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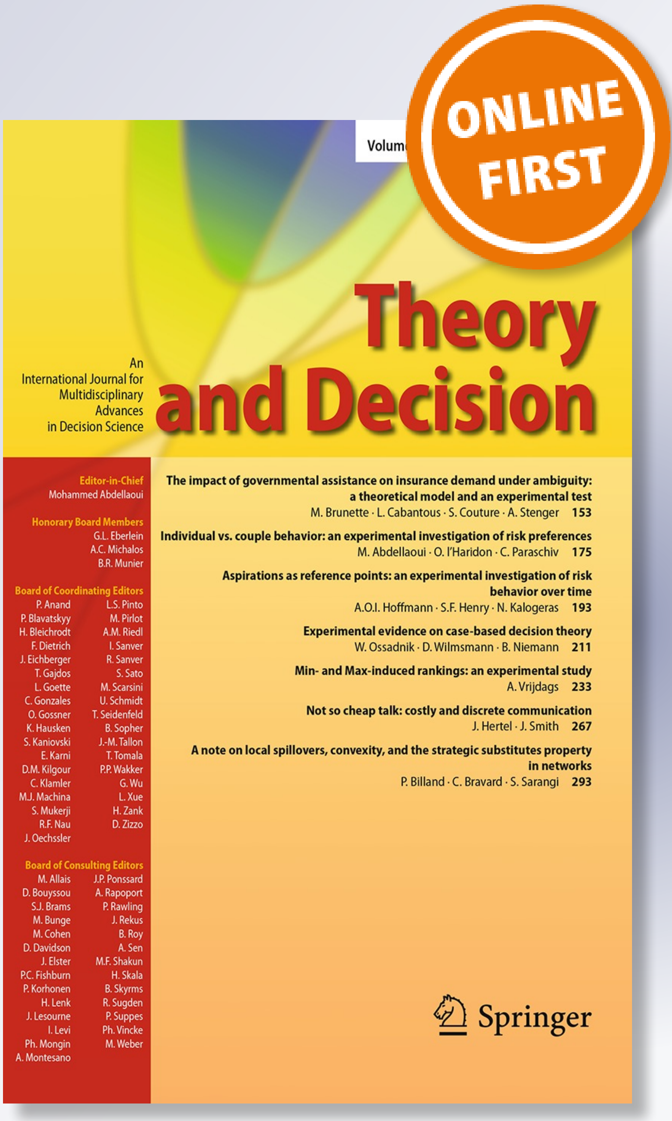

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The bipolar Choquet integral representation

Salvatore Greco · **Fabio Rindone**

Abstract Cumulative Prospect Theory is the modern version of Prospect Theory and it is nowadays considered a valid alternative to the classical Expected Utility Theory. Cumulative Prospect theory implies *Gain-Loss Separability*, i.e., the separate evaluation of losses and gains within a mixed gamble. Recently, some authors have questioned this assumption of the theory, proposing new paradoxes where the Gain-Loss Separability is violated. We present a generalization of Cumulative Prospect Theory which does not imply Gain-Loss Separability and is able to explain the cited paradoxes. On the other hand, the new model, which we call the *bipolar Cumulative Prospect Theory*, genuinely generalizes the original Prospect Theory of Kahneman and Tversky, preserving the main features of the theory. We present also a characterization of the *bipolar Choquet Integral* with respect to a *bi-capacity* in a discrete setting.

Keywords Cumulative Prospect Theory · Gains-loss separability · bi-Weighting function · Bipolar Choquet integral

JEL Classification D81 · C60

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1 Introduction

Cumulative Prospect Theory (CPT) (Tversky and Kahneman 1992) is the modern version of Prospect Theory (PT) (Kahneman and Tversky 1979) and it is nowadays considered a valid alternative to the classical Expected Utility Theory (EUT) of Von Neumann and Morgenstern (1944). CPT has generalized EUT, preserving the descriptive power of the original PT and capturing the fundamental idea of Rank Dependent Utility (RDU) (Quiggin 1982) and of Choquet Expected Utility (CEU) (Schmeidler 1986, 1989; Gilboa 1987). In recent years CPT has obtained increasing space in applications in several fields: in business, finance, law, medicine, and political science (e.g., Benartzi and Thaler 1995; Barberis et al. 2001; Camerer 2000; Jolls et al. 1998; McNeil et al. 1982; Quattrone and Tversky 1988). Despite the increasing interest in CPT—in the theory and in the practice—some critiques have been recently proposed: Levy and Levy (2002), Blavatskyy (2005), Birnbaum (2005), Baltussen et al. (2006), Birnbaum and Bahra (2007), Wu and Markle (2008), Schade et al. (2010). In our opinion, the most relevant of these critique concerns the *Gain-Loss Separability* (GLS), i.e., the separate evaluation of losses and gains. More precisely, let $P = (x_1, p_1; \dots; x_n, p_n)$ be a *prospect* giving the outcome $x_i \in \mathbb{R}$ with probability p_i, $i = 1, \dots, n$ and let $P^+(P^-)$ be the prospect obtained from P by substituting all the losses (gains) with zero. GLS means that the evaluation of P is obtained as sum of the value of P^+ and P^-, i.e., $V(P) = V(P^+) + V(P^-)$. Wu and Markle (2008) refer to the following experiment: 81 participants gave their preferences as it is shown below (read $H \succ L$ "the prospect H is preferred to the prospect L".)

$$H = \begin{pmatrix} 0.50\ chance \\ at\ \$4,200 \\ 0.50\ chance \\ at\ \$-3,000 \end{pmatrix} \succ \begin{pmatrix} 0.75\ chance \\ at\ \$3,000 \\ 0.25\ chance \\ at\ \$-4,500 \end{pmatrix} = L$$

[52%] [48%]

$$H^+ = \begin{pmatrix} 0.50\ chance \\ at\ \$4,200 \\ 0.50\ chance \\ at\ \$0 \end{pmatrix} \prec \begin{pmatrix} 0.75\ chance \\ at\ \$3,000 \\ 0.25\ chance \\ at\ \$0 \end{pmatrix} = L^+$$

[15%] [85%]

$$H^- = \begin{pmatrix} 0.50\ chance \\ at\ \$0 \\ 0.50\ chance \\ at\ \$-3,000 \end{pmatrix} \prec \begin{pmatrix} 0.75\ chance \\ at\ \$0 \\ 0.25\ chance \\ at\ \$-4,500 \end{pmatrix} = L^-$$

[37%] [63%]

As can be seen, the majority of participants preferred H to L, but, when the two prospects were split in their respective positive and negative parts, a relevant majority prefers L^+ to H^+ and L^- to H^-. Thus, GLS is violated and CPT cannot explain such a pattern of choice. In the sequel, we will refer to this experiment as the "Wu-Markle paradox."

In the CPT model the GLS implies the separation of the domain of the gains from that of the losses, with respect to a subjective *reference point*. This separation, technically, depends on a characteristic *S-shaped utility function*, steeper for losses than for gains, and on two different *weighting functions*, which distort, in different way, probabilities relative to gains and losses. We aim to generalize CPT, maintaining the S-shaped utility function, but replacing the two weighting functions with a *bi-weighting function*. This is a function with two arguments, the first corresponding to the probability of a gain and the second corresponding to the probability of a loss of the same magnitude. We call this model the bipolar Cumulative Prospect Theory (bCPT). The bCPT will allow gains and losses within a *mixed prospect* to be evaluated conjointly. Let us explain our motivation. The basic one, stems from the data in Wu and Markle (2008) and in Birnbaum and Bahra (2007). Both papers, following a rigorous statistical procedure, reported systematic violations of GLS. Moreover, if we look through the Wu-Markle paradox shown above, we understand that the involved probabilities are very clear, since they are the three quartiles 25, 50, and 75%. Similarly, the involved outcomes have the "right" size: neither so small to give rise to indifference nor so great to generate unrealism. Now suppose to look at the experiment in the other sense, from non mixed prospects to mixed ones. The two preferences $L^+ \succ H^+$ and $L^- \succ H^-$, under the hypothesis of GLS, should suggest that L should be strongly preferred to H. Surprisingly enough, $H \succ L$. What happened? Clearly, the two preferences $L^+ \succ H^+$ and $L^- \succ H^-$ did not interact positively and, on the contrary, the trade-off between H^+, H^- and L^+, L^- was in favor of H. These data, systematically replicated, seem to suggest that a sort of Gain Loss Hedging (GLH) appears in the passage from prospects involving only gains or losses to mixed ones. When the GLH phenomenon is intense enough to reverse the preferences, i.e. ($L^+ \succ H^+$ and $L^- \succ H^-$) and also $H \succ L$, then GLS is violated. Thus, the first motivation of the paper is to show how bCPT is able to capture, at least partially, these erroneous predictions of CPT. A second motivation for proposing bCPT, stems from the consideration that, in evaluating mixed prospects, it seems very natural to consider a trade-off between possible gains and losses. This, corresponds to assume that people are more willing to accept the risk of a loss having the hope of a win and, on the converse, are more careful with respect to a possible gain having the risk of a loss. Psychologically, the evaluation of a possible loss could be mitigated if this risk comes together with a possible gain. For example, the evaluation of the loss of \$3,000 with a probability 0.5 in the prospect $H = (0\ , 0.5; \$-3,000,\ 0.5)$ could be different from the evaluation of the same loss within the prospect $L = (\$4,200\ , 0.5;\ \$-3,000\ , 0.5)$, where the presence of the possible gain of \$4, 200 could have a mitigation role. Why should be the overall evaluation of a prospects only be the sum of its positive and negative part? The last motivation has historical roots and involves the revolution given to the development of PT. Since when the theory has been developed (Kahneman and Tversky 1979), a basic problem has been to distinguish gains from losses. However, in the evolution

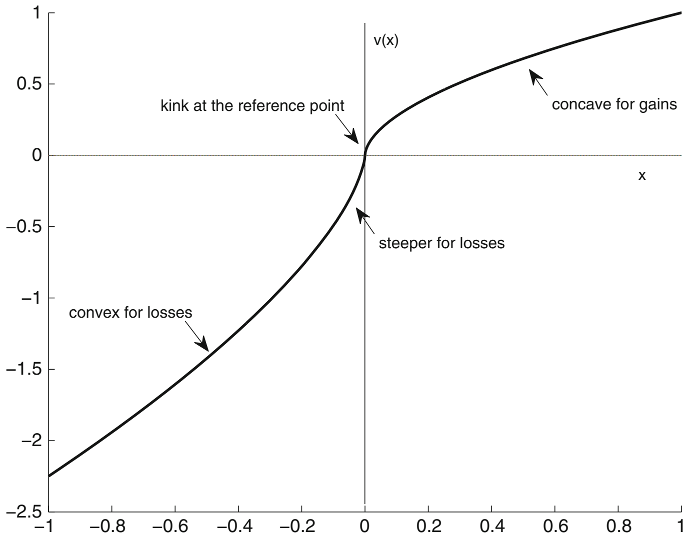

Fig. 1 CPT utility function

of decisions under risk and uncertainty, the majority of data, regarded non-mixed prospects (see, e.g., Allais 1953; Ellsberg 1961; Kahneman and Tversky 1979). Many authors pointed out that the mixed case is still a little understood domain (Luce 1999, 2000; Birnbaum and Bahra 2007; Wu and Markle 2008).

This paper is organized as follows. In Sect. 2 we describe the bCPT, starting from the CPT. In Sect. 3 we present several bi-weighting functions, generalizing well know weighting functions of CPT. Section 4 is devoted to the relationship between CPT and bCPT. In Sect. 5 we extend bCPT to uncertainty. Our main result, the characterization of the bipolar Choquet integral, is developed in Sect. 6. We conclude in Sect. 7. Some proofs, depending on the importance, are presented in the main text, while the remaining proofs are presented in Appendix. The Appendix also contains tests of bCPT on the previous data reported in the literature about the GLS violation.

2 From CPT to bCPT

2.1 Two different approaches

The most important idea in CPT is the concept of gain-loss asymmetry: people perceive possible outcomes as either gains or losses with respect to a *reference point*, rather than as absolute wealth levels. The characteristic S-shaped utility function[1] is null at the reference point, concave for gains and convex for losses, steeper for losses than for gains (see Fig. 1).

[1] Which the authors called *value function*

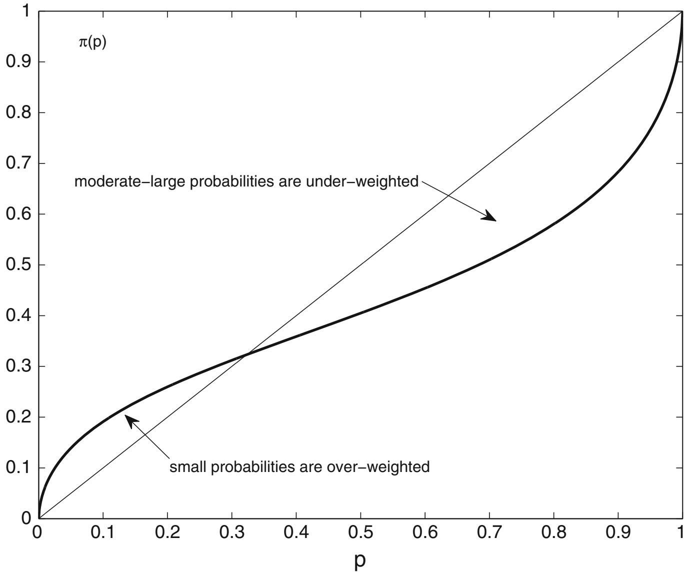

Fig. 2 CPT weighting function

The other important idea in CPT is the notion of probability distortion: people overweight very small probabilities and underweight average and large ones. This probability transformation is mathematically described by means of a *weighting function*, that is a strictly increasing function $\pi : [0; 1] \to [0; 1]$ satisfying the conditions $\pi(0) = 0$, $\pi(1) = 1$. The typical inverse S-shape weighting function graph is shown in Fig. 2.

If in CPT two different weighting functions have the role to transform the probabilities attached to gains and losses, in our model, we have a two variables bi-weighting function. This has, in the first argument, the probability of a gain with a utility greater or equal than a given level l and in the second argument the probability of a symmetric loss, which utility is not smaller than $-l$. The final result is a number within the closed interval $[-1; 1]$. Formally, let us set

$$A = \{(p, q) \in [0; 1]X[0; 1] \text{ such that } p + q \leq 1\},$$

that is, in the $p - q$ plane, the triangle which vertices are $O \equiv (0, 0)$, $P \equiv (1, 0)$ and $Q \equiv (0, 1)$.

Definition 1 We define a bi-weighting function as any function

$$\omega(p, q) : A \to [-1; 1]$$

satisfying the following conditions:

– $\omega(p, q)$ is increasing in p and decreasing in q (bi-monotonicity)
– $\omega(1, 0) = 1$, $\omega(0, 1) = -1$ and $\omega(0, 0) = 0$.

Let us note that if π_- and π_+ are two weighting functions, then a *separable bi-weighting function* is defined by means of $\omega(p,q) = \pi_+(p) - \pi_-(q)$, for all $(p,q) \in A$. Let $P = (x_1, p_1; \dots; x_n, p_n)$ be a lottery assigning the outcome $x_j \in \mathbb{R}$ with probability p_j, a utility function $u(\cdot) : \mathbb{R} \to \mathbb{R}$, two weighting functions π_-, π_+ and a bi-weighting function ω. Using an integral representation we can represent CPT and bCPT, respectively, as

$$V_{CPT}(P) = \int_0^{+\infty} \pi_+\left(\sum_{i:u(x_i)\geq t} p_i\right) dt - \int_0^{+\infty} \pi_-\left(\sum_{i:u(x_i)\leq -t} p_i\right) dt. \tag{1}$$

$$V_{bCPT}(P) = \int_0^{+\infty} \omega\left(\sum_{i:u(x_i)\geq t} p_i, \sum_{i:u(x_i)\leq -t} p_i\right) dt. \tag{2}$$

In our opinion, both formulations (1), (2) genuinely generalize the original PT of Kahneman and Tversky, preserving the main features of the theory. The only difference is that, in (1) we get a separate evaluation of gains and losses, whereas in (2) we get a conjoint evaluation. As we will soon see, the two formulas coincide in a non-mixed context, i.e., when the outcomes involved in the choice process are only gains or only loss. However, in the mixed case the two formulas can differ.

3 The bi-weighting function

In this section we propose some generalizations of well known weighting functions. They coincide with the original gain weighting function, π_+, if $q = 0$, and with the opposite loss weighting function, $-\pi_-$, if $p = 0$.

3.1 The Kahneman–Tversky bi-weighting function

The first and most famous weighting function was proposed in Tversky and Kahneman (1992):

$$\pi(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{\frac{1}{\gamma}}}.$$

The parameter γ can be chosen differently for gains and losses and the authors estimated $\gamma = 0.61$ for gains and $\gamma = 0.69$ for losses. For this weighting function we propose the following bipolar form

$$\omega(p,q) = \frac{p^\gamma - q^\delta}{[p^\gamma + (1-p)^\gamma]^{\frac{1}{\gamma}} + \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1}. \tag{3}$$

As the original KT weighting function is non-monotonic for γ sufficiently close to zero—see Rieger and Wang (2006), Ingersoll (2008)—so it is the case of (3) when γ and δ are sufficiently close to zero. The Next proposition 1 establishes the parameter limitations preserving the bi-monotonicity of (3).

Proposition 1 *The Kahneman, Tversky bi-weighting function with parameters* $1/2 < \gamma, \delta < 1$, *is increasing in* p *and decreasing in* q.

3.2 The Latimore, Baker and Witte bi-weighting function

In Lattimore et al. (1992), Goldstein and Einhorn (1987) was introduced the following weighting function (with $\gamma, \alpha > 0$):

$$\pi(p) = \frac{\alpha p^{\gamma}}{\alpha p^{\gamma} + (1-p)^{\gamma}}, \tag{4}$$

which is known as *linear in log odd form* (Gonzalez and Wu 1999). We propose the following bipolar form of the (4):

$$\omega(p, q) = \frac{\alpha(p^{\gamma} - q^{\delta})}{\alpha p^{\gamma} + (1-p)^{\gamma} + \alpha q^{\delta} + (1-q)^{\delta} - 1}. \tag{5}$$

The next Proposition 2 establishes the parameter limitations allowing for the bi-monotonicity of (5). These limitations include many of previous parameter estimations given for the (4) (see Table 1, from Bleichrodt and Pinto (2000)).

Proposition 2 *The Latimore, Baker and Witte bi-weighting function with* $\alpha > 1/2$ *and* $0 < \gamma, \delta \leq 1$, *is increasing in p and decreasing in q.*

3.3 The Prelec bi-weighting function

One of the most famous alternatives to the classical weighting function of Kahneman and Tversky is the Prelec weighting function (Prelec 1998):

$$\pi(p) = e^{-\beta(-\ln p)^{\alpha}}, \tag{6}$$

Table 1 Recent estimations of parameters for the (4)

Authors	α	γ
Tversky and Fox (1995)	0.77	0.79
Wu and Gonzalez (1996)	0.84	0.68
Gonzalez and Wu (1999)	0.77	0.44
Abdellaoui (2000) (gains)	0.65	0.60
Abdellaoui (2000) (losses)	0.84	0.65
Bleichrodt and Pinto (2000)	0.816	0.550

where $\beta \approx 1$ is variable for gains and for losses and $0 < \alpha < 1$. The Prelec weighting function is undefined for $p = 0$ but it is extended by continuity to the value of zero. We propose the following bi-weighting form of the (6):

$$\omega(p,q) = \begin{cases} \frac{p^\gamma - q^\delta}{|p^\gamma - q^\delta|} e^{-\beta(-\ln|p^\gamma - q^\delta|)^\alpha} & \forall (p,q) \in A \mid p^\gamma - q^\delta \neq 0 \\ 0 & \forall (p,q) \in A \mid p^\gamma - q^\delta = 0 \end{cases}. \tag{7}$$

The (7) is extended by continuity when $p^\gamma - q^\delta = 0$. Note that $|p^\gamma - q^\delta| \in [0, 1]$ and, consequently, the logarithm is non-positive. The next Proposition 3 establishes the parameters limitations allowing for the bi-monotonicity of (7). Without loss of generality, we choose $\beta = 1$.

Proposition 3 *The Prelec bi-weighting function with $\beta = 1$, $\gamma, \delta > 0$ and $0 < \alpha < 1$ is increasing in p and decreasing in q.*

3.4 The inverse S-shape of the bi-weighting function

A typical feature of the weighting function described in Tversky and Kahneman (1992) is the inverse S-shape in the plane. Let us consider and plot the bi-polarized form of the KT weighting function, preserving the original parameters estimation $\gamma = .61$ and $\delta = .69$

$$\omega(p,q) = \frac{p^{0.61} - q^{0.69}}{\left[p^{0.61} + (1-p)^{0.61}\right]^{\frac{1}{0.61}} + \left[q^{0.69} + (1-q)^{0.69}\right]^{\frac{1}{0.69}} - 1} \tag{8}$$

The typical inverse S-Shape is generalized from the plane to the space (see Fig. 3). Clearly, we are interested to the part of this plot such that $p + q \leq 1$.

3.5 Stochastic dominance and bCPT

The bi-monotonicity of the bi-weighting function, ensures the bCPT model satisfies the Stochastic Dominance Principle. This means that, if prospect P stochastically dominates prospect Q, then $V_{bCPT}(P) \geq V_{bCPT}(Q)$. The following theorem establishes this result.

Theorem 1 *Let us suppose that prospects are evaluated with the bipolar CPT, then Stochastic Dominance Principle is satisfied.*

Proof Let us consider $P = (x_1, p_1; \ldots; x_n, p_n)$ and $Q = (y_1, q_1; \ldots; y_m, q_m)$ such that P stochastically dominates Q. This means that for all $t \in \mathbb{R}$

$$\sum_{i:x_i \geq t} p_i \geq \sum_{i:y_i \geq t} q_i \quad \text{or equivalently} \quad \sum_{i:x_i \leq t} p_i \leq \sum_{i:y_i \leq t} q_i. \tag{9}$$

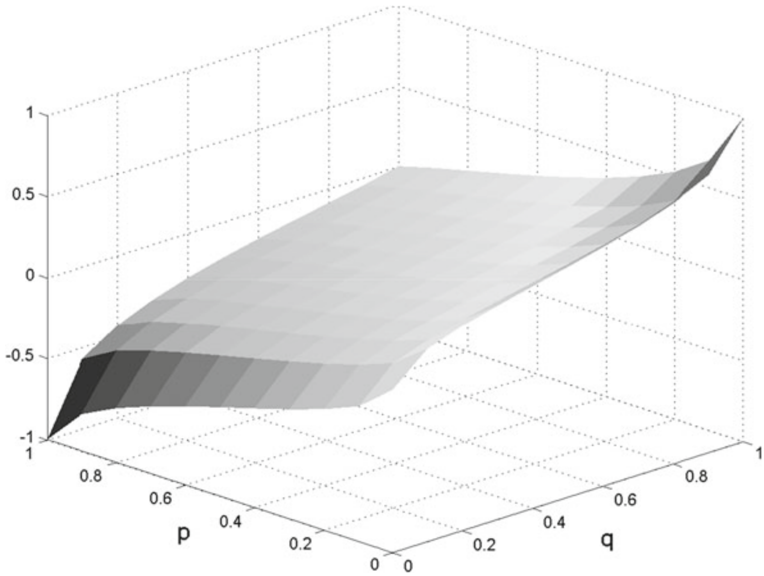

Fig. 3 bi-CPT weighting function

By the stochastic dominance of P over Q, we have that for all $t \in \mathbb{R}^+$

$$\sum_{i:u(x_i)\geq t} p_i \geq \sum_{i:u(y_i)\geq t} q_i \quad \text{and} \quad \sum_{i:u(x_i)\leq -t} p_i \leq \sum_{i:u(y_i)\leq -t} q_i. \tag{10}$$

From (10), considering the bi-monotonicity of $\omega(\cdot,\cdot)$, we have that for all $t \in \mathbb{R}^+$

$$\omega\left(\sum_{i:u(x_i)\geq t} p_i, \sum_{i:u(x_i)\leq -t} p_i\right) \geq \omega\left(\sum_{i:u(y_i)\geq t} q_i, \sum_{i:u(y_i)\leq -t} q_i.\right) \tag{11}$$

We conclude that $V_{bCPT}(P) \leq V_{bCPT}(Q)$ by monotonicity of the integral.

On the other hand, in the absence of the bi-monotonicity of the bi-weighting function, we are able to build preferences violating the Stochastic Dominance Principle. Indeed, let us suppose the bi-weighting function $\omega(\cdot,\cdot)$ is not [increasing in p and decreasing in q], i.e., that there exist $(p,q), (\widetilde{p},\widetilde{q}) \in [0,1]^2$ such that

$$\begin{cases} p \geq \widetilde{p} \\ q \leq \widetilde{q} \\ (p-\widetilde{p})^2 + (q-\widetilde{q})^2 > 0 \\ \omega(p,q) < \omega(\widetilde{p},\widetilde{q}) \end{cases}$$

Let us consider $x > 0$ and $y < 0$ such that $u(x) = -u(y)$ and the two lotteries $R = (x, p; y, q)$ and $S = (x, \widetilde{p}; y, \widetilde{q})$. Even if R stochastically dominates S, it would results

$$V_{bCPT}(R) = \omega(p, q) \cdot u(x) < \omega(\widetilde{p}, \widetilde{q}) \cdot u(x) = V_{bCPT}(S).$$

4 The relationship between CPT and bCPT

The relation between CPT and bCPT can be formalized in two basic propositions. The proofs are quite simple; however, the importance in the paper of these propositions is great.

Proposition 4 *For non-mixed prospects (containing only gains or losses), the bCPT model coincides with the CPT model.*

Proof From a bi-weighting function, ω, we can define the two weighting function $\pi_+(p) = \omega(p, 0)$ and $\pi_-(q) = -\omega(0, q)$ for all $p,\ q \in [0, 1]$. Vice versa, starting from two weighting functions, π_+ and π_- we can define the separable bi-weighting function $\omega(p, q) = \pi_+(p) - \pi_-(q)$ for all $(p, q) \in A$. Thus, for the same utility function u and a prospect $P = (x_1, p_1; \ldots; x_n, p_n)$ assigning non-negative (non-positive) outcome $x_j \in \mathbb{R}^+$ with probability p_j, $j = 1, \ldots, n$, it is easily checked that the value $V_{bCPT}(P)$ computed with respect to ω coincides with the value $V_{CPT}(P)$ computed with respect to π_+ (π_-).

Proposition 4 states that CPT and bCPT are the same model for non-mixed prospects. This fact is, for us, of great importance, since CPT has been widely tested in situations involving only gains or only losses, as remembered for instance in Wu and Markle (2008): *"In the last 50 years, a large body of empirical research has investigated how decision makers choose among risky gambles. Most of these findings can be accommodated by prospect theory... However, the majority of the existing empirical evidence has involved single-domain gambles."*

Proposition 5 *If the prospects are evaluated with the bCPT model with a separable bi-weighting function, then the representation coincides with that obtained with the CPT model. On the converse, if the prospects are evaluated with the CPT model, than the representation coincides with that obtained with the bCPT model with a separable bi-weighting function.*

Although the proof of Proposition 5 is trivial, the content is crucial. Indeed Proposition 5 establishes that CPT can be considered a special case of bCPT, provided that we use a separable bi-weighting function. In other words there exists a (separable) bi-weighting function $\omega(p, q) = \pi_+(p) - \pi_-(q)$ such that $V_{bCPT}(P) = V_{CPT}(P)$ for all prospects P. This fact is relevant in order to provide a preference foundation for the model, since bCPT will need a less restrictive set of axioms with respect to CPT.

4.1 BCPT and the Wu–Markle paradox

Let us reconsider the Wu–Markle paradox described in the introduction. The paradox consists in the GLS violation, contrary to the prediction of CPT. Wu and Markle (2008) suggested to use the same model, CPT, with a different parametrization for mixed prospects and those involving only gains or losses: *"Our study indicates that mixed gamble behavior is described well by an S-shaped utility function and an inverse S-shaped probability weighting function. However, gain-loss separability fails, and hence different parameter values are needed for mixed gambles than single-domain gambles..."*

Despite these conclusions, we are able to explain their paradox using bCPT, without changing the parameters in the passage from non-mixed prospects to mixed ones. If we use the bCPT with the KT bi-weighting function,

$$\omega(p,q) = \frac{p^{0.61} - q^{0.69}}{\left[p^{0.61} + (1-p)^{0.61}\right]^{\frac{1}{0.61}} + \left[q^{0.69} + (1-q)^{0.69}\right]^{\frac{1}{0.69}} - 1}$$

and the classical KT power utility function[2]

$$u(x) = \begin{cases} x^{.88} & \text{if} x \geq 0 \\ -2.25\,|-x|^{.88} & \text{if} x < 0 \end{cases},$$

we obtain

$$\begin{array}{ll} V_{bCPT}(H) = -443.24 & > V_{bCPT}(L) = -453.76 \\ V_{bCPT}(H^+) = 649.19 & < V_{bCPT}(L^+) = 652.26 \\ V_{bCPT}(H^-) = -1,172.45 & < V_{bCPT}(L^-) = -1,083.04. \end{array}$$

These results agree with the preference relation $\succsim$. The most influential paper showing systematic violation of GLS is Wu and Markle (2008). Similar results are, for example, in Birnbaum and Bahra (2007). In the Appendix 2, we show in detail how bCPT seems to naturally capture the essence of the phenomenon.

5 Extension of bCPT to uncertainty

5.1 Bi-capacity and the bipolar Choquet integral

In order to extend bCPT to the field of uncertainty, we need to generalize the concept of capacity and Choquet integral with respect to a capacity. Let S be a non-empty set of states of the world and Σ an algebra of subsets of S (the events). Let $\mathcal{B}$ denote the set of bounded real-valued $\Sigma-$measurable functions on S and $\mathcal{B}_0$ the set of simple (i.e. finite valued) functions in $\mathcal{B}$. A function $\nu : \Sigma \rightarrow [0,1]$ is a normalized capacity on Σ if $\nu(\emptyset) = 0,\ \nu(S) = 1$ and $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. Choquet (1953) defined an integration operation with respect to ν. Given a non-negative valued function $f \in \mathcal{B}$ and a capacity $\nu : \Sigma \rightarrow [0,1]$, the Choquet integral of f with respect to ν is

[2] Both for $\omega(p,q)$ and $u(x)$ we use the original parameters, estimated in Tversky and Kahneman (1992)

$$\int_S f(s)\,d\nu =: \int_0^\infty \nu\left(\{s \in S : f(s) \geq t\}\right) dt.$$

Successively, Schmeidler (1986) extended this definition by defining the Choquet integral of a real valued function $f \in \mathcal{B}$ with respect to ν as

$$\int_S f(s)\,d\nu =: \int_{-\infty}^0 [\nu\left(\{s \in S : f(s) \geq t\}\right) - 1]\,dt + \int_0^\infty \nu\left(\{s \in S : f(s) \geq t\}\right) dt. \quad (12)$$

Obviously, for a non-negative function, f, the first summand in (12) is zero, while for a non-positive function, f, it is zero the second summand. Moreover, let us note that, following formula (12) we have that

$$\int_S f(s)\,d\nu = \int_S f_+(s)\,d\nu + \int_S f_-(s)\,d\nu,$$

where f_+ denotes the positive part of f, i.e. $f_+(s) = f(s)$ if $f(s) \geq 0$ and $f_+(s) = 0$ if $f(s) < 0$ and f_- denotes the negative part of f, i.e. $f_-(s) = f(s)$ if $f(s) \leq 0$ and $f_-(s) = 0$ if $f(s) > 0$. It follows that, since the Choquet integral (12) is at the basis of CEU (Schmeidler 1986, 1989; Gilboa 1987), also CEU implies GLS. Thus, we need an extended version of the Choquet integral, which does not implies GLS.

Let us consider the set of all the couples of disjoint events

$$\mathcal{Q} = \left\{(A, B) \in 2^S \times 2^S : A \cap B = \emptyset\right\}.$$

Definition 2 (Grabisch and Labreuche 2005a,b; Greco et al 2002) A function $\mu_b : \mathcal{Q} \to [-1, 1]$ is a bi-capacity on S if

- $\mu_b(\emptyset, \emptyset) = 0$, $\mu_b(S, \emptyset) = 1$ and $\mu_b(\emptyset, S) = -1$;
- $\mu_b(A, B) \leq \mu_b(C, D)$ for all $(A, B), (C, D) \in \mathcal{Q}$ such that $A \subseteq C \wedge B \supseteq D$.

Definition 3 (Grabisch and Labreuche 2005a,b; Greco et al 2002) The bipolar Choquet integral of a simple function $f \in \mathcal{B}_0$ with respect to a bi-capacity μ_b is given by:

$$\int_S f(s)\,d\mu_b =: \int_0^\infty \mu_b(\{s \in S : f(s) > t\}, \{s \in S : f(s) < -t\})dt.$$

5.2 Two different approaches

Since we are working with simple acts $f \in \mathcal{B}_0$, it follows that an uncertain *act* can be expressed as a vector $f = (x_1, s_1; \cdots ; x_n, s_n)$, where x_i will be obtained if the state s_i will occur. The dual capacity of a capacity $\nu : \Sigma \to [0, 1]$ is defined as $\widehat{\nu}(A) = 1 - \nu(N \setminus A)$ for all $A \in \Sigma$. Let be given a utility function $u(\cdot) : \Re \to \mathbb{R}$, two capacities (one for gains, one for losses) $\nu_+ : \Sigma \to [0, 1]$ and $\nu_- : \Sigma \to [0, 1]$ and a bi-capacity $\mu_b : \mathcal{Q} \to [-1, 1]$. The evaluation of $f = (x_1, s_1; \cdots ; x_n, s_n)$ in CPT and bCPT is

$$V_{CPT}(f) = \int_S u\left[f_+(s)\right] d\nu_+ + \int_S u\left[f_-(s)\right] d\widehat{\nu}_-$$
$$= \int_0^{\infty} \nu_+\left(\left\{s_j : u(x_j) \geq t\right\}\right) dt - \int_{-\infty}^{0} \nu_-\left(\{s_i : u((x_i) \leq t\}\right) dt. \quad (13)$$

$$V_{bCPT}(P) = \int_S u\left[f(s)\right] d\mu_b = \int_0^{+\infty} \mu_b\left(\{s_i : u(x_i) > t\}, \{s_i : u(x_i) < -t\}\right) dt. \quad (14)$$

In CPT we sum the Choquet integral of $u(f_+)$ with respect to ν_+ with the Choquet integral of $u(f_-)$ with respect to $\widehat{\nu}_-$, by getting a separate evaluation of gains and losses. In bCPT we calculate the bipolar Choquet integral of $u(f)$ with respect to μ_b getting a conjointly evaluation of gains and losses.

5.3 Link between CPT and bCPT

Grabisch and Labreuche (2005a) define a bi-capacity μ_b of CPT type (or separable) if there exist two capacities $\nu_+ : \Sigma \to [0, 1]$ and $\nu_- : \Sigma \to [0, 1]$, such that for all $(A, B) \in \mathcal{Q}$, $\mu_b(A, B) = \nu_+(A) - \nu_-(B)$. As in a risk-context, the two situations where CPT and bCPT coincide will occur for non mixed acts or using a separable bi-capacity.

Proposition 6 *For non-mixed acts, the bCPT model coincides with the CPT model.*

Proof Let us suppose that $\mu_b : \mathcal{Q} \to [-1, 1]$ is a bi-capacity, then we can define two capacities ν_+ and ν_- as follows: for all $E \in \Sigma$, $\nu_+(E) = \mu_b(E, \emptyset)$ and $\nu_-(E) = -\mu_b(\emptyset, E)$. If $f \in \mathcal{B}_0$ is such that $f(s) \geq 0$ for all $s \in S$, then it is easily checked that $\int_S f(s)\, d\mu_b = \int_S f(s)\, d\nu_+$ and $\int_S -f(s)\, d\mu_b = \int_S -f(s)\, d\nu_-$.

Proposition 7 *The bCPT model with a separable bi-capacity coincides with the CPT model.*

Proof of proposition 7 is trivial (not the consequence), based on the fact (see Grabisch and Labreuche (2005b)) that for a separable bi-capacity $\mu_b(A, B) = \nu_+(A) - \nu_-(B)$ and for all $f \in \mathcal{B}_0$ we get $\int_S f(s) d\mu_b = \int_S f_+(s) d\nu_+ + \int_S f_-(s) d\widehat{\nu_-} = \int_S f_+(s) d\nu_+ - \int_S (-f)_+(s) d\nu_-$.

In the remaining part of this paper we will face the problem of the preference foundation of bCPT. As we have just seen, the main concept to extend bCPT from the field of risk to that of uncertainty is the bipolar Choquet integral with respect to a bi-capacity. We will present a fairly simple characterization of the bipolar Choquet integral.

6 Axiomatic characterization of the bipolar Choquet integral

Let us identify $(A, B) \in \mathcal{Q}$ with the double-indicator function $(A, B)^* \in \mathcal{B}_0$,

$$(A, B)^*(s) = \begin{cases} 1 & \text{if } s \in A \\ -1 & \text{if } s \in B \\ 0 & \text{if } s \notin A \cup B. \end{cases}$$

Since $\int_S (A, B)^* \mu_b = \int_0^1 \mu_b(A, B) dt = \mu_b(A, B)$ then, the functional $\int_S \mu_b$, i.e. the bipolar Choquet integral, can be considered as an extension of the bi-capacity μ_b from $\mathcal{Q}$ to $\mathcal{B}_0$.

Definition 4 (Grabisch and Labreuche 2005b) $f, g : S \to \mathbb{R}$ are absolutely co-monotonic and cosigned (a.c.c.) if their absolute values are co-monotonic, i.e.,

$$(\,|f(s)| - |f(t)|\,) \cdot (\,|g(s)| - |g(t)|\,) \geq 0 \quad \forall s, t \in S$$

and, moreover, they are co-signed, i.e., $f(s) \cdot g(s) \geq 0 \quad \forall s \in S$.

The next proposition 8 lists some basic properties of the bipolar Choquet integral (for additional properties see also Grabisch et al. (2009)).

Proposition 8 (Grabisch and Labreuche 2005b) *The bipolar Choquet integral* $\int_S \mu_b$ *satisfies the following properties*

- *(P1) Monotonicity.*

$$f(s) \geq g(s) \quad \forall s \in S \quad \Rightarrow \quad \int_S f(s) \mu_b \geq \int_S g(s) \mu_b;$$

- *(P2) Positive homogeneity. For all* $a > 0$, *and* $f,\ a \cdot f \in \mathcal{B}_0$

$$\int_S a \cdot f(s) \mu_b = a \cdot \int_S f(s) \mu_b;$$

- *(P3) Bipolar idem-potency. For all* $\lambda > 0$

$$\int_S \lambda(S,\emptyset)^* \mu_b = \lambda \quad and \quad \int_S \lambda(\emptyset,S)^* \mu_b = -\lambda;$$

- *(P4) Additivity for acts a.c.c. If* $f, g \in \mathcal{B}_0$ *are a.c.c., then*

$$\int_S f(s) + g(s)\mu_b = \int_S f(s)\mu_b + \int_S g(s)\mu_b.$$

The following Theorem 2 characterizes the bipolar Choquet integral $\int_S \mu_b$. We present the proof in the main text, due to the importance of the theorem.

Theorem 2 *Let* $J : \mathcal{B}_0 \to \mathbb{R}$ *satisfy*

- $J((S,\emptyset)^*) = 1$ *and* $J((\emptyset,S)^*) = -1$;
- *(P1) Monotonicity;*
- *(P4) Additivity for a.c.c. acts;*

then, by assuming $\mu_b(A,B) = J\left((A,B)^*\right) \quad \forall (A,B) \in \mathcal{Q}$*, we have*

$$J(f) = \int_S f(s) d\mu_b \quad \forall f \in \mathcal{B}_0.$$

Proof First of all let us note that $(\mu_b(A,B) = J\left((A,B)^*\right), \forall (A,B) \in \mathcal{Q})$ defines a bi-capacity. It is easily checked that if $(A,B), (C,D) \in \mathcal{Q}$ with $A \subseteq C$ and $B \subseteq D$, then the functions $(A,B)^*$ and $(C,D)^*$ are a.c.c.. Moreover, let us suppose that the three functions $f, g, h \in \mathcal{B}$ are pairwise a.c.c. then also the function $f + g$ is a.c.c. with f, g, and h. Indeed, $f + g$ is obviously cosigned with f, g, and h, while the fact that $|f+g| = |f| + |g|$ is co-monotonic with $|f|$, $|g|$, and $|h|$ is a direct consequence of the definition of co-monotonicity. Indeed, suppose that f, g, h are pairwise co-monotonic and let us prove that $f + g$ is co-monotonic with h. For this it is sufficient to note that for all $r, s \in S$ we have $[(f+g)(s) - (f+g)(r)](h(s) - h(r)) = (f(s) - f(r))(h(s) - h(r)) + (g(s) - g(r))(h(s) - h(r)) \geq 0$. As consequence if J is a functional defined on $\mathcal{B}$ and additive for a.c.c. acts and if f, g, and h are pairwise a.c.c., then $J(f+g+h) = J(f) + J(g) + J(h)$. Now we are able to prove the theorem. Let $f \in B_0$ be a simple function with image $f(S) = \{x_1, x_2, \ldots, x_n\}$. Let $(\cdot) : N \to N$ be a permutation of indices in $N = \{1, 2, \ldots, n\}$ such that $|x_{(1)}| \leq |x_{(2)}| \leq \cdots \leq |x_{(n)}|$. f can be written as sum of double-indicator functions, i.e.,

$$f = \sum_{i=1}^{n} \left(\left|x_{(i)}\right| - \left|x_{(i-1)}\right|\right) \left(A(f)_{(i)}, B(f)_{(i)}\right)^*$$

where $A(f)_{(i)} = \left\{s \in S : f(s) \geq \left|x_{(i)}\right|\right\}$, $B(f)_{(i)} = \left\{s \in S : f(s) \leq -\left|x_{(i)}\right|\right\}$ and $\left|x_{(0)}\right| = 0$. Observe that the simple functions $\left(A(f)_{(i)}, B(f)_{(i)}\right)^*$ for

$i = 1, 2, \ldots, n$ are pairwise a.c.c., and, consequently, also the simple functions $\left(|x_{(i)}| - |x_{(i-1)}|\right)\left(A(f)_{(i)}, B(f)_{(i)}\right)^*$ for $i = 1, 2, \ldots, n$ are pairwise a.c.c.. On the basis of this observation, applying (P4), homogeneity and the definition of $\mu_b(A, B)$ we get the thesis as follows:

$$\begin{aligned} J(f) &= J\left[\sum_{i=1}^{n}\left(|x_{(i)}| - |x_{(i-1)}|\right)\left(A(f)_{(i)}, B(f)_{(i)}\right)^*\right] \\ &= \sum_{i=1}^{n} J\left[\left(|x_{(i)}| - |x_{(i-1)}|\right)\left(A(f)_{(i)}, B(f)_{(i)}\right)^*\right] \\ &= \sum_{i=1}^{n}\left(|x_{(i)}| - |x_{(i-1)}|\right) J\left[\left(A(f)_{(i)}, B(f)_{(i)}\right)^*\right] \\ &= \sum_{i=1}^{n}\left(|x_{(i)}| - |x_{(i-1)}|\right)\mu_b\left(A(f)_{(i)}, B(f)_{(i)}\right) = \int\limits_S f d\mu_b \end{aligned}$$

Remark 1 The properties (P2), i.e., the positive homogeneity, (P3) the bipolar idempotency, are not among the hypothesis of Theorem 2 since they are implied by additivity for absolutely co-monotonic and cosigned acts (P4) and monotonicity (P1).

Remark 2 The fact that the functional, $\int_S d\mu_b$, is additive for a.c.c. functions, means that in the bCPT model the weakened version of independence axiom will be true for a.c.c. acts.

7 Concluding remarks

In bCPT, gains and losses within a mixed prospect are evaluated conjointly and not separately, as in CPT. This permits to account for situations in which CPT fails, due to gain-loss separability, such as the "Wu–Markle paradox" In this paper we propose a natural generalization of CPT, which, fundamentally: a) totally preserve CPT in non-mixed cases; b) allows for GLS violation in mixed case. The main concept to get an axiomatic foundation of bCPT, in decision under uncertainty, is the bipolar Choquet integral, about which, we have presented a fairly simple characterization. A full axiomatization of the model, in terms of preferences foundation, will be the aim for future researches.

Appendix 1

Proof of proposition 1 It results that $f(x) = \left[x^\delta + (1-x)^\delta\right]^{\frac{1}{\delta}} \geq 1$ for all $x \in [0, 1]$ and $\delta \in [0, 1]$. Indeed the function f is continuous in the closed interval [0, 1], with $f(0) = f(1) = 1$, while $f'(x)$ is positive in]0, 1/2[and negative in]1/2, 1[. In fact:

$$f'(x) = \left[x^\delta + (1-x)^\delta\right]^{\frac{1}{\delta}-1}\left[x^{\delta-1} - (1-x)^{\delta-1}\right] \geq 0$$

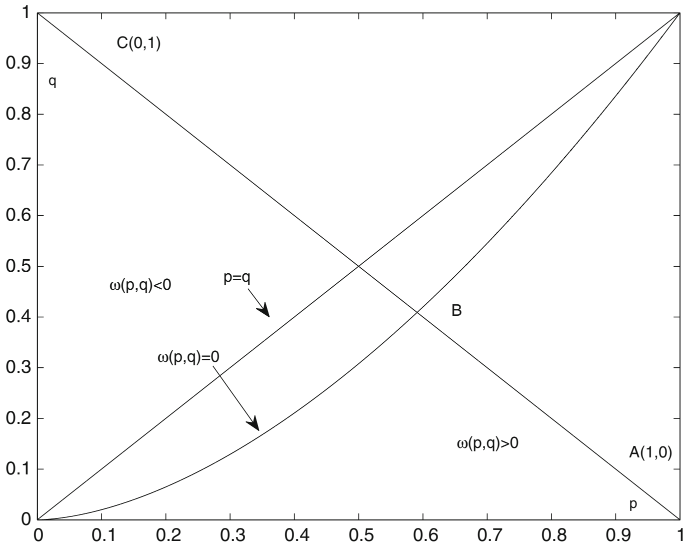

Fig. 4 the KT bi-weighting function domain; in the case $\gamma > \delta$, the curve $q = p^{\gamma/\delta}$ is convex

$$\Leftrightarrow \left[x^{\delta-1} - (1-x)^{\delta-1}\right] \geq 0 \Leftrightarrow 1 \geq \left(\frac{x}{1-x}\right)^{1-\delta} \Leftrightarrow x \leq \frac{1}{2}$$

It follows that in (3) the denominator is positive and the sign depends on $p^\gamma - q^\delta$. Let us consider a point (p, q) belonging to the zero curve $\omega(p, q) = 0 \Leftrightarrow p^\gamma - q^\delta = 0$, that is the $\widehat{OB}$ curve in Fig. 4. If the first coordinate p increases the point (p, q) will belong to the domain in which the function (3) is positive (OAB "triangle"), while if the second coordinate q increases the point (p, q) will belong to the domain in which the function (3) is negative (OBC "triangle"). Thus, starting from the zero curve $p^\gamma - q^\delta = 0$ the function (3) is increasing in p and decreasing in q. Now it is sufficient to prove that $\omega(p, q)$ is increasing in p and decreasing in q within the two triangles, i.e. where $\omega(p, q) > 0$ (< 0) and $p, q > 0$. If $\omega(p, q) > 0$, and then if $p^\gamma - q^\delta > 0$ and since the function $\ln(x)$ is strictly increasing, it is sufficient to prove that $\ln[\omega(p, q)]$ is increasing in p and decreasing in q. By differentiating w. r. t. the first variable:

$$\frac{\partial \ln[\omega(p,q)]}{\partial p} = \frac{\gamma p^{\gamma-1}}{p^\gamma - q^\delta} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \times \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}} + \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1} \tag{15}$$

If $1/2 \leq p < 1 \rightarrow \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \leq 0$ and (15) is positive. Suppose $0 < p < 1/2$, then the first summand in (15) is positive and the second is negative.

We have the following decreasing sequence:

$$\frac{\partial \ln[\omega(p,q)]}{\partial p} = \frac{\gamma p^{\gamma-1}}{p^\gamma - q^\delta} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \times \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}} + \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1}$$

Since

$$\frac{\gamma p^{\gamma-1}}{p^\gamma - q^\delta} > \frac{\gamma p^{\gamma-1}}{p^\gamma}$$

$$\geq \frac{\gamma p^{\gamma-1}}{p^\gamma} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \cdot \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}} + \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1}$$

Since from

$$\left\{\left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1 \geq 0 \rightarrow \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}} + \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1} \leq \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}}} \rightarrow -\left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}} + \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1} -\left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}}}\right\}$$

$$\geq \frac{\gamma p^{\gamma-1}}{p^\gamma} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \cdot \frac{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}}}$$
$$= \frac{\gamma p^{\gamma-1}}{p^\gamma} - \left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \cdot \frac{1}{p^\gamma + (1-p)^\gamma}$$

Since

$$-\left(\frac{1}{p}\right)^{1-\gamma} \leq -\left[\left(\frac{1}{p}\right)^{1-\gamma} - \left(\frac{1}{1-p}\right)^{1-\gamma}\right] \leq 0$$

$$\geq \frac{\gamma \left(\frac{1}{p}\right)^{1-\gamma}}{p^\gamma} - \frac{\left(\frac{1}{p}\right)^{1-\gamma}}{p^\gamma + (1-p)^\gamma} = \left(\frac{1}{p}\right)^{1-\gamma} \cdot \left[\frac{\gamma}{p^\gamma} - \frac{1}{p^\gamma + (1-p)^\gamma}\right]$$

Now, in order to prove that the partial derivative (15) is non negative, it is sufficient to show that the quantity in the last square bracket is non negative, i.e.

$$\frac{\gamma}{p^\gamma} - \frac{1}{p^\gamma + (1-p)^\gamma} = \frac{\gamma\left[p^\gamma + (1-p)^\gamma\right] - p^\gamma}{p^\gamma\left[p^\gamma + (1-p)^\gamma\right]} \geq 0 \Leftrightarrow \gamma\left[p^\gamma + (1-p)^\gamma\right] - p^\gamma \geq 0$$
$$\Leftrightarrow \gamma(1-p)^\gamma \geq (\gamma)p^\gamma \Leftrightarrow \left(\frac{1-p}{p}\right)^\gamma \geq \frac{1-\gamma}{\gamma} \Leftrightarrow \frac{1-p}{p} \geq \left(\frac{1-\gamma}{\gamma}\right)^{\frac{1}{\gamma}}$$

Since we are under the limitation $0 < p < 1/2$ the first term is greater than 1 and the last inequality is true if

$$\left(\frac{1-\gamma}{\gamma}\right)^{\frac{1}{\gamma}} \leq 1 \Leftrightarrow \gamma \geq \frac{1}{2}$$

and this is ensured by the hypothesis of proposition 1.
Thus we have proved that if $\omega(p, q) > 0$ then the function $\omega(p, q)$ is increasing in p. An analogous proof gives that, if $\omega(p, q) < 0$, then the function is decreasing in q, i.e., the function $-\omega(p, q)$ is increasing in q. For this it is sufficient to exchange p with q and γ with δ and to repeat the previous passages. Now, in the case $\omega(p, q) > 0$ we turn out our attention to the first derivative of $\ln[\omega(p, q)]$ with respect to q

$$\frac{\partial \ln[\omega(p,q)]}{\partial q} = \frac{-\delta q^{\delta-1}}{p^\gamma - q^\delta} - \left[\left(\frac{1}{q}\right)^{1-\delta} - \left(\frac{1}{1-q}\right)^{1-\delta}\right] \cdot$$
$$\cdot \frac{\left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}-1}}{\left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}} + \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} - 1} \tag{16}$$

If $\left[\left(\frac{1}{q}\right)^{1-\delta} - \left(\frac{1}{1-q}\right)^{1-\delta}\right] \geq 0 \Leftrightarrow q \leq 1/2$ then the (16) is negative. Supposing $q > 1/2$, the first summand in (16) is negative and the second is positive. Note that if $\gamma \geq \delta$, the curve which equation is $p^\gamma - q^\delta = 0$ coincides with the graph of the function $q = p^{\frac{\gamma}{\delta}}$ that is convex, like in Fig. 4, and within the domain

$$A^+ = \left\{(p, q) \in [0; 1] \times [0; 1] \text{ such that } p + q \geq 1 \text{ and } p^\gamma - q^\delta\right\}$$

it is impossible that $q > 1/2$ and so we have finished the proof. On the other hand, if $\gamma < \delta$ the graph of the function $q = p^{\frac{\gamma}{\delta}}$ is concave and within the domain A^+ there are points such that $q > 1/2$. For these reasons, from here we will suppose $q > 1/2$ and $\gamma < \delta$ and we will refer to Fig. 5.

From a sequence of increases it results:

$$\frac{\partial \ln[\omega(p,q)]}{\partial q}$$

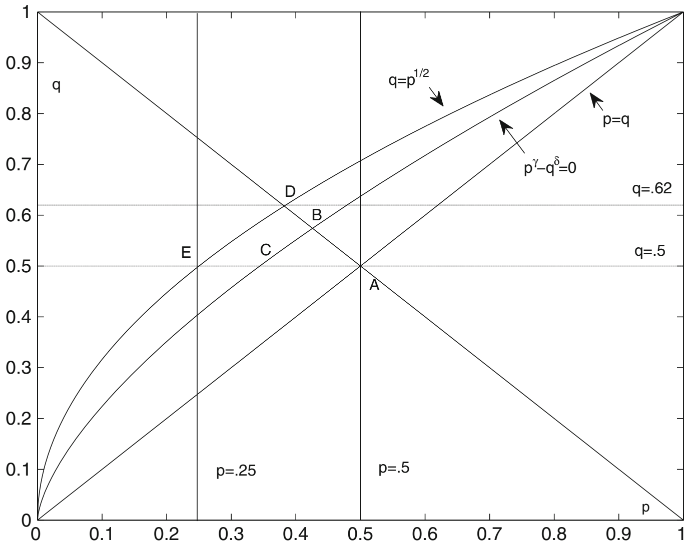

Fig. 5 if $\gamma < \delta$, the curve $\widehat{OB}: q = p^{\gamma/\delta}$ is concave and its most accentuate curvature is that of $\widehat{OD}: q = \sqrt{p}$. The point $A(.5, .5)$ is the intersection between the lines $p = q$ and $p + q = 1$; the point B is the intersection between $p^\gamma - q^\delta = 0$ and $p + q = 1$; the point C is the intersection between $p^\gamma - q^\delta = 0$ and $q = .5$; the point $D(.38, .62)$ is the intersection between $q = \sqrt{p}$ and $p + q = 1$; the point $E(.25, .5)$ is the intersection between $q = \sqrt{p}$ and $q = .5$

Since from

$$\Bigg\{1/2 < \gamma, \delta \leq 1 \rightarrow \left[p^\gamma + (1-p)^\gamma\right]^{\frac{1}{\gamma}} - 1 \geq 0 \text{ and } \left[q^\delta + (1-q)^\delta\right]^{\frac{1}{\delta}} \geq 1 \rightarrow$$

$$\frac{\left[q^\delta+(1-q)^\delta\right]^{\frac{1}{\delta}-1}}{\left[p^\gamma+(1-p)^\gamma\right]^{\frac{1}{\gamma}}+\left[q^\delta+(1-q)^\delta\right]^{\frac{1}{\delta}}-1} \leq \frac{\left[q^\delta+(1-q)^\delta\right]^{\frac{1}{\delta}-1}}{\left[q^\delta+(1-q)^\delta\right]^{\frac{1}{\delta}}} = \frac{1}{q^\delta+(1-q)^\delta} \leq 1\Bigg\}$$

$$\leq \frac{-\delta q^{\delta-1}}{p^\gamma - q^\delta} - \left[\left(\frac{1}{q}\right)^{1-\delta} - \left(\frac{1}{1-q}\right)^{1-\delta}\right]$$

$$= q^{\delta-1}\left[\frac{-\delta}{p^\gamma - q^\delta} + \left(\frac{q}{1-q}\right)^{1-\delta} - 1\right]$$

Then it is sufficient to prove that

$$\frac{-\delta}{p^\gamma - q^\delta} + \left(\frac{q}{1-q}\right)^{1-\delta} - 1 \leq 0 \Leftrightarrow \left(\frac{q}{1-q}\right)^{1-\delta} \leq 1 + \frac{-\delta}{p^\gamma - q^\delta}$$

and this will follow from:

$$\frac{q}{1-q} \leq 1 + \frac{-\delta}{p^\gamma - q^\delta}$$

since

$$q > \frac{1}{2} \Rightarrow \frac{q}{1-q} > 1 \Rightarrow \left(\frac{q}{1-q}\right)^{1-\delta} \Rightarrow \frac{q}{1-q}$$

now we need to prove that

$$\frac{q}{1-q} \leq 1 + \frac{-\delta}{p^{\gamma} - q^{\delta}} \tag{17}$$

Under the restrictions we are working with, it is possible to elicit some limitations of the variables p, q, γ and δ. We have supposed $p^{\gamma} - q^{\delta} > 0$, $q > 1/2$ and $\delta > \gamma$, that in Fig. 5 delimit the area ABC. Since the curvature of $p^{\gamma} - q^{\delta} = 0$ is more accentuated when larger is the difference between γ and δ, a limit is, for us, the curve $p^{0.5} - q^{1} = 0$, i.e. $q = \sqrt{p}$, which delimits the area ADE containing the area ABC. This consideration allows us to elicit some sure limitations for p and q: the "highest" point is the intersection between $q = \sqrt{p}$ and $p + q = 1$, that is $D(0.38; 0.62)$; the most "left-placed" point is the intersection between $q = \sqrt{p}$ and $q = 0.5$, that is $E(0.25; 0.5)$; we elicit $0.25 < p < 0.5$ and $0.5 < q < 0.62$. Consider the function $p^{\gamma} - q^{\delta}$, by differentiating, we can prove that it is increasing in p and δ and decreasing in q and γ, and then, using the elicited parameter limitations we have

$$p^{\gamma} - q^{\delta} \leq \left(\frac{1}{2}\right)^{0.5} - \left(\frac{1}{2}\right)^{1}$$

which in turn implies

$$1 + \frac{-\delta}{p^{\gamma} - q^{\delta}} \geq 1 + \frac{\delta}{\left(\frac{1}{2}\right)^{0.5} - \left(\frac{1}{2}\right)^{1}} \tag{18}$$

Finally, the quantity $q/(1-q)$ is increasing in q and then by using the sup limitation of q it follows that

$$\frac{q}{1-q} \leq \frac{0.62}{1-0.62} \tag{19}$$

Using (18) and (19) the (17) is true if it is true that:

$$\frac{0.62}{1-0.62} \leq 1 + \frac{\delta}{\left(\frac{1}{2}\right)^{0.5} - \left(\frac{1}{2}\right)^{1}}$$

which gives $\delta > 0.131$ that is within our limitations. Similarly, by exchanging p with q and γ with δ it follows that $\omega(p, q)$ is increasing in p when $\omega(p, q) < 0$. □

Proof of proposition 2 For $x \in [0, 1]$, $\alpha > 1/2$ and $\gamma \in]0, 1]$ it results $f(x) = \alpha x^{\gamma} + (1-x)^{\gamma} \geq min\{1, \alpha\} > 1/2$. Since this function is continuous in the closed interval [0, 1], with $f(0) = 1$, $f(1) = \alpha$ and the second derivative is non-positive from zero to one:

$$f''(x) = \gamma(\gamma-1)\alpha x^{\delta-2} + \gamma(\gamma-1)(1-x)^{\delta-2} \leq 0$$

It follows that, in (5), the denominator is positive under the limitation $\alpha > 1/2$. Within its domain the first derivative of the (5) with respect to p is :

$$\frac{\partial\omega(p,q)}{\partial p} = \alpha\gamma \frac{(1-p)^{\gamma-1}\left(p^{\gamma-1}-q^{\delta}\right) + p^{\gamma-1}\left[2\alpha q^{\delta} + (1-q)^{\delta} - 1\right]}{\left[\alpha p^{\gamma} + (1-p)^{\gamma} + \alpha q^{\delta} + (1-q)^{\delta} - 1\right]^2} \quad (20)$$

Having chosen $\gamma \leq 1$ the term $p^{\gamma-1} \geq 1$ for all $p \in]0, 1]$ and since $q^{\delta} \leq 1$ then $p^{\gamma-1} - q^{\delta} \geq 0$. On the other hand $(2\alpha q^{\delta} + 1 - q)^{\delta} - 1 \geq 0$ since for $x \in [0, 1]$, $\alpha > 1/2$ and $0 < \delta \leq 1$ the function $f(x) = 2\alpha x^{\delta} + (1-x)^{\delta} \geq min\{1, 2\alpha\} \geq 1$ since it is continuous in the closed interval [0, 1], with $f(0) = 1$, $f(1) = 2\alpha$ and the second derivative is non-positive from zero to one:

$$f''(x) = \gamma(\gamma-1)2\alpha x^{\delta-2} + \gamma(\gamma-1)(1-x)^{\delta-2} \leq 0$$

Then (20) is non-negative and the (5) is increasing in p.
In the same manner it can be checked that the first derivative with respect to q

$$\frac{\partial\omega(p,q)}{\partial q} = \alpha\delta \frac{(1-q)^{\delta-1}\left(p^{\gamma}-q^{\delta-1}\right) - q^{\delta-1}\left[2\alpha p^{\gamma} + (1-p)^{\gamma} - 1\right]}{\left[\alpha p^{\gamma} + (1-p)^{\gamma} + \alpha q^{\delta} + (1-q)^{\delta} - 1\right]^2} \quad (21)$$

is non-positive and then the (5) is decreasing in q. □

Proof of Proposition 3 Let us consider a point (p, q) belonging to the zero curve $\omega(p, q) = 0 \Leftrightarrow p^{\gamma} - q^{\delta} = 0$ that is the $\widehat{OB}$ curve in Fig. 4. If p increases the point (p, q) will belong to the domain in which the function is positive (OAB "triangle"), while if q increases the point (p, q) will belong to the domain in which the function is negative (OBC "triangle"). Now it is sufficient to prove that $\omega(p, q)$ is increasing in p and decreasing in q within the two triangle, i.e. where $\omega(p, q) > 0$ or $\omega(p, q) < 0$ and $p, q > 0$. If $w(p, q) > 0$ and then if $p^{\gamma} - q^{\delta} > 0$ the (7) becomes: $\omega(p, q) = e^{-\left[-\ln(p^{\gamma}-q^{\delta})\right]^{\alpha}}$ and by differentiating w. r. t. the two variables:

$$\frac{\partial\omega(p,q)}{\partial p} = e^{-\left[-\ln(p^{\gamma}-q^{\delta})\right]^{\alpha}} \alpha \left[-\ln\left(p^{\gamma}-q^{\delta}\right)\right]^{\alpha-1} \frac{\gamma p^{\gamma-1}}{p^{\gamma}-q^{\delta}} > 0$$
$$\frac{\partial\omega(p,q)}{\partial p} = e^{-\left[-\ln(p^{\gamma}-q^{\delta})\right]^{\alpha}} \alpha \left[-\ln\left(p^{\gamma}-q^{\delta}\right)\right]^{\alpha-1} \frac{-\delta q^{\delta-1}}{p^{\gamma}-q^{\delta}} < 0$$

This proves the property within the triangle OBA, where $\omega(p, q) > 0$. Similarly if $p^{\gamma} - q^{\delta} < 0$ the (7) becomes: $\omega(p, q) = -e^{-\left[-\ln(-p^{\gamma}+q^{\delta})\right]^{\alpha}}$ and by differentiating w. r. t. the two variables:

$$\frac{\partial\omega(p,q)}{\partial p} = -e^{-\left[-\ln(-p^{\gamma}+q^{\delta})\right]^{\alpha}}\alpha\left[-\ln\left(p^{\gamma}-q^{\delta}\right)\right]^{\alpha-1}\frac{-\gamma p^{\gamma-1}}{-p^{\gamma}+q^{\delta}} > 0$$
$$\frac{\partial\omega(p,q)}{\partial p} = -e^{-\left[-\ln(-p^{\gamma}+q^{\delta})\right]^{\alpha}}\alpha\left[-\ln\left(p^{\gamma}-q^{\delta}\right)\right]^{\alpha-1}\frac{\delta q^{\delta-1}}{-p^{\gamma}+q^{\delta}} < 0$$

We conclude that the Prelec bi-weighting function is increasing in its first argument and decreasing in the second, for all the parameter values. □

Appendix 2

Recent literature denouncing GLS

As discussed in the paper, this study aims to generalize CPT, allowing gains and losses within a mixed prospect to be evaluated conjointly, rather than separately. In the following we shall focus our attention on two recent papers: Wu and Markle (2008) and Birnbaum and Bahra (2007). Both of them report systematic violations of GLS. CPT and all the models it generalizes, such as EUT, cannot account for such a pattern of choice. We show how bCPT is able to capture, at least partially, these errata predictions.

Wu and Markle (2008)

In Table 2 we reproduce Table 1 of page 1326 in Wu and Markle (2008). As in the Wu–Markle paradox, two binary prospects, H and G, containing a gain and a loss were confronted trough the preferences of participants. Then, also the positive and negative part of these prospects were confronted. The second and third column describe the two prospects H and G in terms of outcomes and relative percentages. The fourth column gives the percentage of participants whose choice was respectively H, H_+, H_- over G, G_+, G_-. The fifth column, titled "preferences", gives the preferred prospect in percentage. In this column we report both prospects (like GH) when the percentage of choice was close to 50%. Finally, in the sixth column we report the choice predicted by bCPT, used with the specification of parameters given below. In many cases (tests 6,7, 10-18) the respondents preferred (in percentage) H to G while, splitting the prospects into their respective positive and negative part, the preferences were reversed. These are the phenomenons of GLS and GLH. To test bCPT we have used the KT bi-weighing function (3) with parameters $\gamma = 0.9$ and $\delta = 0.89$ and the KT power utility function

$$u(x) = \begin{cases} x^{\alpha_+} & \text{if } x \geq 0 \\ -\lambda(-x)^{\alpha_-} & \text{if } x < 0 \end{cases} \tag{22}$$

with parameters $\lambda = 1.77$, $\alpha_+ = 0.68$, and $\alpha_- = 0.79$.

As can be seen in Table 2 on the last two columns, the predictions of bCPT are in the same directions of the preferences in all the pure positive choices except that in

Table 2 application of bCPT to the data in Wu and Markle (2008)

Test	H gamble				G gamble				Choice %			Preferences			bCPT		
	g	p	l	$1-p$	g'	p'	l'	$1-p'$	H	H+	H−						
1	150	0.3	−25	0.7	75	0.8	−60	0.2	22	10	17	G+	G−	G	G+	G−	G
2	1,800	0.05	−200	0.95	600	0.3	−250	0.7	21	17	15	G+	G−	G	G+	G−	G
3	1,000	0.25	−500	0.75	600	0.5	−700	0.5	28	12	20	G+	G−	G	G+	G−	H
4	200	0.3	−25	0.7	75	0.8	−100	0.2	33	18	22	G+	G−	G	G+	G−	G
5	1,200	0.25	−500	0.75	600	0.5	−800	0.5	43	21	25	G+	G−	G	G+	G−	H
6	750	0.4	−1,000	0.6	500	0.6	−1,500	0.4	51	26	25	G+	G−	HG	G+	G−	H
7	4,200	0.5	−3,000	0.5	3,000	0.75	−6,000	0.25	52	15	37	G+	G−	HG	G+	G−	H
8	4,500	0.5	−1,500	0.5	3,000	0.75	−3,000	0.25	48	17	47	G+	G−	GH	G+	G−	H
9	4,500	0.5	−3,000	0.5	3,000	0.75	−6,000	0.25	58	17	55	G+	H−	H	G+	G−	H
10	1,000	0.3	−200	0.7	400	0.7	−500	0.3	51	48	28	G+	G−	HG	G+	G−	H
11	4,800	0.5	−1,500	0.5	3,000	0.75	−3,000	0.25	54	33	44	G+	G−	H	G+	G−	H
12	3,000	0.01	−490	0.99	2,000	0.02	−500	0.98	59	42	36	G+	G−	H	G+	H−	H
13	2,200	0.4	−600	0.6	850	0.75	−1,700	0.25	52	38	42	G+	G−	HG	H+	H−	H
14	2,000	0.2	−1,000	0.8	1,700	0.25	−1,100	0.75	58	34	48	G+	G−	H	G+	H−	H
15	1,500	0.25	−500	0.75	600	0.5	−900	0.5	51	51	33	GH+	G−	HG	H+	H−	H
16	5,000	0.5	−3,000	0.5	3,000	0.75	−6,000	0.25	65	43	43	G+	G−	H	G+	G−	H
17	1,500	0.4	−1,000	0.6	600	0.8	−3,500	0.2	59	48	41	G+	G−	H	G+	H−	H
18	2,025	0.5	−875	0.5	1,800	0.6	−1,000	0.4	72	52	42	G+	G−	H	G+	G−	H
19	600	0.25	−100	0.75	125	0.75	−500	0.25	58	55	44	H+	G−	H	H+	H−	H
20	5,000	0.1	−900	0.9	1,400	0.3	−1,700	0.7	40	47	53	G+	HG−	G	G+	G−	H
21	700	0.25	−100	0.75	125	0.75	−600	0.25	71	59	48	H+	H−	H	H+	G−	H
22	700	0.5	−150	0.5	350	0.75	−400	0.25	63	58	48	H+	GH−	H	H+	H−	H
23	1,200	0.3	−200	0.7	400	0.7	−800	0.3	70	59	50	H+	H−	H	G+	H−	H

Table 2 continued

Test	H gamble				G gamble				Choice %			Preferences			bCPT		
	g	p	l	$1-p$	g'	p'	l'	$1-p'$	H	H+	H−						
24	5,000	0.5	−2,500	0.5	2,500	0.75	−6,000	0.25	79	54	54	H+	H−	H	H+	H−	H
25	800	0.4	−1,000	0.6	500	0.6	−1,600	0.4	58	64	51	H+	H−	H	G+	H−	H
26	5,000	0.5	−3,000	0.5	2,500	0.75	−6,500	0.25	71	61	59	H+	H−	H	H+	H−	H
27	700	0.25	−100	0.75	100	0.75	−800	0.25	73	58	64	H+	H−	H	H+	H−	H
28	1,500	0.3	−200	0.7	400	0.7	−1,000	0.3	75	59	63	H+	H−	H	H+	G−	H
29	1,600	0.25	−500	0.75	600	0.5	−1,100	0.5	73	60	69	H+	H−	H	H+	H−	H
30	2,000	0.4	−800	0.6	600	0.8	−3,500	0.2	65	66	63	H+	H−	H	H+	H−	H
31	2,000	0.25	−400	0.75	600	0.5	−1,100	0.5	80	63	69	H+	H−	H	H+	H−	H
32	1,500	0.4	−700	0.6	300	0.8	−3,500	0.2	78	64	68	H+	H−	H	H+	H−	H
33	900	0.4	−1,000	0.6	500	0.6	−1,800	0.4	70	74	61	H+	H−	H	H+	H−	H
34	1,000	0.4	−1,000	0.6	500	0.6	−2,000	0.4	78	71	70	H+	H−	H	H+	H−	H

$\alpha_+ = 0.68, \alpha_- = 0.79, \delta = 0.89\ \lambda = 1.77, \gamma = 0.9$

tests 13, 23, and 25, in all the pure negative choices except in tests 9, 12-15, 17, and 19 and in all the mixed choices except in tests 3, 5, and 20. But, what we think is very interesting, is that bCPT is able to explain the reversed preferences, totally in tests 6, 7, 10, 11, 16, and 18 and partially in test 12, 14, 15, and 17. The model seems able to naturally capture, totally or partially, the GLH.

Birnbaum-Bahra

Birnbaum and Bahra (2007) reported systematic violations of two behavioral properties implied by CPT. One, is the just discussed GLS and the other, is the property known as coalescing: "*coalescing is the assumption that if there are two probability-consequences branches in a gamble leading to the same consequence, they can be combined by adding their probabilities.*" For example, the three-branch gamble $A = (\$100, 25\%; \$100, 25\%; \$0, 50\%)$ should be equivalent to the two-branch gamble $A' = (\$100, 50\%; \$0, 50\%)$. Our model is not able to accommodate for violation of coalescing; however, we briefly discuss this point. Birnbaum and Bahra tested violation of coalescing presenting to the participants the gambles in terms of a container holding exactly 100 marbles of different colors. So, according to coalescing, $B' =$ (25 red \$100; 75 white \$0) should be considered equivalent to $B =$ (25 red \$100; 25 white \$0; 50 *white* \$0). We doubt that the prospects presented in these form are equivalent to the form $B' = (25\%\ \$100; 75\%\ \$0)$ should be considered equivalent to $B = (25\%\ \$100; 25\%\ \$0; 50\%\ \$0)$. In fact, a person facing B could ask himself what is the reason that the first 25 white marbles were not summed to the second 50 white marbles. It is admissible that she could think if they differ in some way, e.g., in size. In any case, she will have an additional information, or doubt to process and this could generate errors. As focused from the authors in Wu and Markle (2008), the examples in Birnbaum and Bahra (2007) to underline the GLS violation, are less simple than theirs, but our model is able to accommodate for these violations too. The only we need is to modify the parameter γ from the value of 0.9, used to accommodate the majority of data in Wu and Markle (2008), to the value of 0.74. Next, we report the part of the Table 5 at page 1022 in Birnbaum and Bahra (2007) that, in the words of the same authors, form a test for the GLS. Each gamble "*is described in terms of a container holding exactly 100 marbles of different colors, from which one marble would be drawn at random, and the color of that marble would determine the prize*". In brackets the percentages of each choose.

$$F = \begin{pmatrix} 25\ black \\ to\ win\ \$100 \\ \\ 25\ white \\ to\ win\ \$0 \\ \\ 50\ pink \\ to\ lose\ \$50 \end{pmatrix} \succ \begin{pmatrix} 50\ blue \\ to\ win\ \$50 \\ \\ 25\ white \\ to\ lose\ \$0 \\ \\ 25\ red \\ to\ lose\ \$100 \end{pmatrix} = G$$

[76%] [24%]

$$F^{+} = \begin{pmatrix} 25\ black \\ to\ win\ \$100 \\ \\ 25\ white \\ to\ win\ \$0 \\ \\ 50\ white \\ to\ win\ \$0 \end{pmatrix} \prec \begin{pmatrix} 25\ blue \\ to\ win\ \$50 \\ \\ 25\ blue \\ to\ win\ \$50 \\ \\ 50\ white \\ to\ win\ \$0 \end{pmatrix} = G^{+}$$

[29%] [71%]

$$F^{-} = \begin{pmatrix} 50\ white \\ to\ lose\ \$0 \\ \\ 25\ pink \\ to\ lose\ \$50 \\ \\ 25\ pink \\ to\ lose\ \$50 \end{pmatrix} \prec \begin{pmatrix} 50\ white \\ to\ lose\ \$0 \\ \\ 25\ white \\ to\ lose\ \$0 \\ \\ 25\ red \\ to\ lose\ \$100 \end{pmatrix} = G^{-}$$

[35%] [65%]

As can be seen, F is preferred to G, but splitting the prospects in their positive/negative parts a relevant majority prefers G^{+} to F^{+} and G^{-} to F^{-}. In order to evaluate these prospects, we substitute the respective probabilities to the colors, as the authors did, by dividing for 100 any number of color. Using the bipolar CPT with the bi-polarized KT weighting functions with parameters $\gamma = 0.74$, $\delta = 0.89$ and the KT power utility function (22) with parameters $\lambda = 1.77$, $\alpha_{+} = 0.68$ and $\alpha_{-} = 0.79$, the numerical evaluation of prospects agree with the preference relation $\succsim$

$$\begin{aligned} V_{bCPT}(F) &= -11.07 \geq V_{bCPT}(G) = -11.11 \\ V_{bCPT}(F^{+}) &= 6.67 \leq V_{bCPT}(G^{+}) = 6.71 \\ V_{bCPT}(F^{-}) &= -19.28 \leq V_{bCPT}(G^{-}) = -18.25 \end{aligned}$$

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