

Cosmological phase space analysis of the $F(X) - V(\phi)$ scalar field and bouncing solutionsJosue De-Santiago,^{1,2,3,*} Jorge L. Cervantes-Cota,^{3,†} and David Wands^{2,‡}¹*Universidad Nacional Autónoma de México, 04510, D. F., México*²*Institute of Cosmology and Gravitation, University of Portsmouth, Dennis Sciamia Building, Portsmouth, PO1 3FX, United Kingdom*³*Departamento de Física, Instituto Nacional de Investigaciones Nucleares, México*

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We analyze the dynamical system defined by a universe filled with a barotropic fluid plus a scalar field with modified kinetic term of the form $\mathcal{L} = F(X) - V(\phi)$. After a suitable choice of variables that allows us to study the phase space of the system, we obtain the critical points and their stability. We find that some of them reduce to the ones defined for the canonical case when $F(X) = X$. We also study the field energy conditions to have a nonsingular bounce.

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I. INTRODUCTION

Scalar fields play an important role in cosmological models because, due to their simplicity and adaptability, they can account for different interesting phenomena. They are some of the most popular choices for modeling cosmological scenarios such as inflation [1] and dark energy [2], and they also have been studied in the context of dark matter models [3], bounce cosmology [4], and different unification models of those phenomena [5].

The proposal that the Lagrangian could be a general function of the kinetic term $X = -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2$ and the field ϕ was introduced in cosmology first in [6] in the context of inflation and then used for dark energy models in [7]. Different particular forms of the Lagrangian $\mathcal{L} = p(X, \phi)$ have been studied for different reasons [2].

In this paper we study the general class of models with sum-separable Lagrangian $\mathcal{L} = F(X) - V(\phi)$. Several aspects of this type of scalar field have been studied in the literature. Phenomenology of the inflation models arising from them [8,9], topological defects [10], supersymmetry extension [11], boson stars [12], unification models of dark matter and dark energy [5,13], and unification of dark energy, dark matter, and inflation [14,15]. This model also offers the possibility of being understood as a vacuum energy density V coupled with a barotropic fluid [16]. It reduces to the canonical scalar field when $F(X) = X$.

In Sec. II we study the system of autonomous differential equations related to this class of scalar fields. This method is equivalent to the one used for canonical scalar fields in [17] and allows us to identify the general behavior of the cosmological solutions associated with the present Lagrangian. This method has been applied to a wide range of cosmological models, for example [18–26]. It can be used to determine the presence and stability of solutions of

cosmological interest, such as those with de Sitter phases or with scaling behaviors.

One possible application for this type of Lagrangian is the generation in the early universe of a nonsingular bounce, in which the state of the Universe goes from collapsing to expanding for $a \neq 0$. The bouncing models have been proposed as alternatives to inflation [27,28] and as a way to evade a singular big bang, as in the pre-big bang scenario [29], in the ekpyrotic universe [30–33], or in other multifield models [34].

In Sec. III we study how, when the density of this field is allowed to be negative, it can drive a bounce. As the variables defined for previous the dynamical system analysis are not suitable to study this phenomenon, we redefine the system as in Ref. [34] in order to study this case and obtain the conditions to accomplish a bouncing behavior. To obtain the bounce, the scalar field has to violate the null energy condition (NEC) possibly giving rise to instabilities. Some works have been made trying to erase these instabilities with ghost condensate scalar fields [31,35,36] (however, see Ref. [37]). Here we will only consider the dynamics of homogeneous cosmologies.

II. AUTONOMOUS SYSTEM FOR $\mathcal{L} = F(X) - V(\phi)$

For a spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology filled with a scalar field with a Lagrangian of the form $\mathcal{L} = F(X) - V(\phi)$, where

$$X = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi, \quad (1)$$

and a matter component with density ρ_m and equation of state $p_m = \omega_m \rho_m$, the equations of motion are

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} [2XF_X - F + V + \rho_m], \quad (2)$$

$$\dot{H} = -\frac{1}{2M_{\text{Pl}}^2} [2XF_X + (1 + \omega_m)\rho_m], \quad (3)$$

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where H is the Hubble factor. These equations can be combined to imply the conservation of the total energy momentum tensor. We can suppose additionally the conservation of the scalar field and barotropic fluid energy momentum tensors separately, which is the case when there is no interchange of energy between the two components. In that case the barotropic component satisfies the equation $\rho_m \propto a^{-3(1+\omega_m)}$ for a constant ω_m , where a is the scale factor, and the scalar field satisfies

$$\frac{d}{dN}(2XF_X - F + V) + 6XF_X = 0, \quad (4)$$

where the subindex X means differentiation with respect to that variable. The time differentiation here has been changed to $dN = d \log a$, a variable that for an expanding FLRW model can be used as the independent variable instead of the cosmological time, with the relation $dN = H dt$.

In order to obtain the autonomous system we define the variables

$$x = \frac{\sqrt{2XF_X - F}}{\sqrt{3}M_{\text{Pl}}H}, \quad y = \frac{\sqrt{V}}{\sqrt{3}M_{\text{Pl}}H}, \quad (5)$$

where x^2 is proportional to the kinetic part of the energy density

$$\rho_k = 2XF_X - F, \quad (6)$$

and y^2 to the potential part of the energy density $\rho_V = V$. They are equivalent to the ones used in the analysis for canonical scalar fields [17]. We will also need to define the auxiliary variables

$$\sigma = -\frac{M_{\text{Pl}}V_\phi}{V} \sqrt{\frac{2X}{3|2XF_X - F|}} \text{sign}(\phi), \quad (7)$$

$$\omega_k = \frac{F}{2XF_X - F}, \quad (8)$$

where the former corresponds to the change in time of the potential, as can be seen if we write it as

$$\sigma = -\frac{M_{\text{Pl}}}{\sqrt{3|\rho_k|}} \frac{d \log V}{dt}, \quad (9)$$

and the latter corresponds to the equation of state for the kinetic part of the Lagrangian, as the kinetic part of the pressure is $P_k = F$. In the case of a canonical scalar field $F(X) = X$ and the auxiliary variables turn out to be $\omega_k = 1$ and $\sigma = \sqrt{2/3}\lambda$ for $V \propto e^{-\lambda\phi/M_{\text{Pl}}}$ as defined in Ref. [17].

The equation of state of the scalar field can be obtained in terms of the new variables as

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\omega_k x^2 - y^2}{x^2 + y^2}. \quad (10)$$

The evolution equations for the first two variables (x and y) can be written as

$$\frac{dx}{dN} = \frac{2XF_{XX} + F_X}{2\sqrt{3}(2XF_X - F)M_{\text{Pl}}H} \frac{dX}{dN} - x \frac{\dot{H}}{H^2}, \quad (11)$$

$$\frac{dy}{dN} = \frac{V_\phi \dot{\phi}}{2\sqrt{3}VM_{\text{Pl}}H^2} - y \frac{\dot{H}}{H^2}. \quad (12)$$

The common \dot{H}/H^2 factor can be obtained from Eq. (3) dividing by H^2 and replacing the original for the new variables

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}[(1 + \omega_m)(1 - y^2) + x^2(\omega_k - \omega_m)], \quad (13)$$

where we made use of the Eq. (2) in the new variables

$$x^2 + y^2 + \Omega_m = 1. \quad (14)$$

Now in order to calculate the first term in the evolution Eq. (12) we only have to substitute the values of the new variables

$$\frac{V_\phi \dot{\phi}}{2\sqrt{3}VM_{\text{Pl}}H^2} = -\frac{3}{2}\sigma xy. \quad (15)$$

For the first term in Eq. (11), we use the continuity Eq. (4) that can be written as

$$\frac{dX}{dN} = -\frac{3F}{(2XF_{XX} + F_X)\omega_k} \left(\omega_k + 1 - \frac{\sigma y^2}{x} \right), \quad (16)$$

so that the evolution Eqs. (11) and (12) become, in terms of the new variables,

$$\begin{aligned} \frac{dx}{dN} &= \frac{3}{2}[\sigma y^2 - x(\omega_k + 1)] \\ &\quad + \frac{3}{2}x[(1 + \omega_m)(1 - y^2) + x^2(\omega_k - \omega_m)], \end{aligned} \quad (17)$$

$$\frac{dy}{dN} = -\frac{3}{2}\sigma yx + \frac{3}{2}y[(1 + \omega_m)(1 - y^2) + x^2(\omega_k - \omega_m)]. \quad (18)$$

The evolution equations for the variables ω_k and σ can be obtained using the Eq. (16) and the definition of X . But we have to define new auxiliary variables that depend on the second-order derivatives of the Lagrangian potentials. The evolution equations are

$$\frac{d\omega_k}{dN} = 3 \frac{2\Xi\omega_k + \omega_k - 1}{2\Xi + 1} \left(\omega_k + 1 - \frac{\sigma y^2}{x} \right), \quad (19)$$

$$\begin{aligned} \frac{d\sigma}{dN} &= -3\sigma^2x(\Gamma - 1) + \frac{3\sigma(2\Xi(\omega_k + 1) + \omega_k - 1)}{2(2\Xi + 1)(\omega_k + 1)} \\ &\quad \times \left(\omega_k + 1 - \frac{\sigma y^2}{x} \right), \end{aligned} \quad (20)$$

where the auxiliary variables are defined as

$$\Xi = \frac{XF_{XX}}{F_X}, \tag{21}$$

$$\Gamma = \frac{VV_{\phi\phi}}{V_{\phi}^2}. \tag{22}$$

The new second-order derivative variables Γ , Ξ will have evolution equations in terms of the dynamical variables and new third-order derivative variables, and so on. In order to truncate this succession of equations we can consider fixing the functions $F(X)$ and $V(\phi)$.

The first of these assumptions is to choose the potential related variable Γ as a constant. For it to happen, we need

$$V(\phi) = V_0(\phi - \phi_0)^{1/(1-\Gamma)} \tag{23}$$

for $\Gamma \neq 1$, or

$$V(\phi) = V_0 e^{-\lambda\phi/M_{\text{Pl}}} \tag{24}$$

for $\Gamma = 1$. The second assumption is to consider the case in which

$$F(X) = AX^\eta, \tag{25}$$

where A and η are constants, in this case $\omega_k = 1/(2\eta - 1)$ and Eq. (19) is trivially satisfied. In the following we will use these assumptions.

The dynamical system will be reduced to Eq. (17) for the evolution of x , Eq. (18) for the evolution of y , and an equation for the evolution of σ that due to the choice of F as a power law becomes

$$\frac{d\sigma}{dN} = -3\sigma^2 x(\Gamma - 1) + \frac{3\sigma(1 - \omega_k)}{2(1 + \omega_k)} \left(\omega_k + 1 - \frac{\sigma y^2}{x} \right). \tag{26}$$

A. Critical points

The autonomous system of equations written above can be analyzed if we consider its critical points, in which Eqs. (17) and (18) are equal to zero, corresponding to x and y constant. In the first instance we will not consider the evolution Eq. (26), but if the critical points (x_0, y_0) depend on σ they will not be truly fixed unless we ensure that σ is constant.

The variables x^2 and y^2 correspond to the fraction of the energy density contained in kinetic and potential energy of the scalar field, as can be seen from (14). The condition of constancy for the critical points implies that these variables have a constant contribution to the total energy density, which can happen in three scenarios: (i) if $x^2 + y^2$ is equal to one, meaning that all the energy density comes from the scalar field, (ii) if they are zero, meaning no contribution, or (iii) if they are between zero and one, corresponding to what is also known as a scaling solution, meaning that the energy density of the field scales at the same rate as that of matter. The three behaviors are of cosmological interest

and are present for the critical points of generally defined parameters x and y if they satisfy Eq. (14).

To present the critical points, we have labeled with Latin letters those that reduce in the canonical case to the ones studied in [38], and with Greek letters to the ones with no correspondence.

For $x_a = 0$ and $y_a = 0$, this corresponds to the scalar field not contributing to the energy density of the Universe.

If $x = 0$ and $y \neq 0$, the equation of state becomes $\omega_\phi = -1$ that is an interesting case from the cosmological point of view due to the possibility to describe dark energy or inflation phenomena. In this case the evolution equation for x reduces to

$$\frac{dx}{dN} = \frac{3}{2}\sigma y^2, \tag{27}$$

which requires $\sigma = 0$, a potential that doesn't change with time. On the other hand, the evolution equation for y reduces to

$$\frac{dy}{dN} = \frac{3}{2}y(1 + \omega_m)(1 - y^2), \tag{28}$$

that can be zero for $\omega_m = -1$ or $y = 1$. Both cases correspond to a FLRW model filled with fluid with equation of state -1 .

- (i) In the first case $x_\alpha = 0$ and $y_\alpha = 1$, the Friedmann equation in the new variables (14) implies that the matter field has zero energy density and the only component of the model is the ϕ field.
- (ii) In the second case $x_\beta = 0$ and y_β is arbitrary, then there are contributions from the barotropic fluid as well as from the scalar field.

If $y = 0$ and $x \neq 0$, the potential energy is zero and the equation of state reduces to $\omega_\phi = \omega_k$. The energy density of the field is stored in the kinetic part. The evolution equation for y vanishes and the one for x reduces to

$$\frac{dx}{dN} = \frac{3}{2}x(\omega_k - \omega_m)(x^2 - 1), \tag{29}$$

that can become zero in two cases:

- (i) For $x_b = 1, y_b = 0$ this corresponds to the density of the model coming entirely from the kinetic part of the field ϕ .
- (ii) For $\omega_m = \omega_\phi$, with $y_\gamma = 0$ and x_γ arbitrary, the equation of state of the field is the same as the equation of state of the matter. It corresponds to a kinetically driven scaling solution. These types of solutions are important in cosmology, because in the case of dark energy they have been proposed to alleviate the coincidence problem [2]. This case, however, is not completely what in the literature is called a scaling solution in the sense that it can only reproduce a constant equation of state of the matter when the Lagrangian of the field satisfies

$$F(X) = AX^{(1+\omega_m)/2\omega_m}. \quad (30)$$

For example, if the energy density of the matter satisfies a relativistic equation of state, we need $F(X) = AX^2$ such that $\omega_k = 1/3$. The latter happens in the unified dark matter models based in Scherrer's Lagrangian $F(X) = F_0 + F_m(X - X_0)^2$. It is known [39] that for high energies in which $X \gg X_0$ the model can have a radiation-like behavior and this is because F_0 and X_0 can be disregarded, approximating to (30).

The last case is when both x and y are different from zero. From (17) and (18), we can see that the critical points satisfy

$$x = \frac{1}{2\sigma}(\omega_k + 1 \pm \sqrt{(\omega_k + 1)^2 - 4\sigma^2 y^2}). \quad (31)$$

In this case there are two different critical points, the first one has the form

$$x_c = \frac{\sigma}{\omega_k + 1}, \quad y_c = \frac{\sqrt{(\omega_k + 1)^2 - \sigma^2}}{\omega_k + 1}. \quad (32)$$

It corresponds to a cosmology filled with the scalar field, as can be seen from (14) which in this case corresponds to $x_c^2 + y_c^2 = 1$ with zero matter density. The equation of state of the system will be

$$\omega_\phi = \frac{\sigma^2}{1 + \omega_k} - 1. \quad (33)$$

The second nonzero critical point corresponds to

$$x_d = \frac{\omega_m + 1}{\sigma}, \quad y_d = \frac{\sqrt{(\omega_m + 1)(\omega_k - \omega_m)}}{\sigma}, \quad (34)$$

where the equation of state in this case is $\omega_\phi = \omega_m$, in other words corresponding to a scaling solution. In the canonical case with exponential potential we will recover the scaling solution of Ref. [17]. The fraction of the total energy density stored in the scalar field will be

$$x_d^2 + y_d^2 = \frac{(1 + \omega_k)(1 + \omega_m)}{\sigma^2}. \quad (35)$$

It is interesting to point out that, except for the canonical case, the Lagrangians studied here with ω_k and Γ constants cannot be reduced to the case $\mathcal{L} = Xg(Xe^{\lambda\phi})$, which is considered in Ref. [40] as the general form for a scalar field with scaling solutions. The difference from the case studied there is that we are not considering a coupling between the field and the barotropic fluid as in their case.

The points defined in (32) and (34) depend explicitly on σ , that in general is an evolving quantity. It means that, unless the variable σ is also fixed, those points won't be critical points of the system. Setting the evolution equation for σ equal to zero gives us the condition $\Gamma = \Gamma_0$ with

$$\Gamma_0 = \frac{3 + \omega_k}{2(1 + \omega_k)}. \quad (36)$$

This relation between the derivatives of potential and kinetic terms in the Lagrangian has to be accomplished in order to have the critical points (32) and (34). For the canonical case, as $\omega_k = 1$ then the critical points are fixed only for $\Gamma = 1$, that from (24) corresponds to the exponential potential, as expected. For the noncanonical case as $\omega_k \neq 1$ then there will be a relation between the exponent in the kinetic and the one in the potential term of the form $\mathcal{L} = AX^\eta - B(\phi - \phi_0)^n$ with

$$\eta = \frac{n}{2 + n}, \quad (37)$$

if this relation is not satisfied, the critical points won't be truly fixed. In the Appendix we show that when the system satisfies this relation, it is invariant under a set of symmetry transformations, which turn allows to reduce the number of degrees of freedom. The same symmetry invariance happens for the canonical scalar field with exponential potential as proved in Ref. [41].

The stability of the critical points can be analyzed by the matrix of the derivatives of the right-hand side of Eqs. (17) and (18). Analyzing the eigenvalues of the matrix we obtain the results of Table I.

The critical point (a) presents a behavior of unstable node for $\omega_k < \omega_m$ which can drive the scalar field density towards bigger values even if it starts with small density. The saddle point behavior that was already obtained in the canonical case is recovered here when $\omega_m < \omega_k$. Point (α) corresponds to slow roll behavior as $\sigma = 0$ and the potential dominates, and it can be a saddle point or a stable node depending on the equation of state of the kinetic part. Point (b) can be stable, unstable, or a saddle point. In the canonical case the stable behavior is not obtained. Points (c) and (d) have the same stability behavior as in the canonical case except that the conditions get modified by ω_k as stated in the table. The lines (β) and (γ) are obtained when the equation of state of matter is the same as that of the kinetic part or the potential part of the Lagrangian and can be stable or unstable. The cosmological relevance of these solutions is further discussed in the conclusions.

B. Critical points at infinity

The former critical points a, b, c, α , β , and γ correspond to the situation in which the dynamical variables are finite. This is always the case for x and y as Eq. (14) requires both variables to be smaller or equal than one. However, for σ we can see from the definition (7) that it can become infinity as ρ_k the kinetic energy density or V the potential tend to zero. To study this case, we make a change of the variable σ to $\Sigma = 1/\sigma$ and study the possibility of it becoming zero.

TABLE I. Stability and existence of the critical points assuming $-1 \leq \omega_m \leq 1$. The points labeled with Latin letters reduce in the canonical case to the ones already studied in the literature [38], the points with Greek letters are new.

	x	y	Existence	Stability	Ω_ϕ	ω_ϕ
(a)	0	0	Always	Unstable node for $\omega_k < \omega_m$ Saddle point for $\omega_m < \omega_k$	0	-
(α)	0	1	$\sigma = 0$	Saddle point for $\omega_k < -1$ Stable node for $\omega_k > -1$	1	-1
(b)	1	0	Always	Unstable node for $\omega_k > \{\omega_m, \sigma - 1\}$ Stable node for $\omega_k < \{\omega_m, \sigma - 1\}$ Otherwise saddle point	1	ω_k
(c)	$\frac{\sigma}{1+\omega_k}$	$\sqrt{\frac{(1+\omega_k)^2 - \sigma^2}{(1+\omega_k)^2}}$	$\Gamma = \Gamma_0$ $\sigma(1+\omega_k) > 0$ $\sigma^2 > (1+\omega_k)^2$	Saddle point for $\sigma^2 > (1+\omega_k)(1+\omega_m)$ Otherwise stable node	1	$\frac{\sigma^2}{1+\omega_k} - 1$
(d)	$\frac{1+\omega_m}{\sigma}$	$\frac{\sqrt{(\omega_k - \omega_m)(1+\omega_m)}}{\sigma}$	$\Gamma = \Gamma_0$ $\omega_m < \omega_k$ $\sigma > \sqrt{(1+\omega_k)(1+\omega_m)}$	Stable node for $\frac{\sigma^2(8-\omega_k+9\omega_m)}{8(1+\omega_k)(1+\omega_m)^2} < 1$ Stable spiral otherwise	$\frac{(1+\omega_k)(1+\omega_m)}{\sigma^2}$	ω_m
(β)	0	Arbitrary	$\sigma = 0$ and $\omega_m = -1$	Stable line for $\omega_k > -1$ Unstable otherwise	y^2	-1
(γ)	Arbitrary	0	$\omega_k = \omega_m$	Stable line for $x\sigma > 1 + \omega_k$ Unstable otherwise	x^2	ω_m

Considering ω_k and Γ constants, the evolution Eqs. (17), (18), and (26) become

$$\frac{dx}{dN} = \frac{3}{2} \left[\frac{y^2}{\Sigma} - x(\omega_k + 1) \right] + \frac{3}{2} x [(1 + \omega_m)(1 - y^2) + x^2(\omega_k - \omega_m)], \quad (38)$$

$$\frac{dy}{dN} = -\frac{3xy}{2\Sigma} + \frac{3}{2} y [(1 + \omega_m)(1 - y^2) + x^2(\omega_k - \omega_m)], \quad (39)$$

$$\frac{d\Sigma}{dN} = 3x(\Gamma - 1) - \frac{3(1 - \omega_k)}{2(1 + \omega_k)} \left((\omega_k + 1)\Sigma - \frac{y^2}{x} \right). \quad (40)$$

In order to have a critical point at $\Sigma = 0$ it's required that the terms y^2/Σ and xy/Σ each vanish. These factors can be computed considering that as ω_k and Γ are constants the kinetic term is a power-law $F = AX^\eta$, and the potential term is either a power-law $V = B\phi^n$ or an exponential $V = Ce^{-\lambda\phi/M_{\text{pl}}}$. For the power-law potential, the variable Σ has the expression

$$\Sigma = -\frac{1}{nM_{\text{pl}}} \sqrt{\frac{3(2\eta - 1)A}{2}} \text{sign}(\dot{\phi}) \phi X^{(\eta-1)/2}, \quad (41)$$

and the factors

$$\frac{y^2}{\Sigma} = -\frac{nB}{3M_{\text{pl}}H^2} \sqrt{\frac{2}{3(2\eta - 1)A}} \text{sign}(\dot{\phi}) \phi^{n-1} X^{(1-\eta)/2}, \quad (42)$$

$$\frac{xy}{\Sigma} = -\frac{n}{3M_{\text{pl}}H^2} \sqrt{\frac{2B}{3}} \text{sign}(\dot{\phi}) \phi^{(n-2)/2} X^{1/2}. \quad (43)$$

In order to have these two terms equal to zero at the same time as $\Sigma \rightarrow 0$, we require $\phi = 0$ and $n > 2$. With these conditions, the variable y becomes zero, too, and the evolution equations for x and y get reduced to

$$\frac{dx}{dN} = \frac{3}{2} x(x^2 - 1)(\omega_k - \omega_m), \quad (44)$$

$$\frac{dy}{dN} = 0, \quad (45)$$

which implies that the critical points occur when $x = 0$, $x = 1$, or $\omega_k = \omega_m$. These three cases correspond to the already studied critical points (a), (b), and (γ), respectively.

The second case happens when the potential is an exponential $V = Be^{-\lambda\phi/M_{\text{pl}}}$. In this case, however, $y = \sqrt{V/\rho_c}$ cannot become zero, except for the trivial case in which $B = 0$. This means that y^2/Σ diverges as $\Sigma \rightarrow 0$, which implies that (39) cannot be zero and there is no critical point at infinity.

C. Analysis of the phase space

In this subsection we plot the phase space defined by the Eq. (17), (18), and (26) for several potentials. An important case happens when the potential satisfies Eq. (36). For example, this occurs for a Lagrangian of the form $\mathcal{L} = AX^2 + B/\phi^4$. In this case the extra parameters have

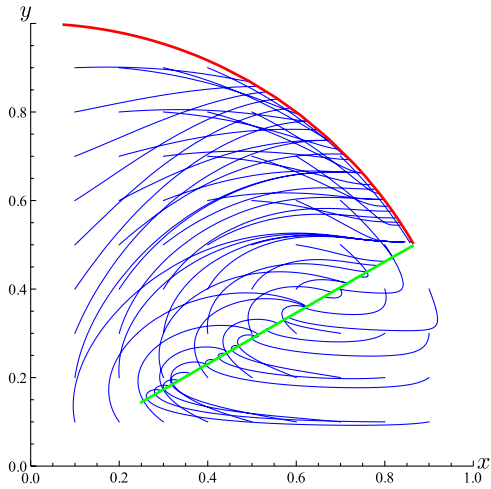


FIG. 1 (color online). Projection of the phase space along the (x, y) plane for a Lagrangian of the form $\mathcal{L} = AX^2 + B/\phi^4$, with initial condition $\sigma = 1.5$. The solutions tend to the critical points (c) and (d) studied in Table I. As σ changes, the critical points lie on the light green line for the (c) and the red segment of circle for (d). This behavior can be better seen in Figs. 2 and 3 corresponding to different projections of the same system. The critical point curves are plotted with the same colors.

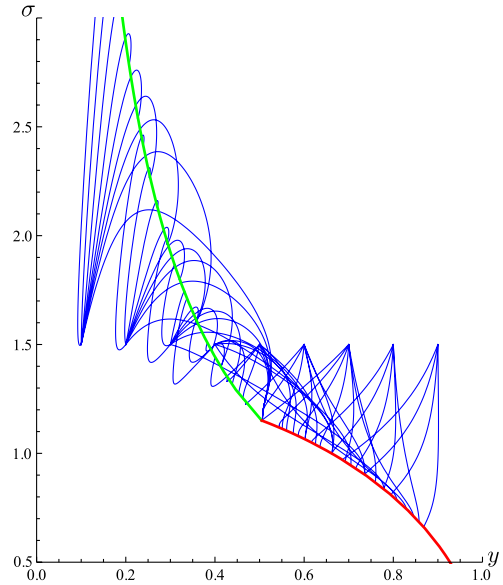


FIG. 3 (color online). Projection of the phase space along the (x, σ) plane for a Lagrangian of the form $\mathcal{L} = AX^2 + B/\phi^4$. The system is the same as in Figs. 1 and 2. The solutions where chosen to start in $\sigma = 1.5$ and they evolve towards different directions depending on the values of x and y . See the explanation in Fig. 1.

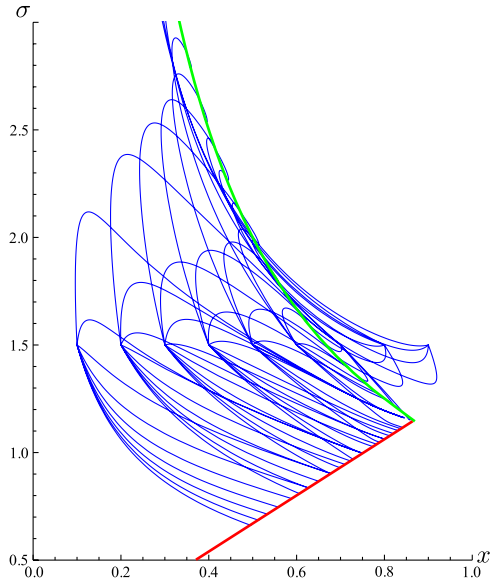


FIG. 2 (color online). Projection of the phase space along the (x, σ) plane for a Lagrangian of the form $\mathcal{L} = AX^2 + B/\phi^4$. The system is the same as in Figs. 1 and 3. The solutions where chosen to start in $\sigma = 1.5$ and for a constant x they evolve in different directions due to the different values in y . See the explanation in Fig. 1.

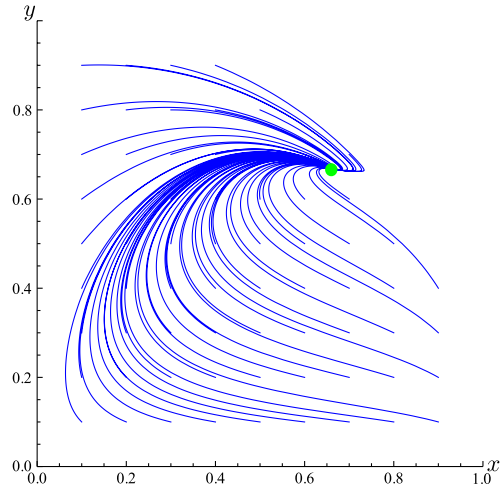


FIG. 4 (color online). Phase space for the canonical scalar field with exponential potential and $\sigma = 1.5$. The phase space for this system is two-dimensional, unlike in the case of nonexponential potentials or noncanonical kinetic terms with three dimensional phase spaces. The solutions tend to the critical point (d), scaling solution.

values of $\omega_k = 1/3$ and $\Gamma = 5/4$. In Figs. 1–3 we plotted the two-dimensional projections of this system for $\omega_m = 0$ for an initial condition of $\sigma = 1.5$. We can see how the solutions approach to the critical points (c) and (d) from Table I depending on the values of σ . The critical points lie

on a curve in the three-dimensional phase space, and due to the evolution of σ , the solutions tend to different points in those curves.

As we have stated, for the canonical scalar field the condition (36) implies an exponential potential. In that case σ is a constant determined by the exponent in the potential, as $V \propto e^{-\sqrt{3}\sigma\phi/(\sqrt{2}M_{\text{Pl}})}$. The phase space in that

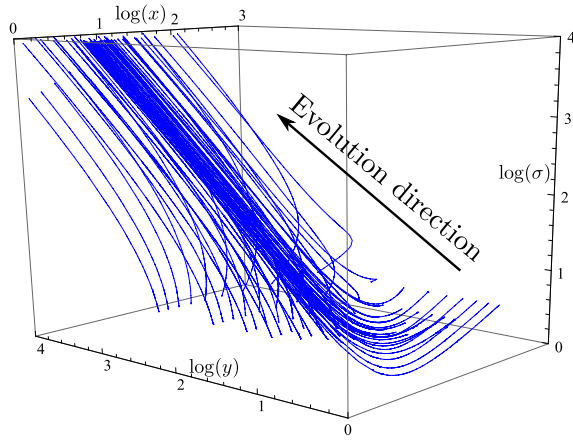


FIG. 5 (color online). Phase space for the Lagrangian $\mathcal{L} = AX^2 + B\phi^2$. The solutions don't tend to any critical point. They evolve towards high values of σ because this parameter is proportional to ϕ^{-1} for this Lagrangian.

case is effectively two-dimensional. In Fig. 4 we can see the behavior for $\sigma = 1.5$ with $\omega_m = 0$. In this case the solutions approach only to one point, as there is not evolution in σ .

When the condition (36) is not satisfied, we do not have the critical points (c) and (d), but we can still plot the system. For example, for the Lagrangian $\mathcal{L} = AX^2 + B\phi^2$ in which case the values of the auxiliary parameters are $\omega_k = 1/3$ and $\Gamma = 1/2$. In Fig. 5 we plotted the three-dimensional phase space system for solutions that start with $\sigma = 1.5$. We can see that the system evolves towards big values of σ , this happens because $\sigma \propto 1/\phi$ in this case, and the system goes towards small values of ϕ . In fact, it crosses $\phi = 0$ in a finite time, in which the variable σ is not useful to describe the system. We can see that the solutions do not tend to any critical point.

III. BOUNCE COSMOLOGY

In this section we consider a nonsingular bounce ($a \neq 0$) in a FLRW cosmology filled with the scalar field $\mathcal{L} = F(X) - V(\phi)$ and a barotropic fluid with constant equation of state ω_m . For a bounce to happen, we need the evolution of the scale factor to go from decreasing to increasing as a function of time. In terms of the derivative of the scale factor this implies that at the bounce it has to satisfy $\dot{a}(t_b) = 0$ and $\ddot{a}(t_b) > 0$. The first condition can be translated in terms of the Hubble parameter as $H_b = 0$, but this means that the dynamical variables defined in Eq. (5) in the last section will diverge. Besides that, the independent variable N that we have used to parametrize the evolution of the system is no longer well defined at the bounce, as $d/dN = H^{-1}d/dt$. Those complications arise from the fact that our choice of dynamical variables was adjusted to study a cosmology with increasing a . Accordingly, in order to study a bouncing FLRW metric,

we need to define new variables adapted to the current problem.

Also we have to note that the total energy density of the model is zero at the bounce, which can be seen from the Friedmann equation (2). If we suppose that the energy density of the barotropic component is positive $\rho \propto a^{-3(1+\omega_m)}$ then the energy density of the field has to be negative, something that will be considered in the definition of the dynamical variables below.

Now let us define the new set of variables

$$\tilde{x} = \frac{\sqrt{3}M_{\text{Pl}}H}{\sqrt{|\rho_k|}}, \quad \tilde{y} = \sqrt{\left|\frac{V}{\rho_k}\right|} \text{sign}(V), \quad (46)$$

where ρ_k is given in Eq. (6) and the absolute values come from the fact that we are interested in the behavior of both, positive and negative energy densities. The new independent variable defined in analogy to N is

$$d\tilde{N} = \sqrt{\frac{|\rho_k|}{3M_{\text{Pl}}^2}} dt. \quad (47)$$

The evolution equations for the above variables can be then written as

$$\begin{aligned} \frac{d\tilde{x}}{d\tilde{N}} &= -\frac{3}{2}[(\omega_k - \omega_m)\text{sign}(\rho_k) + (1 + \omega_m)(\tilde{x}^2 - \tilde{y}|\tilde{y}|)] \\ &\quad + \frac{3}{2}\tilde{x}[(\omega_k + 1)\tilde{x} - \sigma\tilde{y}|\tilde{y}|\text{sign}(\rho_k)], \\ \frac{d\tilde{y}}{d\tilde{N}} &= \frac{3}{2}\tilde{y}[-\sigma + (\omega_k + 1)\tilde{x} - \sigma\tilde{y}|\tilde{y}|\text{sign}(\rho_k)]. \end{aligned} \quad (48)$$

In general we also need to evolve σ , its evolution equation can be obtained from Eq. (20), transforming to the new variables as

$$\begin{aligned} \frac{d\sigma}{d\tilde{N}} &= -3\sigma^2(\Gamma - 1) + \frac{3\sigma(2\Xi(\omega_k + 1) + \omega_k - 1)}{2(2\Xi + 1)(\omega_k + 1)} \\ &\quad \times ((\omega_k + 1)\tilde{x} - \sigma\tilde{y}^2). \end{aligned} \quad (49)$$

The above variables are well behaved only for $\rho_k \neq 0$, so neither of the possible cases of purely potential bounce, nor a change in sign for ρ_k after the bounce will be studied here.

A. Conditions for a bounce

In the following we work with the phase space defined by the set of equations (48). Due to the relation between the dynamical variables (\tilde{x}, \tilde{y}) and (x, y) from the previous section, the critical points of both systems coincide. It is easy to check that the points of Table I are also critical points of the new system with the transformation $\tilde{x} = 1/x$ and $\tilde{y} = y/x$, except for those with $x = 0$ in which the new variables diverge.

In this subsection we will use Eq. (48) to study the evolution of the systems near the bounce. In general we have to consider also Eq. (50) to make a representation of

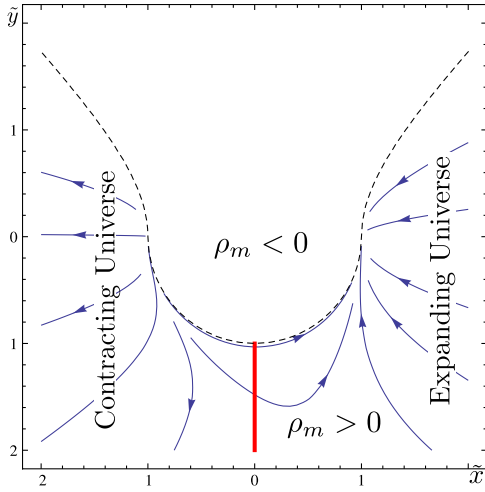


FIG. 6 (color online). Schematic projection of the phase space for a noncanonical nonphantom system with $\rho_k > 0$ and $\omega_k = -5$ and $\sigma \sim \sqrt{2/3}$. The bounce occurs when the solutions cross the vertical (thick, red) line.

the three-dimensional phase space as in Fig. 5 of the previous section; however, we will not consider this equation because we are interested in the behavior only close to the bounce and the variable σ won't evolve much during this short time. For this reason, the plots of the phase spaces of Figs. 6 and 7, which will be studied with more detail in this section, correspond only to schematic representations of the phase space near the bounce. For Figs. 8 and 9, the representation corresponds to the actual phase space because for those Lagrangians σ is a constant.

Besides Eq. (48), the system has to satisfy the Friedmann constraint (2), which translates into

$$\tilde{x}^2 - \tilde{y}|\tilde{y}| - \tilde{\Omega}_m = 1 \times \text{sign}(\rho_k), \quad (50)$$

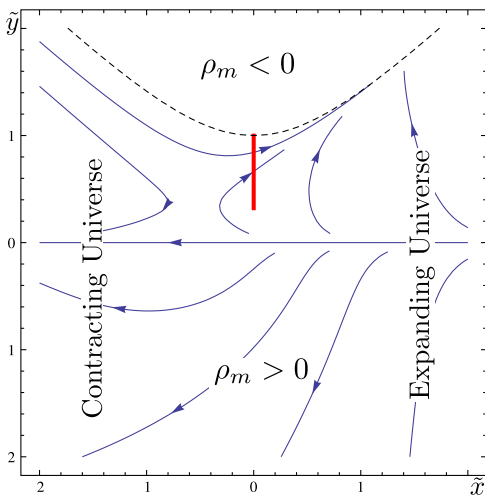


FIG. 7 (color online). Schematic projection of the phase space for a field with $\rho_k < 0$, $\omega_k = 1/6$, $\sigma \sim -\sqrt{2/3}$ and $\omega_m = 1/3$. The bounce occurs when the solutions cross the vertical (thick, red) line. There is no purely kinetic ($\rho_V = 0$) bounce.

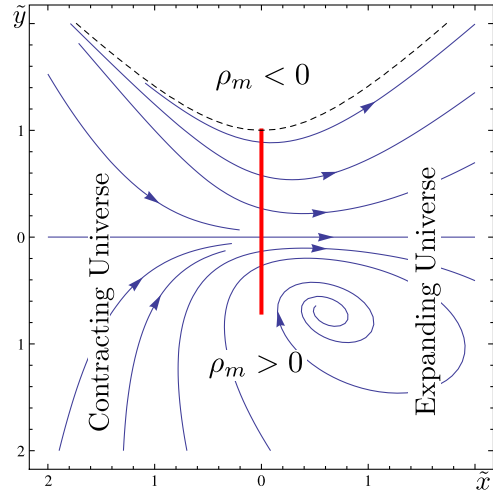


FIG. 8 (color online). Phase space (\tilde{x}, \tilde{y}) for the special case of a phantom system $F(X) = -X$ and $V \propto e^{-\lambda\phi/M_{\text{pl}}}$ (such that σ is constant) plus a barotropic radiation component $\omega_m = 1/3$. The bounce occurs when the solutions cross the vertical (thick, red) line. The spiral in the graph corresponds to the critical point (d) studied in the previous section.

where $\tilde{\Omega}_m = \rho_m/|\rho_k|$ corresponds to a dimensionless density parameter for the barotropic fluid component. As $\tilde{\Omega}_m$ is assumed to be nonnegative, we obtain the expression

$$\tilde{x}^2 - \tilde{y}|\tilde{y}| \geq 1 \times \text{sign}(\rho_k), \quad (51)$$

which defines the allowed regions of the phase space. For the nonphantom case $\rho_k > 0$, the Friedmann constraint becomes

$$\tilde{x}^2 - \text{sign}(V)\tilde{y}^2 \geq 1, \quad (52)$$

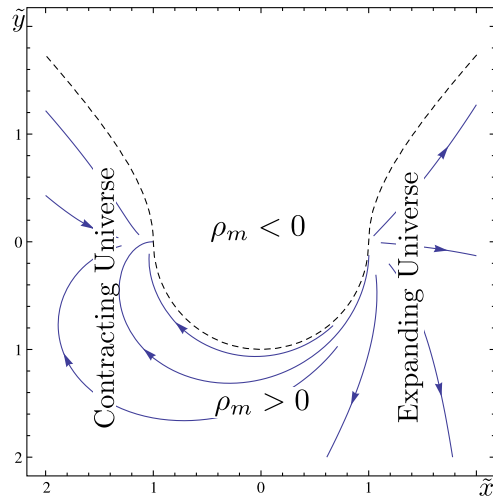


FIG. 9 (color online). Phase space for the special case of a canonical scalar field with potential $V \propto e^{-\lambda\phi/M_{\text{pl}}}$ (such that σ is constant) plus a barotropic radiation component $\omega_m = 1/3$. All the solutions that cross $\tilde{x} = 0$ have negative $d\tilde{x}/d\tilde{N}$ which corresponds to recollapse. The bounce is not possible.

which for \tilde{y} positive corresponds to the region inside the branches of the hyperbola $\tilde{x}^2 - \tilde{y}^2 = 1$, and for \tilde{y} negative to the region outside the circle defined by $\tilde{x}^2 + \tilde{y}^2 = 1$, as seen in Fig. 6. In the $\rho_k < 0$ case the condition (51) translates into

$$\tilde{x}^2 - \text{sign}(V)\tilde{y}^2 \geq -1, \quad (53)$$

which for $\tilde{y} > 0$ corresponds to the region below the hyperbola $\tilde{y}^2 - \tilde{x}^2 = 1$. For $\tilde{y} < 0$, this condition is satisfied for all the values, as we can see in Fig. 8.

From the definitions (46), we can see that

- (i) $\tilde{x} > 0$ corresponds to the regime of an expanding cosmology,
- (ii) $\tilde{x} < 0$ corresponds to a contracting cosmology,
- (iii) $\tilde{x} = 0$ corresponds either to a bounce, a recollapse, or a static cosmology.

For the case of $\tilde{x} = 0$, we can use the information contained in the derivative to study whether we are dealing with a bounce or a recollapse:

- (i) $\frac{d\tilde{x}}{d\tilde{N}} > 0$ corresponds to a bounce,
- (ii) $\frac{d\tilde{x}}{d\tilde{N}} < 0$ corresponds to a recollapse,
- (iii) $\frac{d\tilde{x}}{d\tilde{N}} = 0$ gives not enough information and one has to consider higher derivatives or analyze the neighboring phase space.

To see which of the above cases occurs in the phase space of our system, we use $\tilde{x} = 0$ in the evolution equations (48). In particular, for the evolution of \tilde{x} , we obtain

$$\frac{d\tilde{x}}{d\tilde{N}} = -\frac{3}{2}[(\omega_k - \omega_m)\text{sign}(\rho_k) - (1 + \omega_m)\tilde{y}|\tilde{y}|]. \quad (54)$$

As we stated above this expression has to be positive for a bounce, which implies a condition in the parameter \tilde{y} as

$$\tilde{y} > \sqrt{\left| \frac{\omega_k - \omega_m}{1 + \omega_m} \right| \text{sign}(\rho_k(\omega_k - \omega_m))}. \quad (55)$$

In addition, we also have the condition (51) for the case $\tilde{x} = 0$

$$\tilde{y} \leq -1 \times \text{sign}(\rho_k). \quad (56)$$

To analyze the above conditions, we first suppose $\rho_k > 0$. In this case the inequality (56) transforms to $\tilde{y} \leq -1$ and (55) to $\tilde{y}^2 < (\omega_m - \omega_k)/(1 + \omega_m)$. For the two conditions to be satisfied, in an interval of \tilde{y} is necessary to have $\omega_k < -1$. For example, in the case of a canonical scalar field, one has $F(X) = X$ and consequently $\rho_k > 0$, but as $\omega_k = 1$ the system of a barotropic component and a canonical scalar field cannot give rise to a bounce, as already shown in [42]. This can be seen in the phase space of Fig. 9 in which all the solutions that cross the \tilde{y} axis move from positive to negative values of \tilde{x} , corresponding to recollapse.

For $\rho_k < 0$, the conditions for the bounce become

$$\sqrt{\left| \frac{\omega_k - \omega_m}{1 + \omega_m} \right|} \text{sign}(\omega_m - \omega_k) < \tilde{y} \leq 1, \quad (57)$$

which can be satisfied for an interval of \tilde{y} as long as $\omega_k > -1$. The original phantom field with $F(X) = -X$ satisfies $\omega_k = 1$ and, as shown in Fig. 8, can have a bounce behavior.

In order to have a purely kinetic bounce, in other words one with $\tilde{y} = 0$ the conditions above state that the density of the scalar field ρ_k has to be negative and $\omega_k > \omega_m$. Figure 7 shows a case in which the later is not accomplished and then there is no purely kinetic bounce.

The two conditions in the previous paragraph can be generalized. First, to obtain a bounce, one needs the total energy density of the field to be negative in order to compensate for the positive barotropic energy density in the Friedmann equation

$$H^2 = \frac{1}{3M_{\text{Pl}}^2}(\rho_\phi + \rho_m) = 0, \quad (58)$$

where $\rho_\phi = \rho_V + \rho_k$ is the total energy density in the field. Moreover, the total equation of state of the field ω_ϕ has to be bigger than that of the barotropic fluid in order to have a positive energy density for $a > a_{\text{bounce}}$, as seen in Fig. 10. Otherwise, we will be dealing with a system that exhibits positive energy density only for $a < a_{\text{bounce}}$ corresponding to a recollapse.

The above two conditions are in fact the same as those in the expressions (55) and (56) in terms of the dynamical variables. For the first one, the negativity of the energy density $\rho_k + \rho_V$ can be translated as $1 \times \text{sign}(\rho_k) + \rho_V/|\rho_k| < 0$ or from the definitions of the variables (46) as

$$\tilde{y} < -1 \times \text{sign}(\rho_k), \quad (59)$$

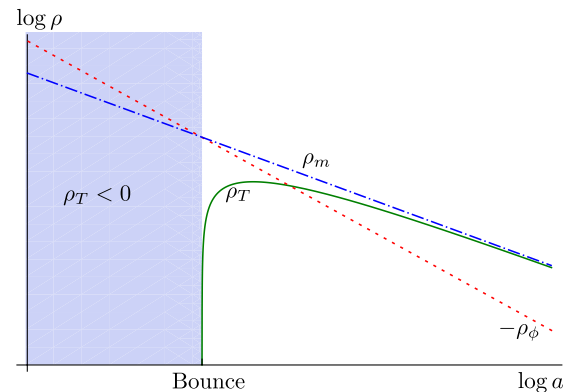


FIG. 10 (color online). The densities of the barotropic fluid (blue, dashed-dotted line), the scalar field (red, dotted line), and the total density of the Universe (green, continuous line), respectively, as a function of the scale factor. The total energy density tends to zero at the bounce and for smaller values of a is negative, which is forbidden.

which corresponds to expression (56). The condition on the total equation of state of the field, in terms of the dynamical variables can be written as

$$\frac{\omega_k - \tilde{y}|\tilde{y}|\text{sign}(\rho_k)}{1 + \tilde{y}|\tilde{y}|\text{sign}(\rho_k)} > \omega_m, \quad (60)$$

which can be transformed into (55) after some algebra and using the expression (59).

The conditions on the field to have a negative energy density and an equation of state greater than that of matter implies a violation of the NEC that states that $\rho_\phi + p_\phi$ be positive, as seen in Fig. 11. In the last years extensive literature has been produced studying fields that violate the NEC. The main reason for that interest is because the current measurements of the dark energy equation of state slightly favor models with $\omega_{\text{de}} < -1$ [2]. However, fields violating the NEC might have several types of instabilities, for example, imaginary sound speed which results in an increase of inhomogeneities in small periods of time [43,44], or decay of the vacuum into negative energy particles of the field plus positive energy particles [45–47]. The inclusion of higher order terms in the Lagrangian has been proposed as a method to obtain particles with positive energy in the so-called Ghost condensate models [31,35,36,48]; however, usually these extra terms add new stability problems to the models, and it is not clear if there is a well behaved high energy theory to account for them [49]. Due to those problems, a recent

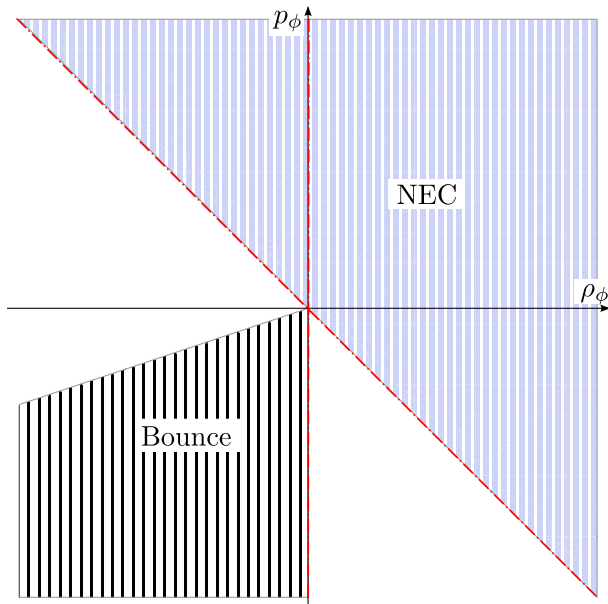


FIG. 11 (color online). The bottom left region corresponds in the ρ - p plane to the part which can drive a bounce, with $\rho < 0$ and $\omega_\phi > \omega_m$ with $\omega_m = 1/3$. The upper right region is the one that satisfies the null energy condition. The dashed-dotted lines cannot be crossed by k -essence Lagrangians like the ones considered here [43].

series of works has been published studying fields in which the introduction of certain symmetries can ensure the stability of the model in spite of breaking the NEC [50–52]. However, those models, so-called Galileons, have dynamics which was not studied in this paper.

IV. CONCLUSIONS

As we have seen, the system of equations (2) and (3) for the Lagrangian $\mathcal{L} = F(X) - V(\phi)$ can be rewritten in terms of the dynamical variables (5) as (17) and (18). This system allows us to understand the dynamical behavior of the Universe under different initial conditions. The critical points and their stability are summarized in Table I. This system is naturally adapted to study Lagrangians with kinetic terms of the type $F(X) \propto X^\eta$ and potentials $V(\phi) \propto \phi^{1/(\Gamma-1)}$ or $V(\phi) \propto e^{-\lambda\phi}$ such as those studied for k inflation in [9]. The canonical case and its critical points are recovered for $F(X) = X$.

In general, the critical points (α), (β), (γ), (c), and (d) are present only for particular choices of the Lagrangian $\mathcal{L} = F(X) - V(\phi)$, which happens for the canonical scalar field in which the points (c) and (d) are only present for exponential potentials. The conditions for their existence are summarized in Table I.

The point (α) corresponds to the slow roll scenario in which the potential dominates ($y = 1$) and its derivative is zero ($\sigma = 0$). The case with $\omega_k < -1$ is interesting for inflationary models as it corresponds to a saddle point, offering an explanation of how the Universe could enter in the slow roll regime and exit eventually. For that case, it is also necessary to study the dynamical behavior of σ to understand the conditions for it to evolve towards zero, something that was not analyzed in this paper.

The potential dominated line (β) has the cosmologically interesting behavior of an equation of state of -1 ; however, it requires that the barotropic fluid has the same behavior, something that is very restrictive.

The kinetic dominated line (γ) corresponds to critical points of the system only when $\omega_k = \omega_m$, for example, if $F(X) \propto X^2$ when the barotropic fluid is radiation. It can happen for example in the purely kinetic unified model studied in Ref. [39], in which the proposed Lagrangian behaves as a radiation fluid for high energies. An interesting extension to this purely kinetic model is the addition of a potential term to the Lagrangian, which could leave the kinetic dominated line stable at early times, setting the initial conditions necessary for a later evolution as dark matter plus dark energy if the potential becomes flat at late times; see also Ref. [15].

The scalar field dominated solution (c) and the scaling solution (d) are not in general critical points of the system except for the case in which the potentials in the Lagrangian satisfy the particular relation (36). This relation for the canonical scalar field means that the potential has to be exponential, and for the noncanonical field means

that the Lagrangian has to satisfy (37). As in the canonical case, the scaling solution corresponds to a stable node or a stable spiral, and the scalar field dominated solution behaves as a stable node or saddle point. If condition (36) is not satisfied, even if $dx/dN = 0$ and $dy/dN = 0$ for a particular time, the variables will evolve because the time dependence on σ will drive $x' \neq 0$ and $y' \neq 0$ as time passes. However, in the cases in which (36) is satisfied we can obtain scaling solutions despite the fact that the Lagrangian cannot be reduced to the form $\mathcal{L} = Xg(Xe^{\lambda\phi})$, which is studied in Ref. [40] as the general form of scalar fields with scaling solutions; but we are not considering here an interaction with the matter component as in that case.

In order to study a bouncing cosmology we had to redefine the dynamical variables to some more suited to the problem as (46). We obtained the conditions (55) and (56) necessary for a bounce. In the phase space it is seen as the possibility to have a crossing of the \tilde{y} axis from the negative to the positive \tilde{x} region.

The dynamical variables (\tilde{x}, \tilde{y}) and (x, y) are related by the transformation $\tilde{x} = 1/x$ and $\tilde{y} = y/x$ which means that the critical points of both dynamical systems coincide when both are valid. This happens when ρ_k and ρ_V are positive; otherwise, the variables x, y are not defined, and when $x \neq 0$. It can be seen that the points of Table I are also critical points of the new system except for those with $x = 0$.

We split the analysis of the bouncing system in two cases, ρ_k negative (phantom scalar field) and ρ_k positive, and obtained that in order to have a bounce we need $\omega_k > -1$ for the first case and $\omega_k < -1$ for the second one.

For a canonical scalar field, we know that a negative potential can lead to a crossing of the \tilde{x} axis ($H = 0$) only for recollapse, and not for a bounce. Here we showed that for certain values of ω_k a bounce is possible even for ρ_k positive, giving the possibility of a potentially driven bounce. We also showed that the conditions (55) and (56) obtained in terms of the dynamical variables can be ultimately understood as $\rho_\phi < 0$ and $\omega_\phi > \omega_m$, better seen from Fig. 10 as the conditions to have zero energy density at the bounce and positive energy density immediately after and immediately before it.

We showed that the field has to violate the null energy condition in order to account for the bounce, as seen in Fig. 11. This is a well-known result that can have implications concerning the stability of the field. In this paper we did not deal with the inhomogeneous perturbations; however, it has been argued that this type of Lagrangians have both classical and quantum stability problems when they violate the NEC [37,43]. All the former arguments make us conclude that possibly fields as simple as F - V are not good candidates to violate NEC and therefore to produce a bounce. The study of other types of fields might be in

order, but it escapes the purpose of the present paper where only the homogeneous dynamics of the fields was considered.

ACKNOWLEDGMENTS

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APPENDIX: SYMMETRY FOR PARTICULAR LAGRANGIANS

The critical points (b) and (d) from Table I exist only for canonical scalar fields with exponential potential or for scalar fields whose Lagrangians are of the form

$$\mathcal{L} = AX^\eta - B(\phi - \phi_0)^n \quad (\text{A1})$$

with

$$\eta = \frac{n}{2+n}. \quad (\text{A2})$$

In these cases the system presents a symmetry that allows the number of degrees of freedom to be reduced to two, and the dynamical system to be described only by x and y . For the canonical scalar field with exponential potential, this symmetry was described in [41].

The equations of motion (2)–(4) plus the continuity equation for the barotropic component can be written for a Lagrangian of the form (A1) as

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} [(2\eta - 1)AX^\eta + B\phi^n + \rho_m], \quad (\text{A3})$$

$$H \frac{dH}{dN} = -\frac{1}{2M_{\text{Pl}}^2} [2\eta AX^\eta + (1 + \omega_m)\rho_m], \quad (\text{A4})$$

$$\frac{d\rho_m}{dN} = -3(1 + \omega_m)\rho_m, \quad (\text{A5})$$

$$\frac{d}{dN} ((2\eta - 1)AX^\eta + B\phi^n) = -6\eta AX^\eta, \quad (\text{A6})$$

where, for simplicity, we considered $\phi_0 = 0$. Here ϕ , X , and ρ_m are the independent variables and the transformation

$$\phi \rightarrow \xi^{2\eta} \phi, \quad X \rightarrow \xi^{2n} X, \quad \rho_m \rightarrow \xi^{2n\eta} \rho_m \quad (\text{A7})$$

will leave invariant the equations of motion as long as the Hubble parameter also transforms as $H \rightarrow \xi^n H$, but its transformation is already determined by the relation

$$X = \frac{1}{2} \left(H \frac{d\phi}{dN} \right)^2, \quad (\text{A8})$$

which implies that H transforms as $\xi^{n-2\eta}H$. In order to have the correct transformation relation for the Hubble parameter then it is needed that $n\eta = n - 2\eta$ which is equivalent to the relation (A2), only in that case the transformation (A7) will represent a symmetry of the system leaving invariant the equations of motion.

The presence of the symmetry transformation (A7) when (A2) holds means that the number of degrees of freedom in the equations of motion can be reduced by one. For this, a set of variables invariant under the transformation needs to be defined, in this case x and y are already invariant. Any dynamical variable can be written in terms of those two variables, for example, σ satisfies the relation

$$\sigma = s \left(\frac{x}{y} \right)^{2/n}, \quad (\text{A9})$$

where s is a constant defined by the parameters in the Lagrangian as

$$s \equiv -\sqrt{\frac{2}{3}} M_{\text{Pl}} n B^{1/n} (A(2\eta - 1))^{-1/2\eta}. \quad (\text{A10})$$

From this relation, the dynamical system can be rewritten as

$$\frac{dx}{dN} = \frac{3}{2} \left[sxy \left(\frac{y}{x} \right)^{2/(\omega_k+1)} - x(\omega_k + 1) \right] + \frac{3}{2} x [(1 + \omega_m)(1 - y^2) + x^2(\omega_k - \omega_m)], \quad (\text{A11})$$

$$\frac{dy}{dN} = -\frac{3}{2} s x^2 \left(\frac{y}{x} \right)^{2/(\omega_k+1)} + \frac{3}{2} y [(1 + \omega_m)(1 - y^2) + x^2(\omega_k - \omega_m)], \quad (\text{A12})$$

corresponding to only two equations for two variables.

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