Sophisticated Preference Aggregation*

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Abstract

A Sophisticated Social Welfare Function (SSWF) is a mapping from profiles of individual preferences into a sophisticated preference which is a pairwise weighted comparison of alternatives. We characterize Pareto optimal and pairwise independent SSWFs in terms of oligarchies that are induced by some power distribution in the society. This is a fairly large class ranging from dictatoriality to anonymous aggregation rules. Our results generalize the impossibility theorem of Arrow (1951) and the oligarchy theorem of Gibbard (1969).

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1 Introduction

It is possible to have a more general perspective of the preference aggregation problem by incorporating elements of ambiguity into individual and/or social preferences. As there are various ways of conceiving ambiguity, there are also various ways of generalizing the aggregation model of Arrow (1951) through ambiguous preferences.

Two major strands of the literature emerge: One of these models a preference as a fuzzy binary relation and the other has a probabilistic conception of preferences. Our analysis belongs to the latter strand.[1] We introduce the concept of a *sophisticated preference* which is a weighted pairwise comparison of alternatives that allows some kind of a mixed feeling in comparing any given pair of alternatives. To be more concrete, suppose an individual is asked whether she likes Paris or Istanbul. A sophisticated preference allows an answer of the following type: "I like Paris more than Istanbul in some respect but I like Istanbul more than Paris in other respects". The answer is also required to quantify the "rate" at which Istanbul is better than Paris and vice versa. Moreover, these are normalized rates which add up to unity. In other words, a sophisticated preference σ assigns to each ordered pair (x, y) of alternatives some $\sigma(x, y)$ belonging to the interval $[0, 1]$ such that $\sigma(x, y) + \sigma(y, x) = 1$.[2] Sophisticated preferences generalize the standard notion of a preference when $\sigma(x, y) = 1$ is interpreted as x being preferred to y in its usual sense.

We consider *sophisticated social welfare functions (SSWFs)* which aggregate vectors of (non-sophisticated) preferences into a sophisticated preference. We propose two intepretations of our model. One of these is from a social choice perspective which aims to represent the existing preferences in a society. Here, a vector of preferences is seen as the list of preferences that different individuals of the society have. These are aggregated into a sophisticated preference which is a representation of the various opinions prevailing in the society. Our second interpretation is from an individual choice perspective where a vector of preferences contains various rankings of alternatives by one given individual, according to various criteria. For example, a new Ph.D. graduate in the job market may rank universities according to different criteria such as their location, their salaries etc. Each of these criteria may result in a different ranking from which the individual has to derive

[1] As Barrett and Salles (2005) mention, there seems to be a debate between these two strands to which this paper does not aim to contribute. One can see Fishburn (1998) and Barrett and Salles (2005) for a survey of the related literatures.

[2] This is where a sophisticated preference technically differs from a fuzzy one which does not require $\sigma(x, y) + \sigma(y, x) = 1$.

an overall preference with possibly mixed feelings.

Given these interpretations, we require a certain consistency of the aggregated outcome, expressed by some transitivity condition imposed over sophisticated preferences[3]: We qualify a sophisticated preference as *transitive* whenever given any three alternatives x, y and z, we have $\sigma(x, y) = 1 \Longrightarrow \sigma(x, z) \geq \sigma(y, z)$. In other words, if x is preferred to y in all respects and r is the "rate" at which y is preferred to z, then the "rate" at which x is preferred to z is at least r. As we will discuss in details, this is a relatively weak transitivity condition whose non-sophisticated counterpart is equivalent to quasi-transitivity.[4] However, given our interpretations of the model, it seems the most appropriate and we do not wish to strengthen it so that its reflection over non-sophisticated preferences becomes equivalent to the usual transitivity condition.[5]

Our setting is closely related to the *collective probabilistic judgement* model of Barberà and Valenciano (1983). In fact, their *collective probabilistic judgement functions* being more general than our SSWFs, their results can be imported to our environment. On the other hand, as further discussed in in Section 4, we present a strong result which does not follow from Barberà and Valenciano (1983): We give a full characterization of Pareto optimal and pairwise independent SSWFs in terms of oligarchies induced by some power distribution in the society. As an oligarchy is any non-empty subsociety whose members share the decision power, this is a fairly large class ranging from dictatoriality (where the oligarchy consists of a single individual) to anonymous SSWFs (where decision power is equally distributed among individuals). In fact, our characterization generalizes two major results of the literature: In case the ranges of Pareto optimal and pairwise independent SSWFs are restricted to non-sophisticated preferences that are linear orders, the oligarchies must contain precisely one individual (thus a dictator) - which is the impossibility theorem of Arrow (1951, 1963). In case the social outcome is restricted to non-sophisticated preferences that are complete and quasitransitive, Pareto optimal and pairwise independent SSWFs are oligarchical in the sense that the oligarchy has full decision power while all

[3]The literature on ambiguous preferences admits a range of transitivity conditions of varying strenght, a list of which can be found in Dubois and Prade (1980) or Dasgupta and Deb (1996).

[4]Quasi-transitivity of a non-sophisticated preference requires x being better than z, whenever x is better than y and y is better than z. This is weaker than the usual transitivity requirement of x being at least as good as z, whenever x is at least as good as y and y is at least as good as z.

[5]Nevertheless, we discuss, at the end of Section 3, how such strenghtenings affect our results.

proper subsets of the oligarchy have equal decision power - a result which is known as the oligarchy theorem of Gibbard (1969).

Section 2 introduces the basic notions. Section 3 states the results. Section 4 makes some concluding remarks.

2 Basic Notions

We consider a finite set of individuals N with $\#N \geq 2$, confronting a finite set of alternatives A with $\#A \geq 3$. A *sophisticated preference* is a mapping $\sigma : A \times A \to [0,1]$ such that for all distinct $x, y \in A$ we have $\sigma(x,y) + \sigma(y,x) = 1$ while $\sigma(x,x) = 0 \ \forall \ x \in A$. Interpreting $\sigma(x,y)$ as the weight by which x is preferred to y, the former condition imposes a kind of completeness over σ while the latter is an irreflexivity requirement.[6] We qualify a sophisticated preference σ as *transitive* iff $\sigma(x,y) = 1 \implies \sigma(x,z) \geq \sigma(y,z) \ \forall \ x,y,z \in A$.[7]

We write Σ for the set of transitive sophisticated preferences. Let $\Pi = \{\pi \in \Sigma : \pi(x,y) \in \{0,1\}$ for all $x,y \in A\}$ be the set of sophisticated preferences which map $A \times A$ into the $\{0,1\}$ doubleton. Note that by interpreting $\pi(x,y) = 1$ as x being preferred to y in its usual meaning and writing $x \ \pi \ y$ whenever $\pi(x,y) = 1$, Π becomes the set of linear orders over A.[8] We assume that individual preferences belong to Π and we write $\pi_i \in \Pi$ for the preference of $i \in N$ over A. A preference profile over A is an n-tuple $\underline{\pi} = (\pi_1, ..., \pi_{\#N}) \in \Pi^N$ of individual preferences.

A *sophisticated social welfare function* (SSWF) is a mapping $\alpha : \Pi^N \to \Sigma$. So $\alpha(\underline{\pi}) \in \Sigma$ is a sophisticated preference over A which, by a slight abuse of notation, we denote $\alpha_{\underline{\pi}}$. Thus $\alpha_{\underline{\pi}}(x,y) \in [0,1]$ stands for the weight that α assigns to $(x,y) \in A \times A$ at $\underline{\pi} \in \Pi^N$.

Given any distinct $x, y \in A$, let $\Pi(x,y) = \{\pi \in \Pi : x \ \pi \ y\}$ be the set of preferences where x is preferred to y. A SSWF $\alpha : \Pi^N \to \Sigma$ is *Pareto Optimal (PO)* iff given any distinct $x, y \in A$ and any $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(x,y)$ for all $i \in N$, we have $\alpha_{\underline{\pi}}(x,y) = 1$. A SSWF $\alpha : \Pi^N \to \Sigma$ is *independent of irrelevant alternatives (IIA)* iff given any distinct $x, y \in A$

[6] Letting $\sigma(x,x) = 0$ is conventional. All our results can be proven by taking $\sigma(x,x) = 1$ or $\sigma(x,x) = \frac{1}{2}$.

[7] Remark that $\sigma(y,z) = 1 \implies \sigma(x,z) \geq \sigma(x,y)$ would be an equivalent statement of transitivity. Moreover, transitivity implies $\sigma(x,y) = \sigma(y,z) = 1 \implies \sigma(x,z) = 1$. It is also worth noting that Condition 1 (Consistency under Complete Rejection) of Barberà and Valenciano (1983), adapted to our framework, is equivalent to transitivity

[8] In other words, for any $\pi \in \Pi$ and any distinct $x, y \in A$, precisely one of $x \ \pi \ y$ and $y \ \pi \ x$ holds while $x \ \pi \ x$ holds for no x in A. Moreover $x \ \pi \ y$ and $y \ \pi \ z$ implies $x \ \pi \ z$ for all $x, y, z \in A$.

and any $\underline{\pi}$, $\underline{\pi}' \in \Pi^N$ with $\pi_i \in \Pi(x,y) \iff \pi'_i \in \Pi(x,y)$ for all $i \in N$, we have $\alpha_{\underline{\pi}}(x,y) = \alpha_{\underline{\pi}'}(x,y)$.[9]

SSWFs satisfying IIA can, as usual, be expressed in terms of pairwise SSWFs. To see this, take any distinct $x, y \in A$ and let $\{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}$ be the set of possible (non-sophisticated) preferences over $\{x,y\}$ where $\genfrac{}{}{0pt}{}{x}{y}$ is interpreted as x being preferred to y and $\genfrac{}{}{0pt}{}{y}{x}$ is y being preferred to x. We denote the set of sophisticated preferences over $\{x,y\}$ as Σ^{xy}.[10] A *pairwise SSWF* (defined over $\{x,y\}$) is a mapping $f : \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^n \to \Sigma^{xy}$. So at each $\underline{r} \in \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^n$, $f(\underline{r}) \in \Sigma^{xy}$ is a sophisticated preference over $\{x,y\}$ which, by a slight abuse of notation, we denote $f_{\underline{r}}$. Given any $\underline{\pi} \in \Pi^N$ and any distinct $x, y \in A$, we write $\underline{\pi}^{xy} \in \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^n$ for the restriction of $\underline{\pi}$ to $\{x,y\}$ where for each $i \in N$, we have $\pi_i^{xy} = \genfrac{}{}{0pt}{}{x}{y}$ iff $x \; \pi_i \; y$.[11] Thus, every SSWF $\alpha : \Pi^N \to \Sigma$ satisfying IIA can equivalently be expressed in terms of a family of pairwise SSWFs $\{f^{xy}\}$ indexed over all distinct pairs $\{x,y\}$ such that given any $\underline{\pi} \in \Pi^N$ and any (distinct) $x, y \in A$ we have $f^{xy}_{\underline{\pi}^{xy}}(x,y) = \alpha_{\underline{\pi}}(x,y)$. Note that f^{xy} and f^{yx} are by definition the same. We extend the notion of sameness to any two pairwise SSWFs $f : \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^n \to \Sigma^{xy}$ and $g : \{\genfrac{}{}{0pt}{}{z}{t}, \genfrac{}{}{0pt}{}{t}{z}\}^n \to \Sigma^{zt}$ with $\{x,y\} \neq \{z,t\}$, by qualifying f and g as the *same* whenever there exists a bijection $\beta : \{x,y\} \longrightarrow \{z,t\}$ such that $f_{\underline{r}}(x,y) = g_{\underline{s}}(\beta(x), \beta(y))$ $\forall \underline{r} \in \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^n$, $\forall \underline{s} \in \{\genfrac{}{}{0pt}{}{z}{t}, \genfrac{}{}{0pt}{}{t}{z}\}^n$ with $r_i = \genfrac{}{}{0pt}{}{x}{y} \iff s_i = \genfrac{}{}{0pt}{}{\beta(x)}{\beta(y)}$ $\forall i \in N$.

f^{xy} and f^{yx} being the same does not mean an equal treatment of x and y which is ensured by the following neutrality condition: We say that $f^{xy} : \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^n \to \Sigma^{xy}$ is *neutral* iff $f^{xy}_{\underline{r}}(x,y) = f^{xy}_{\underline{s}}(y,x)$ $\forall \underline{r}, \underline{s} \in \{\genfrac{}{}{0pt}{}{x}{y}, \genfrac{}{}{0pt}{}{y}{x}\}^n$ with $r_i = \genfrac{}{}{0pt}{}{x}{y} \iff s_i = \genfrac{}{}{0pt}{}{y}{x}$ $\forall i \in N$.

[9] Remark that a *social welfare function (SWF)* - as defined by Arrow (1951)- is a SSWF $\alpha : \Pi^N \to \Pi$. Moreover, for such SSWFs, the definitions of PO and IIA coincide with their original definitions made for SWFs. Hence our framework generalizes the Arrovian aggregation model.

[10] A sophisticated preference is originally defined for a set of alternatives whose cardinality is at least three while it can be easily adapted for doubletons: For every $\sigma \in \Sigma^{xy}$, we have $\sigma(x,y) \in [0,1]$, $\sigma(x,y) + \sigma(y,x) = 1$, $\sigma(x,x) = 0$ and $\sigma(y,y) = 0$.

[11] So $\pi_i^{xy} = \genfrac{}{}{0pt}{}{y}{x}$ if and only if $y \; \pi_i \; x$.

3 Results

Proposition 3.1 *A PO and IIA SSWF* $\{f^{xy}\} : \Pi^N \to \Sigma$ *uses the same neutral pairwise SSWF for each pair* $x, y \in A$.

Proof. Let $\{f^{xy}\} = \alpha$ be a PO and IIA SSWF. We first show that f^{ab} and f^{ac} are the same for any distinct $a, b, c \in A$. Consider the bijection $\beta : \{a, b\} \longrightarrow \{a, c\}$ defined as $\beta(a) = a$ and $\beta(b) = c$. We will show $f^{ab}_{\underline{r}}(a, b) = f^{ac}_{\underline{s}}(a, c)$ for all $\underline{r} \in \{\genfrac{}{}{0pt}{}{a}{b}, \genfrac{}{}{0pt}{}{b}{a}\}^n$ and for all $\underline{s} \in \{\genfrac{}{}{0pt}{}{a}{c}, \genfrac{}{}{0pt}{}{c}{a}\}^n$ with $r_i = \genfrac{}{}{0pt}{}{a}{b} \iff s_i = \genfrac{}{}{0pt}{}{a}{c}$ $\forall i \in N$. In case $r_i = \genfrac{}{}{0pt}{}{a}{b}$ $\forall\, i \in N$, hence $s_i = \genfrac{}{}{0pt}{}{a}{c}$ $\forall\, i \in N$, we have $f^{ab}_{\underline{r}}(a, b) = f^{ac}_{\underline{s}}(a, c) = 1$, by PO. Now consider the case where for some (non-trivial) partition $\{K, N\backslash K\}$ of N we have $r_i = \genfrac{}{}{0pt}{}{a}{b}$ for all $i \in K$ and $r_i = \genfrac{}{}{0pt}{}{b}{a}$ for all $i \in N\backslash K$. To see $f^{ab}_{\underline{r}}(a, b) = f^{ac}_{\underline{s}}(a, c)$, suppose for a contradiction and without loss of generality that $f^{ab}_{\underline{r}}(a, b) > f^{ac}_{\underline{s}}(a, c)$. Take some $\underline{\pi} \in \Pi^N$ such that $\pi_i \in \Pi(a, b) \cap \Pi(b, c)$ for all $i \in K$ and $\pi_i \in \Pi(b, c) \cap \Pi(c, a)$ for all $i \in N\backslash K$. Note that $b\ \pi_i\ c$ holds for all $i \in N$. So by PO we have $\alpha_{\underline{\pi}}(b, c) = 1$ and the transitivity of $\alpha_{\underline{\pi}}$ implies $\alpha_{\underline{\pi}}(b, a) \geq \alpha_{\underline{\pi}}(c, a)$, which in turn implies $\alpha_{\underline{\pi}}(a, b) \leq \alpha_{\underline{\pi}}(a, c)$. As $\underline{\pi}^{ab} = \underline{r}$ and $\underline{\pi}^{ac} = \underline{s}$, we have $f^{ab}_{\underline{r}}(a, b) \leq f^{ac}_{\underline{s}}(a, c)$, giving the desired contradiction. Now take any distinct $a, b, c, d \in A$. f^{ab} and f^{ac} are the same, f^{ac} and f^{cd} are the same, hence f^{ab} and f^{cd} are the same.

We now show the neutrality of f^{ab}. Suppose, for a contradiction, that f^{ab} fails neutrality. So there exists $\underline{r}, \underline{r}' \in \{\genfrac{}{}{0pt}{}{a}{b}, \genfrac{}{}{0pt}{}{b}{a}\}^n$ with $r_i = \genfrac{}{}{0pt}{}{a}{b} \iff r'_i = \genfrac{}{}{0pt}{}{b}{a}$ $\forall i \in N$ while $f^{ab}_{\underline{r}}(a, b) \neq f^{ab}_{\underline{r}'}(b, a)$, say $f^{ab}_{\underline{r}}(a, b) < f^{ab}_{\underline{r}'}(b, a)$, without loss of generality. Let $K = \{i \in N : r_i = \genfrac{}{}{0pt}{}{a}{b}\}$. As f^{ab} and f^{ac} are the same, we have $f^{ac}_{\underline{s}}(a, c) = f^{ab}_{\underline{r}}(a, b)$ for $\underline{s} \in \{\genfrac{}{}{0pt}{}{a}{c}, \genfrac{}{}{0pt}{}{c}{a}\}^n$ with $s_i = \genfrac{}{}{0pt}{}{a}{c} \iff i \in K$. As f^{ab} and f^{bc} are the same, we have $f^{bc}_{\underline{t}}(b, c) = f^{ab}_{\underline{r}'}(b, a)$ for $\underline{t} \in \{\genfrac{}{}{0pt}{}{b}{c}, \genfrac{}{}{0pt}{}{c}{b}\}^n$ with $t_i = \genfrac{}{}{0pt}{}{b}{c} \iff i \in K$. Thus, $f^{ac}_{\underline{s}}(a, c) < f^{bc}_{\underline{t}}(b, c)$. Take some $\underline{\pi} \in \Pi^N$ such that $\pi_i \in \Pi(a, b) \cap \Pi(b, c)$ for all $i \in K$ and $\pi_i \in \Pi(c, a) \cap \Pi(a, b)$ for all $i \in N\backslash K$. Note that $a\ \pi_i\ b$ holds for all $i \in N$. So by PO we have $\alpha_{\underline{\pi}}(a, b) = 1$ and the transitivity of $\alpha_{\underline{\pi}}$ implies $\alpha_{\underline{\pi}}(a, c) \geq \alpha_{\underline{\pi}}(b, c)$. As $\underline{\pi}^{ac} = \underline{s}$ and $\underline{\pi}^{bc} = \underline{t}$, we have $f^{ac}_{\underline{s}}(a, c) \geq f^{bc}_{\underline{t}}(b, c)$, contradicting $f^{ac}_{\underline{s}}(a, c) < f^{bc}_{\underline{t}}(b, c)$. Hence, f^{ab} is neutral. ■

So by Proposition 3.1, any PO and IIA SSWF $\alpha : \Pi^N \to \Sigma$ is *globally neutral*, i.e., it can be expressed in terms of a single neutral pairwise SSWF

$f : \{{x \atop y},{y \atop x}\}^n \to \Sigma^{xy}$. We now show that f must be *monotonic*, i.e., $\forall \underline{r}, \underline{r}' \in \{{x \atop y},{y \atop x}\}^n$ with $r'_i = {x \atop y} \Longrightarrow r_i = {x \atop y}$ $\forall\ i \in N$, we have $f_{\underline{r}}(x,y) \geq f_{\underline{r}'}(x,y)$.

Proposition 3.2 *Take any PO and IIA SSWF $\alpha : \Pi^N \to \Sigma$. If $f : \{{x \atop y},{y \atop x}\}^n \to [0,1]$ is the pairwise SSWF through which α is expressed, then f is monotonic.*

Proof. Take any PO and IIA SSWF α and let f be the pairwise SSWF which expresses α. Suppose f fails monotonicity. So there exists $\underline{r}, \underline{r}' \in \{{x \atop y},{y \atop y}\}^n$ with $r'_i = {x \atop y} \Longrightarrow r_i = {x \atop y}$ $\forall\ i \in N$ while $f_{\underline{r}}(x,y) < f_{\underline{r}'}(x,y)$. Let $K = \{i \in N : r_i = {x \atop y}\}$ and $L = \{i \in N : r'_i = {x \atop y}\}$. Note that $L \subset K$. Take any distinct $a, b, c \in A$ and any $\underline{\pi} \in \Pi^N$ such that $\pi_i \in \Pi(a,b) \cap \Pi(b,c)$ $\forall i \in L$, $\pi_i \in \Pi(a,c) \cap \Pi(c,b)$ $\forall i \in K \backslash L$ and $\pi_i \in \Pi(c,a) \cap \Pi(a,b)$ $\forall i \in N \backslash K$. By PO, we have $\alpha_{\underline{\pi}}(a,b) = 1$. As α and f are equivalent and by the choice of $\underline{\pi}$, we have $\alpha_{\underline{\pi}}(a,c) = f_{\underline{r}}(a,c)$ and $\alpha_{\underline{\pi}}(b,c) = f_{\underline{r}'}(b,c)$. Thus, $\alpha_{\underline{\pi}}(a,c) < \alpha_{\underline{\pi}}(b,c)$, violating the transitivity of $\alpha_{\underline{\pi}}$. ■

We now show that PO and IIA SSWFs fall into a class that we call "oligarchical" SSWFs. We say that a SSWF $\alpha : \Pi^N \to \Sigma$ is *oligarchical* iff there exists a nonempty coalition $O \subseteq N$ (to which we refer as the *oligarchy*) such that for any distinct $x, y \in A$ and any $\underline{\pi} \in \Pi^N$, we have $\alpha_{\underline{\pi}}(x,y) > 0 \iff \exists\ i \in O$ such that $x\ \pi_i\ y$.

Remark 3.1 *We define oligarchy as some kind of veto power given to individuals in a specific group. As each sophisticated preference satisfies $\sigma(x,y) + \sigma(y,x) = 1$, this veto power of individuals suffices to endow the group with full power when unanimity prevails among its members. To state this formally, take any oligarchical and IIA SSWF $\alpha : \Pi^N \to \Sigma$ expressed by the pairwise SSWF $f : \{{x \atop y},{y \atop x}\}^n \to [0,1]$.*[12] *Let $O \subseteq N$ be the oligarchy that f induces. Given any distinct $x, y \in A$ and any $\underline{r} \in \{{x \atop y},{y \atop x}\}^n$, we have $f_{\underline{r}}(x,y) = 1 \iff r_i = {x \atop y}$ for all $i \in O$.*

Theorem 3.1 *Every PO and IIA SSWF is oligarchical.*

Proof: Take any SSWF $\alpha : \Pi^N \to \Sigma$ which satisfies PO and IIA. We say that a coalition $K \subseteq N$ is *decisive* for $x \in A$ over $y \in A \backslash \{x\}$ if and only

[12] Every oligarchical SSWF is PO. Thus, by Proposition 3.1, an oligarchical and IIA SSWF can be expressed by a single pairwise SSWF.

if at some $\underline{\pi} \in \Pi^N$ with $\pi_i \in \Pi(x,y)$ for all $i \in K$ and $\pi_i \in \Pi(y,x)$ for all $i \in N \backslash K$, we have $\alpha_{\underline{\pi}}(x,y) > 0$.[13]

We prove the theorem through five lemmata about the properties of decisive coalitions.

Lemma 3.1 *If $K \subseteq N$ is decisive for some $a \in A$ over some $b \in A \backslash \{a\}$, then given any distinct $x, y \in A$, K is decisive for $x \in A$ over y.*

Proof. Let $K \subseteq N$ be decisive for some $a \in A$ over some $b \in A \backslash \{a\}$.

Claim 1: Given any $x \in A \backslash \{a,b\}$, K is decisive for a over x. To show the claim, take any $x \in A \backslash \{a,b\}$. Consider a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(a,b) \cap \Pi(b,x)$ for all $i \in K$ and $\pi_i \in \Pi(b,x) \cap \Pi(x,a)$ for all $i \in N \backslash K$. As K is decisive for a over b, we have $\alpha_{\underline{\pi}}(a,b) > 0$. By PO, we have $\alpha_{\underline{\pi}}(b,x) = 1$. Suppose $\alpha_{\underline{\pi}}(x,a) = 1$. The transitivity of $\alpha_{\underline{\pi}}$ implies $\alpha_{\underline{\pi}}(b,a) = 1$, contradicting $\alpha_{\underline{\pi}}(a,b) > 0$. Thus, $\alpha_{\underline{\pi}}(x,a) < 1$, which means $\alpha_{\underline{\pi}}(a,x) > 0$, showing that K is decisive for a over x as well.

Claim 2: Given any $x \in A \backslash \{a,b\}$, K is decisive for x over b. To show the claim, take any $x \in A \backslash \{a,b\}$. Consider a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(x,a) \cap \Pi(a,b)$ for all $i \in K$ and $\pi_i \in \Pi(b,x) \cap \Pi(x,a)$ for all $i \in N \backslash K$. As K is decisive for a over b, then $\alpha_{\underline{\pi}}(a,b) > 0$. By PO, we have $\alpha_{\underline{\pi}}(x,a) = 1$ and, by transitivity of $\alpha_{\underline{\pi}}$, we have $\alpha_{\underline{\pi}}(x,b) > 0$, showing that K is decisive for x over b as well.

Now take any distinct $x, y \in A$ and consider the following three exhaustive cases:

CASE 1: $x \in A \backslash \{b\}$. By Claim 2, K is decisive for x over b and by Claim 1 K is decisive for x over y.

CASE 2: $x = b$ and $y \in A \backslash \{a\}$. By Claim 1, K is decisive for a over y and by Claim 2 K is decisive for x over y.

CASE 3: $x = b$ and $y = a$. Take some $z \in A \backslash \{a,b\}$. By Claim 1, K is decisive for a over z; by Claim 2 K is decisive for b over z and by Claim 1 K is decisive for a over b. ∎

We call a coalition $K \subseteq N$ *decisive* iff given any distinct $x, y \in A$, K is decisive for x over y. [14]

Lemma 3.2 *Given any disjoint $K, L \subseteq N$ which are both not decisive, $K \cup L$ is not decisive either.*

Proof. Take any disjoint $K, L \subseteq N$ which are both not decisive. Consider distinct $x, y, z \in A$. Pick a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(x,z) \cap \Pi(z,y)$ for

[13] As α satisfies IIA, the definition can be equivalently stated for all $\underline{\pi} \in \Pi^N$.

[14] Remark that the decisiveness of K does not rule out the decisiveness of $N \backslash K$.

all $i \in K$, $\pi_i \in \Pi(z,y) \cap \Pi(y,x)$ for all $i \in L$ and $\pi_i \in \Pi(y,x) \cap \Pi(x,z)$ for all $i \in N \backslash (K \cup L)$. As K is not decisive, $\alpha_{\underline{\pi}}(x,y) = 0$. As L is not decisive, $\alpha_{\underline{\pi}}(z,x) = 0$. The transitivity of $\alpha_{\underline{\pi}}$ implies $\alpha_{\underline{\pi}}(y,z) = 1$, thus $\alpha_{\underline{\pi}}(z,y) = 0$, showing that $K \cup L$ is not decisive. ■

Lemma 3.3 *Take any $K \subseteq N$ which is decisive. For all $L \subset K$, L or $K\backslash L$ is decisive.*

Proof. Take any $K \subseteq N$ which is decisive and any $L \subset K$. Suppose neither L nor $K\backslash L$ is decisive. But by Lemma 3.2, $L \cup (K \setminus L) = K$ is not decisive either, which contradicts that K is decisive. ■

Lemma 3.4 *If $K \subseteq N$ is decisive then any $L \supseteq K$ is also decisive.*

Proof. Take any $K \subseteq N$ which is decisive and any $L \supseteq K$. Consider distinct $x, y, z \in A$. Pick a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(z,x) \cap \Pi(x,y)$ for all $i \in K$, $\pi_i \in \Pi(z,y) \cap \Pi(y,x)$ for all $i \in L\backslash K$ and $\pi_i \in \Pi(y,z) \cap \Pi(z,x)$ for all $i \in N\backslash L$. As K is decisive, we have $\alpha_{\underline{\pi}}(x,y) > 0$. By PO, we have $\alpha_{\underline{\pi}}(z,x) = 1$. By the transitivity of $\alpha_{\underline{\pi}}$, we have $\alpha_{\underline{\pi}}(z,y) > 0$, showing that L is decisive for z over y, hence by Lemma 3.1 decisive. ■

Let $\Delta \subseteq 2^N$ stand for the set of decisive coalitions.

Lemma 3.5 *There exists $O \in 2^N \backslash \{\emptyset\}$ such that given any $K \in 2^N$, we have $K \in \Delta$ if and only if $K \cap O \neq \emptyset$.*

Proof. By PO, we have $N \in \Delta$. Applying Lemma 3.3 successively and by the finiteness of N, the set $O = \{i \in N : \{i\} \in \Delta\}$ is non-empty. Now take any $K \in 2^N$. If $K \cap O \neq \emptyset$, then $K \supseteq \{i\}$ for some $\{i\} \in \Delta$, so by Lemma 3.4, $K \in \Delta$ as well. If $K \in \Delta$, then again by applying Lemma 3.3 successively and by the finiteness of K, there exists $i \in K$ such that $\{i\} \in \Delta$, hence $i \in O$, establishing that $K \cap O \neq \emptyset$. ■

We complete the proof of Theorem 3.1, by asking the reader to check that the coalition O defined in Lemma 3.5 is the oligarchy which makes α oligarchical. Q.E.D.

Remark that the converse statement of Theorem 3.1 does not hold. For, although an oligarchical SSWF is PO, it need not satisfy IIA.[15] To transform Theorem 3.1 into a full characterization, we need to know more about IIA

[15] To see this, consider the following Example 1 where $N = \{1,2,3\}$ and $A = \{a,b,c\}$. Take any $\underline{\pi} \in \Pi^N$ and any distinct $x, y \in A$. If $\underline{\pi}$ admits a Condorcet winner, i.e., $\exists\ c \in A$ such that for each $z \in A\backslash\{c\}$, $\#\{i \in N : c\ \pi_i\ z\} \geq 2$, then let $\alpha_{\underline{\pi}}(x,y) = \#\{i \in N : x\ \pi_i y\}/3$. If $\underline{\pi}$ admits no Condorcet winner, then let $\alpha_{\underline{\pi}}(x,y) = \frac{1}{2}$. One can check that α exemplifies a SSWF which is oligarchical but not IIA.

and oligarchical SSWFs. So we proceed by showing that under IIA and oligarchical SSWFs, the social outcome depends only on the preferences of the oligarchy members.

Proposition 3.3 *Take any oligarchical and IIA SSWF* $\alpha : \Pi^N \to \Sigma$ *expressed by the pairwise SSWF* $f : \{\begin{matrix} x \\ y \end{matrix}, \begin{matrix} y \\ x \end{matrix}\}^n \to [0,1]$*. Let* $O \subseteq N$ *be the oligarchy that* f *induces. Given any* $\underline{r}, \underline{r}' \in \{\begin{matrix} x \\ y \end{matrix}, \begin{matrix} y \\ x \end{matrix}\}^n$ *with* $r_i = \begin{matrix} x \\ y \end{matrix} \iff r_i' = \begin{matrix} x \\ y \end{matrix}$ $\forall i \in O$*, we have* $f_{\underline{r}} = f_{\underline{r}'}$*.*

Proof. Let α, f, and O be as in the statement of the proposition. Take any $\underline{r}, \underline{r}' \in \{\begin{matrix} x \\ y \end{matrix}, \begin{matrix} y \\ x \end{matrix}\}^n$ with $r_i = \begin{matrix} x \\ y \end{matrix} \iff r_i' = \begin{matrix} x \\ y \end{matrix}$ $\forall i \in O$. Let $O_{\underline{r}}^{xy} = \{i \in O : r_i = \begin{matrix} x \\ y \end{matrix}\}$, $O_{\underline{r}}^{yx} = \{i \in O : r_i = \begin{matrix} y \\ x \end{matrix}\}$, $\overline{O}_{\underline{r}}^{xy} = \{i \in N \backslash O : r_i = \begin{matrix} x \\ y \end{matrix}\}$ and $\overline{O}_{\underline{r}}^{yx} = \{i \in N \backslash O : r_i = \begin{matrix} y \\ x \end{matrix}\}$. Take any distinct $a, b, c \in A$ and pick some $\underline{\pi} \in \Pi^N$ such that $\pi_i \in \Pi(a,c) \cap \Pi(c,b)$ for all $i \in O_{\underline{r}}^{xy}$, $\pi_i \in \Pi(c,b) \cap \Pi(b,a)$ for all $i \in O_{\underline{r}}^{yx} \cup \overline{O}_{\underline{r}}^{yx}$, $\pi_i \in \Pi(a,b) \cap \Pi(b,c)$ for all $i \in \overline{O}_{\underline{r}}^{xy}$. By the choice of π we have $\pi^{ab} = \pi^{ac} = \underline{r}$, implying $\alpha_{\underline{\pi}}(a,c) = \alpha_{\underline{\pi}}(a,b)$. Now take some $\underline{\pi}' \in \Pi^N$ such that $\pi_i' = \pi_i$ $\forall i \in O$ and $a\ \pi_i'\ c \iff a\ \pi_i\ c$ $\forall i \in N \backslash O$. Thus $\alpha_{\underline{\pi}'}(c,a) = \alpha_{\underline{\pi}}(c,a)$. As $c\ \pi_i'\ b$ $\forall i \in O$, by Remark 3.1, we have $\alpha_{\underline{\pi}'}(c,b) = 1$ and the transitivity of $\alpha_{\underline{\pi}'}$ implies $\alpha_{\underline{\pi}'}(c,a) = \alpha_{\underline{\pi}}(c,a) \geq \alpha_{\underline{\pi}'}(b,a)$. Now pick some $\underline{\pi}'' \in \Pi^N$ such that $\pi_i'' = \pi_i'$ $\forall i \in N \backslash O$ and $b\ \pi_i''\ c \iff c\ \pi_i'\ b$ $\forall i \in O$ while $a\ \pi_i''\ x \iff a\ \pi_i'\ x$ $\forall x \in \{b,c\}$ $\forall i \in O$. Note that $a\ \pi_i''\ c \iff a\ \pi_i'\ c$ $\forall i \in O$. Thus $\alpha_{\underline{\pi}''}(c,a) = \alpha_{\underline{\pi}'}(c,a)$. As $b\ \pi_i''\ c$ $\forall i \in O$, by Remark 3.1, we have $\alpha_{\underline{\pi}''}(b,c) = 1$ and the transitivity of $\alpha_{\underline{\pi}''}$ implies $\alpha_{\underline{\pi}''}(b,a) \geq \alpha_{\underline{\pi}''}(c,a) = \alpha_{\underline{\pi}'}(c,a)$. Noting $\alpha_{\underline{\pi}''}(b,a) = \alpha_{\underline{\pi}'}(b,a)$, we establish $\alpha_{\underline{\pi}'}(b,a) = \alpha_{\underline{\pi}}(c,a) = \alpha_{\underline{\pi}}(b,a)$, completing the proof. ■

We define a *power distribution* in the society as a mapping $\omega : 2^N \to [0,1]$ such that $\omega(K) + \omega(N \backslash K) = 1$ for all $K \in 2^N$. We consider *monotonic* power distributions which satisfy $\omega(K) \leq \omega(L)$ for all $K, L \in 2^N$ with $K \subseteq L$ while $\omega(N) = 1$. We qualify a monotonic power distribution ω as *oligarchical* iff $\omega(L) = 0 \implies \omega(K \cup L) = \omega(K)$ $\forall K, L \in 2^N$. Remark that when ω is oligarchical, the set $\{i \in N : \omega(\{i\}) > 0\}$ is non-empty. Moreover, $\omega(K) = 0$ $\forall K \in 2^N$ with $K \cap \{i \in N : \omega(\{i\}) > 0\} = \emptyset$.

Lemma 3.6 *Any oligarchical power distribution* $\omega : 2^N \to [0,1]$ *induces a PO and IIA SSWF* $\alpha : \Pi^N \to \Sigma$ *which is defined as* $\alpha_{\underline{\pi}}(x,y) = \omega(\{i \in N : \pi_i \in \Pi(x,y)\})$ $\forall\ \underline{\pi} \in \Pi^N$*,* $\forall x, y \in A$*. Moreover* α *is oligarchical where* $O = \{i \in N : \omega(\{i\}) > 0\}$ *is the oligarchy.*

Proof. Let ω and α be as in the statement of the lemma. Take any $\underline{\pi} \in \Pi^N$ and any $x, y \in A$. If x and y are not distinct, then $\alpha_{\underline{\pi}}(x, x) = 0$ holds by the irreflexivity of individual preferences and the fact that $\omega(\emptyset) = 0$. If x, y are distinct, then the definition of a power distribution implies $\alpha_{\underline{\pi}}(x, y) \in [0, 1]$ and $\alpha_{\underline{\pi}}(x, y) + \alpha_{\underline{\pi}}(y, x) = 1$. So $\alpha_{\underline{\pi}}$ is a sophisticated preference. To see the transitivity of $\alpha_{\underline{\pi}}$, take any $x, y, z \in A$ with $\sigma(x, y) = 1$. Let $K_1 = \{i \in N : \pi_i \in \Pi(x, y) \cap \Pi(y, z)\}$, $K_2 = \{i \in N : \pi_i \in \Pi(x, z) \cap \Pi(z, y)\}$, $K_3 = \{i \in N : \pi_i \in \Pi(z, x) \cap \Pi(x, y)\}$, $L_1 = \{i \in N : \pi_i \in \Pi(y, x) \cap \Pi(x, z)\}$, $L_2 = \{i \in N : \pi_i \in \Pi(y, z) \cap \Pi(z, x)\}$ and $L_3 = \{i \in N : \pi_i \in \Pi(z, y) \cap \Pi(y, x)\}$. Note that $\{K_1, K_2, K_3, L_1, L_2, L_3\}$ is a partition of N. Moreover, the way ω induces α implies $\alpha_{\underline{\pi}}(x, y) = \omega(K_1 \cup K_2 \cup K_3) = 1$, $\alpha_{\underline{\pi}}(y, z) = \omega(K_1 \cup L_1 \cup L_2)$ and $\alpha_{\underline{\pi}}(x, z) = \omega(K_1 \cup K_2 \cup L_1)$. As $\omega(K_1 \cup K_2 \cup K_3) = 1$, $\omega(L_1 \cup L_2 \cup L_3) = 0$ and by the monotonicity of ω, we have $\omega(L) = 0$ for all $L \subseteq L_1 \cup L_2 \cup L_3$. As ω is oligarchical, $\alpha_{\underline{\pi}}(y, z) = \omega(K_1 \cup L_1 \cup L_2) = \omega(K_1)$ and $\alpha_{\underline{\pi}}(x, z) = \omega(K_1 \cup K_2 \cup L_1) = \omega(K_1 \cup K_2)$ and the monotonicity of ω implies $\alpha_{\underline{\pi}}(x, z) \geq \alpha_{\underline{\pi}}(y, z)$, showing the transitivity of $\alpha_{\underline{\pi}}$. Thus, α is a SSWF. Checking that α is PO, IIA and oligarchical is left to the reader. ■

So every oligarchical power distribution ω generates a PO and IIA SSWF α where at each $\underline{\pi} \in \Pi^N$, the weight by which x is socially preferred to y equals to the power of the coalition of individuals who prefer x to y at $\underline{\pi}$.[16] We refer to α as the $\omega-$oligarchical SSWF with $O = \{i \in N : \omega(\{i\}) > 0\}$ being the corresponding oligarchy.

We now state our central result which is the characterization of PO and IIA SSWFs in terms of $\omega-$oligarchical SSWFs.

Theorem 3.2 *A SSWF $\alpha : \Pi^N \to \Sigma$ is PO and IIA if and only if α is $\omega-$oligarchical for some oligarchical power distribution ω.*

Proof. The "if" part follows from Lemma 3.6. To see the "only if" part, recall that by Proposition 3.1, α can be expressed in terms of a single neutral pairwise SSWF f. On the other hand, f can be expressed in terms of a *value function* $v : 2^N \to [0, 1]$ which is defined for each $K \in 2^N$ as $v(K) = f_{\underline{r}}(x, y)$ where $x, y \in A$ is an arbitrarily chosen distinct pair while $\underline{r} \in \{\begin{smallmatrix} x \\ y \end{smallmatrix}, \begin{smallmatrix} y \\ x \end{smallmatrix}\}$ is such that $r_i = \begin{smallmatrix} x \\ y \end{smallmatrix}$ $\forall i \in K$ and $r_i = \begin{smallmatrix} y \\ x \end{smallmatrix}$ $\forall i \in N \backslash K$. The fact that $f_{\underline{r}}(x, y) + f_{\underline{r}}(x, y) = 1$ for any distinct $x, y \in A$ and any $\underline{r} \in \{\begin{smallmatrix} x \\ y \end{smallmatrix}, \begin{smallmatrix} y \\ x \end{smallmatrix}\}$ results in v being a power

[16] Remark the ω being oligarchical is critical for Lemma 3.6 to hold. To see this let $N = \{1, 2, 3\}$ and consider the monotonic power distribution $\omega(\{i\}) = 0$ $\forall i \in N$ and $\omega(K) = 1$ $\forall K \in 2^N$ with $\#K > 1$. Picking some distinct $x, y, z \in A$, one can check that $\alpha_{\underline{\pi}}$ fails transitivity at $\underline{\pi} \in \Pi^N$ where $\pi_1 \in \Pi(x, y) \cap \Pi(y, z)\}$, $\pi_2 \in \Pi(y, z) \cap \Pi(z, x)\}$ and $\pi_3 \in \Pi(z, x) \cap \Pi(x, y)\}$.

distribution. Moreover, v is a monotonic by Proposition 3.2 and oligarchical by Proposition 3.3. As v and f uniquely determine each other, v is an oligarchical power distribution that induces α. ■

We now give a few examples of $\omega-$oligarchical SSWFs:

- *A dictatorial SSWF* is $\omega-$oligarchical with some $d \in N$ such that $\omega(K) = 1$ for all $K \in 2^N$ with $d \in K$. Remark that Π is the range of dictatorial SSWFs which consequently are social welfare functions as defined by Arrow (1951). In fact, dictatorial SSWFs are the only $\omega-$oligarchical SSWFs which coincide with this standard Arrovian definition - a matter which we discuss in the proof of Theorem 3.4.

- *A Gibbard – oligarchical SSWF* is $\omega-$oligarchical with some $O \in 2^N \backslash \{\varnothing\}$ such that $\omega(K) = 1$ for all $K \in 2^N$ with $O \subseteq K$, $\omega(K) = \frac{1}{2}$ for all $K \in 2^N$ with $K \cap O \neq \varnothing$ but $O \nsubseteq K$, and $\omega(K) = 0$ for all $K \in 2^N$ with $K \cap O = \varnothing$. Remark that in case $\#O > 1$, the range of a *Gibbard – oligarchical* SSWF is $Q = \{\rho \in \Sigma$ such that $\rho : A \times A \to \{0, \frac{1}{2}, 1\}\}$ which is indeed the set of connected, irreflexive and quasi-transitive binary relations over A.[17] It is straightforward to check that what we call *Gibbard – oligarchical* SSWFs are oligarchical social welfare functions as defined by Gibbard (1969).

- The *equal power $\omega-$oligarchical* SSWF is defined by setting $\omega(K) = \frac{\#K}{\#N}$ for all $K \in 2^N$.

Remark that the equal power $\omega-$oligarchical SSWF as well as the *Gibbard–oligarchical* SSWF where N is set as the oligarchy are anonymous SSWFs.[18] In fact, anonymous $\omega-$oligarchical SSWFs can be characterized in terms of the following anonymity condition we impose over power distributions: We say that a power distribution $\omega : 2^N \to [0, 1]$ is *anonymous* iff given any $K, L \in 2^N$ with $\#K = \#L$ we have $\omega(K) = \omega(L)$.

Proposition 3.4 *An $\omega-$oligarchical SSWF $\alpha : \Pi^N \to \Sigma$ is anonymous if and only if ω is an anonymous power distribution.*

[17] We say this by interpreting $\rho(x, y) = 1$ as x being preferred to y and $\rho(x, y) = \frac{1}{2}$ as indifference between x and y, both terms carrying their usual meanings. Write $x \ \rho \ y$ whenever $\rho(x, y) \geq \frac{1}{2}$ and $x \ \rho^* \ y$ whenever $\rho(x, y) = 1$. In this case, for any $\rho \in Q$ and any distinct $x, y \in A$, we have $x \ \rho \ y$ or $y \ \rho \ x$ while $x \ \rho \ x$ holds for no x in A. Moreover $x \ \rho^* \ y$ and $y \ \rho^* \ z$ implies $x \ \rho^* \ z$ for all $x, y, z \in A$.

[18] As usual, we say that a SSWF $\alpha : \Pi^N \to \Sigma$ is *anonymous* iff given any $(\pi_1, ..., \pi_{\#N}) \in \Pi^N$ and any bijection $\psi : N \longleftrightarrow N$, we have $\alpha(\pi_1, ..., \pi_{\#N}) = \alpha(\pi_{\psi(1)}, ..., \pi_{\psi(\#N)})$.

Proof. The "if" part is left to the reader. To show the "only if" part, let ω be such that $\omega(K) \neq \omega(L)$ for some $K, L \in 2^N$ with $\#K = \#L$. Take any distinct $x, y \in A$ and consider a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(x, y)$ for all $i \in K$ and $\pi_j \in \Pi(y, x)$ for all $j \in N \backslash K$. So $\alpha_{\underline{\pi}}(x, y) = \omega(K)$. Now take any bijection $\psi : N \longleftrightarrow N$ with $\{\psi(i)\}_{i \in K} = L$. Let $\underline{\pi}' = \left(\pi_{\psi(1)}, ..., \pi_{\psi(\#N)}\right)$. So $\{i \in N : \pi'_i \in \Pi(x, y)\} = L$, thus $\alpha_{\underline{\pi}'}(x, y) = \omega(L)$, contradicting the anonymity of α. ■

Theorem 3.2 and Proposition 3.4 lead to the following corollary:

Theorem 3.3 *A SSWF $\alpha : \Pi^N \rightarrow \Sigma$ is PO, IIA and anonymous if and only if α is $\omega-$oligarchical for some oligarchical and anonymous power distribution ω.*[19]

We now show how our results lead to the impossibility theorem of Arrow (1951, 1963) and the oligarchy theorem of Gibbard (1969). We start with the former. In fact, the following theorem is a restatement of the Arrovian impossibility.

Theorem 3.4 *A SSWF $\alpha : \Pi^N \rightarrow \Pi$ is PO and IIA if and only if α is $\omega-$oligarchical for some oligarchical power distribution ω inducing an oligarchy O with $\#O = 1$.*

Proof. The "if" part is left to the reader. To show the "only if" part, take any PO and IIA SSWF $\alpha : \Pi^N \rightarrow \Pi$. We know by Theorem 3.2 that α is $\omega-$oligarchical for some oligarchical power distribution ω. Let O be the oligarchy that ω induces. Suppose, for a contradiction, that $\exists$ distinct $i, j \in O$. Fix distinct $x, y \in A$ and consider a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(x, y)$ and $\pi_j \in \Pi(y, x)$. By definition of a $\omega-$oligarchical SSWF, we have $\alpha_{\underline{\pi}}(x, y) > 0$ and $\alpha_{\underline{\pi}}(y, x) > 0$, contradicting that the range of α is Π. ■

The next theorem is a restatement of the oligarchy therorem of Gibbard (1969):

Theorem 3.5 *A SSWF $\alpha : \Pi^N \rightarrow Q$ is PO and IIA if and only if α is $Gibbard-oligarchical$.*

Proof. The "if" part is left to the reader. To show the "only if" part, take any PO and IIA SSWF $\alpha : \Pi^N \rightarrow Q$. We know by Theorem 3.2 that α is $\omega-$oligarchical for some oligarchical power distribution ω. Let O be the oligarchy that ω induces. By the definition of an oligarchy, we have

[19] A power distribution is oligarchical and anonymous only if the oligarchy is N.

$\omega(K) = 1$ for all $K \in 2^N$ with $O \subseteq K$ and $\omega(K) = 0$ for all $K \in 2^N$ with $K \cap O = \varnothing$. Now take any $K \in 2^N$ with $K \cap O \neq \varnothing$ but $O \nsubseteq K$. Fix distinct $x, y \in A$ and consider a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(x, y)$ for all $i \in K$ and $\pi_j \in \Pi(y, x)$ for all $j \in N \backslash K$. By definition of an oligarchy, we have $\alpha_{\underline{\pi}}(x, y) > 0$ and $\alpha_{\underline{\pi}}(y, x) > 0$. As the range of α is Q, it must be the case that $\alpha_{\underline{\pi}}(x, y) = \frac{1}{2}$ and $\alpha_{\underline{\pi}}(y, x) = \frac{1}{2}$, thus leading to $\omega(K) = \frac{1}{2}$, showing that α is $Gibbard - oligarchical$. ■

We close the section by discussing the effects of strengthening transitivity. We say that a sophisticated preference σ is *strongly transitive* iff $\sigma(x, y) = 1$ and $\sigma(y, z) > 0 \implies \sigma(x, z) = 1 \; \forall \; x, y, z \in A$. We write Σ^* for the set of strongly transitive sophisticated preferences. The positive result announced by Theorem 3.2 vanishes under this strengthening.

Theorem 3.6 *A SSWF $\alpha : \Pi^N \to \Sigma^*$ is PO and IIA if and only if α is $\omega-$oligarchical for some oligarchical power distribution ω inducing an oligarchy O with $\#O = 1$.*

Proof. The "if" part is left to the reader. To show the "only if" part, take any PO and IIA SSWF $\alpha : \Pi^N \to \Sigma^*$. We know by Theorem 3.2 that α is $\omega-$oligarchical for some oligarchical power distribution ω. Let O be the oligarchy that ω induces. Suppose, for a contradiction, that $\exists$ distinct $i, j \in O$. Fix distinct $x, y, z \in A$ and consider a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(x, y) \cap \Pi(y, z)$, $\pi_j \in \Pi(z, x) \cap \Pi(x, y)$ and $\pi_k \in \Pi(x, y)$ for all $k \in O \backslash \{i, j\}$. By definition of an oligarchy, we $\alpha_{\underline{\pi}}(x, y) = 1$, $\alpha_{\underline{\pi}}(y, z) > 0$ and $\alpha_{\underline{\pi}}(z, x) > 0$, thus $\alpha_{\underline{\pi}}(x, z) \neq 1$, contradicting that the range of α is Σ^*. ■

Remark that $Q^* = \{\rho \in \Sigma^* \text{ such that } \rho : A \times A \to \{0, \frac{1}{2}, 1\}\}$ is indeed the set of connected, irreflexive and transitive (non-sophisticated) preferences over A. In other words, strong transitivity of sophisticated preferences is reflected to non-sophisticated preferences as the standard transitivity condition. On the other hand, we must not be tempted to think that the positive result announced by Theorem 3.2 is merely due to the use of a relatively weaker transitivity. For, there exists other strengthenings of transitivity which are again reflected to non-sophisticated preferences as transitivity while they still allow for non-dictatorial SSWFs. As a case in point, consider the following condition $T^\Diamond$ to be imposed over sophisticated preferences: $\sigma(x, y) = \sigma(y, z) = \frac{1}{2} \implies \sigma(x, z) = \frac{1}{2} \; \forall \; x, y, z \in A$. Let $\Sigma^\Diamond = \{\sigma \in \Sigma : \sigma$ satisfies $T^\Diamond\}$ be the set of transitive sophisticated preferences that satisfy $T^\Diamond$. In spite of the fact that Σ^* and $\Sigma^\Diamond$ are not subsets of each other, we have $Q^\Diamond = \{\rho \in \Sigma^\Diamond \text{ such that } \rho : A \times A \to \{0, \frac{1}{2}, 1\}\}$ which is also the set of connected, irreflexive and transitive binary relations over A. Nevertheless, the positive result announced by Theorem 3.2 essentialy prevails over $\Sigma^{\Diamond}$, as the following theorem states:

Theorem 3.7 *A SSWF $\alpha : \Pi^N \rightarrow \Sigma^{\Diamond}$ is PO and IIA if and only if α is $\omega-$oligarchical for some oligarchical power distribution ω with $\omega(K) \neq \frac{1}{2}$ $\forall K \in 2^N$.*

Proof. To see the "if" part, take any oligarchical power distribution ω with $\omega(K) \neq \frac{1}{2}$ $\forall K \in 2^N$. We know by Theorem 3.2 that ω induces an $\omega-$oligarchical SSWF $\alpha : \Pi^N \rightarrow \Sigma$. Moreover, as $\omega(K) \neq \frac{1}{2}$ $\forall K \in 2^N$, $\alpha_{\underline{\pi}}$ trivially satisfies condition $T^{\Diamond}$ at each $\underline{\pi} \in \Pi^N$, thus restricting the range of α to $\Sigma^{\Diamond}$. To see the "only if", take any PO and IIA SSWF $\alpha : \Pi^N \rightarrow \Sigma^{\Diamond}$. By Theorem 3.2, α is $\omega-$oligarchical for some oligarchical power distribution ω. Suppose $\omega(K) = \frac{1}{2}$ for some $K \in 2^N$. Fix distinct $x, y, z \in A$ and consider a profile $\underline{\pi} \in \Pi^N$ where $\pi_i \in \Pi(x,y) \cap \Pi(y,z)$ for all $i \in K$ and $\pi_i \in \Pi(z,x) \cap \Pi(x,y)$ for all $i \in N \backslash K$. As α is induced by ω and $\omega(K) = \frac{1}{2}$, we have $\alpha_{\underline{\pi}}(x,z) = \frac{1}{2}$ and $\alpha_{\underline{\pi}}(y,z) = \frac{1}{2}$ while $\alpha_{\underline{\pi}}(x,y) = 1$ by PO, thus contradicting that $\Sigma^{\Diamond}$ is the range of α. ■

4 Concluding Remarks

We show that the class of Pareto optimal and IIA SSWFs coincides with the family of weighted oligarchies, with dictatorial rules at one end and anonymous rules at the other. Thus, it is possible to aggregate profiles of rankings into a sophisticated preference by distributing power equally in the society. Whether this is desirable or not is another matter which depends on the interpretation of the model. Anonymity is certainly defendable under the social choice interpretation where preferences of distinct individuals are aggregated into a social preference. On the other hand, viewing the model as in individual decision making problem where an individual aggregates vectors of rankings according to various criteria into an overall preference, it may make sense to propose an unequal power distribution among criteria - such as a job market candidate who may weigh the salary more than the location of the university. In any case, our findings announce the possibility of designing anonymous aggregation rules while staying within the class Pareto optimal and IIA aggregation rules.[20] This is in contrast to the generally negative findings on aggregating fuzzy preferences, such as Barrett et al. (1986), Dutta (1987) and Banerjee (1994) who establish various fuzzy counterparts of the

[20]Whether this possibility prevails when individual preferences are also allowed to be sophisticated is an open question to pursue.

Arrovian impossibility. In particular Banerjee (1994) shows that aggregation rules that map non-fuzzy preferences into a fuzzy preference admit a dictator whose power depends on the strength of the transitivity condition. Although our positive results are also affected by the choice of the transitivity condition, they do not merely depend on this. As discussed at the end of Section 3, we owe our permissive findings to the ambiguity that the social preference is allowed to exhibit combined with the relatively weak transitivity condition we use.[21]

Our model not only generalizes the framework and results of Arrow (1951) and Gibbard (1969) but also the probabilistic social welfare functions (PSWFs) of Barberà and Sonnenschein (1978) which assign a probability distribution over (non-sophisticated) preferences to each profile of (non-sophisticated) preferences. As every probability distribution over non-sophisticated preferences induces a sophisticated preference but the converse is not true, SSWFs are more general objects than PSWFs. As a result, with the natural adaptation of the definitions, the fact that every PO and IIA PSWF is $\omega-$oligarchical follows from our Theorem 3.2. On the other hand, concluding that every $\omega-$oligarchical PSWF is PO and IIA requires a (sub)additivity condition imposed over the power distribution (see Barberà and Sonnenschein (1978), McLennan (1980), Bandyopadhyay et al. (1982) and Nandeibam (2003)).

On the other hand, the literature admits an environment which is more general than ours: SSWFs are generalized by the probabilistic collective judgement model of Barberà and Valenciano (1983). In fact, all of their results on probabilistic collective judgement functions can be restricted to our framework so as to be stated for SSWFs. Nevertheless our central result -Theorem 3.2- cannot be deduced from Barberà and Valenciano (1983). Moreover, when Theorems 1 and 4 of Barberà and Valenciano (1983) are restricted to our framework, they are implied by our Theorem 3.2. Thus, comparing our findings with those of Barberà and Valenciano (1983), we can pretend to have established a stronger result in a narrower environment.

We close by noting the lack of obvious connection between a sophisticated preference and the choice it induces. While this imposes a barrier in using our positive findings in resolving social choice problems, it also gives an incentive to propose a rational choice theory with sophisticated preferences.

[21] In the introduction, we discuss the appropriateness of our transitivity condition to our interpretations of the model.

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