painlevé equation for which it is known that all solutions are meromorphic functions in the complex plane, see e.g. [25] or the book [4]. Otherwise, in case of $a = 0$, equation (3.10) reduces to

$$s_2'' = \frac{(s_2')^2}{2s_2} + \frac{3}{2}s_2^3 + 4bs_2^2 + 2(b^2 - c)s_2,$$

which can be solved in terms of elliptic functions.

For the second case suppose that $N_\alpha(r, s_2) = O(T(r, s_2))$, however $N_{-\alpha}(r, s_2) = S(r, s_2)$. We will show that in this case s_2 is an admissible solution of a Riccati equation

$$s_2' = -\alpha s_2^2 + u(z)s_2 + v(z). \tag{3.13}$$

Differentiating (3.13) and equating with the right hand side of (3.10) yields the following conditions by comparing coefficients of powers of s_2:

$$u = \alpha a_3, \quad 2\alpha v = 2\alpha a_3' + a_3^2 - 4a_1 \equiv 0.$$

Suppose now that s_2 does not satisfy any Riccati equation admissibly. Then define the function

$$w = s_2' + \alpha s_2^2 - \alpha a_3 s_2, \tag{3.14}$$

which has proximity function $m(r, w) = S(r, s_2)$. At any pole z_0 of s_2 with leading order $\alpha/(z - z_0)$, by employing the expansion (3.11), w is regular. Therefore w can have poles only where s_2 has a pole with leading order $-\alpha/(z - z_0)$, i.e. we also have $N(r, w) = S(r, s_2)$. But that means that $T(r, w) = S(r, s_2)$, therefore (3.14) is a Riccati equation for which s_2 is an admissible solution in contradiction to the assumption. We have therefore proved Theorem 1.2.

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