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We investigate the evolution of scalar metric perturbations across a sudden cosmological transition, allowing for an inhomogeneous surface stress at the transition leading to a discontinuity in the local expansion rate, such as might be expected in a big crunch/big bang event. We assume that the transition occurs when some function of local matter variables reaches a critical value, and that the surface stress is also a function of local matter variables. In particular we consider the case of a single scalar field and show that a necessary condition for the surface stress tensor to be perturbed at the transition is the presence of a non-zero intrinsic entropy perturbation of the scalar field. We present the matching conditions in terms of gauge-invariant variables assuming a sudden transition to a fluid-dominated universe with barotropic equation of state. For adiabatic perturbations the comoving curvature perturbation is continuous at the transition, while the Newtonian potential may be discontinuous if there is a discontinuity in the background Hubble expansion.

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I. INTRODUCTION

The evolution of cosmological perturbations through a sudden transition in the cosmological evolution has been a subject of much interest over the past decade or so. Starting with the original idea due to Gasperini and Veneziano of the Pre Big Bang scenario, which is based on the effective four dimensional low energy string action (for reviews see [1, 2]), through to the more recent ideas due to Steinhardt, Turok and collaborators, which is motivated by brane collisions in M-theory, whether it be through a single collision arising in the Ekpyrotic scenario [3] - [7], or as a result of a series of brane collisions as in the Cyclic scenario [8] - [11], dealing with the propagation of cosmological perturbations through the collision region has generated considerable debate.

Of particular concern has been the question of whether the approximately scale-invariant spectrum of fluctuations as observed in the microwave background anisotropies [12] can be generated in these type of collapse/re-expansion models, as opposed to the case of the Inflationary Universe where its origin is generally accepted. Addressing these issues is not so straightforward. The equations of motion can become singular as the transition region is reached, or can become ill defined in that extra terms may be induced which we do not have control over. Moreover, most attempts to model the behaviour of the propagating modes to date have been based on an effective four-dimensional description of the Universe. In the case of models relying explicitly on the evolution of branes in an extra fifth dimension, such an explanation may have its limitations. Indeed in [10, 13] the authors argue that the extra dimension plays a vital role in determining the evolution of the perturbations through the transition. This could be important, there have been many results published to date, and depending on the approach taken, whether it be an effective four-dimensional or five-dimensional transition, there are claims that the spectrum of perturbations is either consistent with observations or completely inconsistent. In addressing this thorny issue we have to be clear about the assumptions that are being made, because it could well be that the apparent inconsistencies in the published results have arisen because the authors are not in fact discussing the same physical problem.

One of the problems we face in trying to determine the true evolution of the perturbations is to describe the matching conditions across the transition. In this short paper we propose an improved method for achieving this. The system we will be investigating is four dimensional and although we will be primarily interested in cases where there is an abrupt change in the expansion rate at the matching surface, motivated by attempts to match cosmological perturbations through a "bounce" from contraction to expansion, our results may also be applied to other cosmological transitions such as reheating at the end of inflation where this may also be modelled by a sudden transition. We generalise earlier studies of cosmological matching conditions [14, 15, 16] by allowing a discontinuity in the expansion due to a surface stress tensor at the transition, as suggested by Durrer and Vernizzi [17]. We will require continuity of the metric, but not its normal derivatives.

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The method we follow involves introducing a locally defined scalar function $\Sigma(\chi_I)$, where the moduli χ represent any local scalar observable. At the transition, where the matching is described across a spatial hypersurface, Σ reaches a critical value. By expanding about this critical value in terms of the moduli, we derive a set of conditions that have to be satisfied if an abrupt change in the cosmological expansion across the spatial hypersurface is due to a stress tensor that has a finite effect on the expansion even in the limit of an infinitely short transition. We impose causality by requiring that the surface stress tensor is also a local function of the moduli. Related approaches have been adopted by a number of authors [17, 18, 20, 21]. We use the junction conditions due to Israel [22] to obtain a set of equations that relate perturbation variables on either side of the transition, in particular to consider the behaviour across the transition of gauge-invariant quantities such as the longitudinal gauge curvature perturbation, Ψ, the curvature perturbation on uniform-density hypersurfaces, ζ, and the comoving curvature perturbation, $\mathcal{R}$. As a particular example we consider the case of a single scalar field cosmology (i.e. where the moduli are the local values of the field φ and its velocity $\dot\varphi$). By identifying the intrinsic entropy perturbation of the single field (or isocurvature perturbation), S_φ, we find that we require a non-zero entropy perturbation, for the surface stress tensor to be perturbed at the transition. In other words, we see that for the case of purely adiabatic incoming perturbations, there will be no perturbation in the matching stress tensor and it remains uniform over the transition surface. Armed with this we proceed to show how the incoming perturbations on a matching surface can be matched to the outgoing perturbations after the transition (assuming a fluid-dominated cosmology with fixed equation of state), relating the gauge invariant perturbations $\mathcal{R}$, Ψ and the intrinsic entropy perturbation S_φ. As a result, amongst other things, we are able to understand how for the case of purely adiabatic field perturbations the comoving curvature perturbation remains constant across the transition surface independent of what happens to the background expansion rate. On the other hand, we are able to see that Ψ does in general change across the transition surface unless certain specific conditions are satisfied.

The layout of the paper is as follows. In Section II we introduce the idea of the local matching conditions involving matching across a spatial hypersurface (see also [18]). This is followed in section III by the introduction of the scalar metric perturbations and the gauge invariant perturbations Ψ, $\mathcal{R}$ and ζ. The junction conditions and resulting perturbed equations are presented in section IV. In section V we apply our general analysis to the specific case of a single scalar field cosmology, introducing the important role of the intrinsic entropy perturbation and demonstrating how the matching conditions depend crucially on the presence of that term. We conclude in section VI.

II.   LOCAL MATCHING CONDITION

Our aim is to discuss the evolution of cosmological perturbations through a sudden transition in the cosmological evolution. We will work in the idealised limit in which the transition is described by matching across a spatial hypersurface, i.e., an infinitesimally short transition. We expect this to be sufficient to describe the evolution of perturbations on scales large with respect to the actual duration of any transition.

We assume that the transition occurs when some function of local observables, such as a scalar field value or the local energy density, reaches a critical value which we describe through the locally defined scalar function $\Sigma(\chi_I)$. The N "moduli" χ_I schematically represent any local scalar observables. For example in the case of a single scalar field we might consider $\Sigma(\varphi, \dot\varphi)$ where $\dot\varphi$ is the local proper time derivative. We can then define the matching surface as

$$\Sigma(\chi_I) = 0\,. \tag{1}$$

In a spatially homogeneous FRW spacetime the matching surface must coincide with a fixed conformal time η. In a perturbed spacetime this need no longer be the case in general. Any 4-scalar, which is spatially homogeneous to zeroth-order, transforms under a first-order gauge shift as

$$\eta \;\to\; \eta + \xi^0\,, \tag{2}$$
$$\delta\Sigma \;\to\; \delta\Sigma - \Sigma'\xi^0\,. \tag{3}$$

where Σ' (which we require to be non-zero in the neighbourhood of the transition) denotes the zeroth-order derivative with respect to conformal time of $\Sigma(\chi_I)$. In particular we can choose to work in a temporal gauge in which $\delta\Sigma = 0$, i.e., the matching surface coincides with a constant-η hypersurface in the perturbed spacetime.

In practice we will only be interested in the function Σ in the neighbourhood of the transition. All the information we need to describe the homogeneous background transition are the values of the moduli at $\Sigma = 0$, which we will write as the parameters $\chi_I|_\Sigma$.

To describe first-order perturbations we will also need to know the values of the first-derivatives of Σ with respect to each modulus at the transition:

$$\Sigma_I \equiv \left(\frac{\partial\Sigma}{\partial\chi_I}\right)_{\Sigma=0}\,. \tag{4}$$

Thus for a study of the first-order perturbations it will be sufficient to write Σ in the neighbourhood of the transition as a linear function of the moduli

$$\Sigma(\chi_I) = \sum_I \Sigma_I(\chi_I - \chi_I|_\Sigma) . \tag{5}$$

Only the study of higher-order perturbations would require knowledge of the higher-order derivatives at $\Sigma = 0$.

We will be primarily interested in cases where there is an abrupt change in the expansion rate at the matching surface, motivated by attempts to match cosmological perturbations through a "bounce" from contraction to expansion. But our results may also be applied to other cosmological transitions such as reheating at the end of inflation. Any sudden change in the cosmological expansion across a spatial hypersurface must be due to a "singular" stress tensor S_{ij} (singular in the sense that it still has a finite effect on the expansion in the limit of an infinitely short transition). We can split this source term into a trace and tracefree part

$$S_{ij} \equiv P_s q_{ij} + a \left(\partial_i \partial_j - \frac{1}{3} q_{ij} \partial^2 \right) \Pi_s . \tag{6}$$

The assumption that we are dealing with small perturbations about an FRW geometry requires the stress tensor to be isotropic to zeroth-order in the perturbations, i.e., the anisotropic stress potential Π_s is at most first order.

A cosmological bounce requires a violation of some of the usual energy conditions. In particular a bounce at finite value of the scale factor in a flat FRW universe requires violation of the null energy condition, $\rho + P \geq 0$ [25]. We will restrict any such pathologies to the stress tensor S_{ij} that is localised on the matching surface. There is a close analogy with recent studies of brane world geometries where matter sources are assumed to be localised on a lower dimensional spacetime (or 'brane') in a higher dimensional 'bulk'. The difference in our case is that the matching surface is not timelike, but spacelike. Matter is localised on a brane because it is supposed to represent open string modes which at low energies are tied to the brane. In our setting the localised stress on the spacelike brane can include high-energy excitations that are frozen out at low energies away from the matching surface.

The appearance of a localised source on a spacelike surface may appear rather odd if it looks like the spontaneous appearance of some energy-momentum out of nothing, but this need not be so. It can approximate some actual time-evolution of the energy-momentum tensor in the idealised limit where the timescale for this evolution is small - infinitesimally so - with respect to all other time-scales.

Durrer and Vernizzi [17] have emphasized that if the spacelike matching surface itself has an inhomogeneous stress tensor then these inhomogeneities could in principle be imprinted on the outgoing cosmology. While this is true it could amount to simply writing on a pattern of inhomogeneities unless this inhomogeneity can be related to pre-existing inhomogeneities in the incoming cosmology. As a result we will consider only models in which the localised stress tensor is a function of local moduli. This amounts to assuming that P_s and Π_s are scalar functions of the χ_I. In the background homogeneous solution we only need to know the one parameter

$$P_s|_\Sigma \equiv P_s(\chi_I|_\Sigma) , \tag{7}$$

as the assumption of isotropy requires $\Pi(\chi_I|_\Sigma) = 0$, while to first-order we also need to know the $2N$ parameters

$$P_{sI} \equiv \left(\frac{\partial P_s}{\partial \chi_I} \right)_{\Sigma=0} , \tag{8}$$

$$\Pi_{sI} \equiv \left(\frac{\partial \Pi_s}{\partial \chi_I} \right)_{\Sigma=0} . \tag{9}$$

III. SCALAR METRIC PERTURBATIONS

We will write the linearly perturbed 4D metric, using the notation of Ref. [23], as

$$ds^2 = a^2 \left[-(1 + 2\phi)d\eta^2 + 2B_{,i}dx^i d\eta + \{(1 - 2\psi)\delta_{ij} + 2E_{,ij}\} dx^i dx^j \right] . \tag{10}$$

where commas denote partial derivatives with respect to spatial coordinates.

The induced metric on any constant-η hypersurface is simply the spatial part of the metric in this gauge

$$q_{ij} = a^2 \left[(1 - 2\psi)\delta_{ij} + 2E_{,ij} \right] . \tag{11}$$

The extrinsic curvature of the hypersurfaces, K_{ij}, may be split into a trace and tracefree part

$$K_{ij} \equiv \frac{1}{3}\theta q_{ij} + a\left(\partial_i \partial_j - \frac{1}{3}q_{ij}\partial^2\right)\sigma\,, \tag{12}$$

corresponding to the expansion and shear of the orthogonal vector field. (The vorticity vanishes for a hypersurface orthogonal vector field.) For the metric (10) we find [29, 30]

$$a\theta = 3h(1-\phi) - 3\psi' + \partial^2\sigma\,, \tag{13}$$

$$\sigma = E' - B\,, \tag{14}$$

where $h = a'/a$ is the conformal Hubble rate and a prime denotes derivatives with respect to conformal time.

The background expansion rate is given by the Friedmann equation

$$3h^2 = \kappa^2 a^2 \rho\,. \tag{15}$$

For first-order perturbations about an FRW spacetime the Einstein constraint equations yield

$$3h(\psi' + h\phi) - \partial^2(\psi + h\sigma) = -ah\delta\theta - \partial^2\psi = -\frac{\kappa^2}{2}a^2\delta\rho\,, \tag{16}$$

$$\psi' + h\phi = -\frac{a}{3}\delta\theta + \frac{1}{3}\partial^2\sigma = -\frac{\kappa^2}{2}a^2\delta q\,, \tag{17}$$

where δq is the momentum scalar potential.

The scalar metric perturbations ψ, ϕ and σ are invariant under spatial coordinate transformations $x^i \to x^i + \partial^i\xi$, but do transform under temporal gauge transformations:

$$\eta \to \eta + \xi^0\,, \tag{18}$$

$$\psi \to \psi + h\xi^0\,, \tag{19}$$

$$\phi \to \phi - h\xi^0 - \xi^{0\prime}\,, \tag{20}$$

$$\sigma \to \sigma - \xi^0\,. \tag{21}$$

We will be interested in the behaviour across the transition of gauge-invariant quantities such as the longitudinal gauge curvature perturbation, also known as the Newtonian or Bardeen potential,

$$\Psi \equiv \psi + h\sigma\,, \tag{22}$$

the curvature perturbation on uniform-density hypersurfaces

$$-\zeta \equiv \psi + \frac{h}{\rho'}\delta\rho\,, \tag{23}$$

and the comoving curvature perturbation

$$\mathcal{R} \equiv \psi - \frac{h}{\rho + P}\delta q\,, \tag{24}$$

where the momentum scalar potential for a scalar field is given by

$$\delta q = -\frac{\varphi'\delta\varphi}{a^2}\,. \tag{25}$$

It is possible to express ζ and $\mathcal{R}$ directly in terms of the scalar metric perturbations using the Einstein constraint equations (16) and (17), respectively, to eliminate the density and momentum. This gives

$$-\zeta = \psi - \frac{2}{9(1+w)h^2}\left(ah\delta\theta + \partial^2\psi\right)\,, \tag{26}$$

$$\mathcal{R} = \psi - \frac{2}{9(1+w)h^2}\left(ah\delta\theta - h\partial^2\sigma\right)\,. \tag{27}$$

where we define the barotropic equation of state to be $w \equiv P/\rho$ and make use of the continuity equation for the background fluid

$$\rho' + 3h(1+w)\rho = 0 \,. \tag{28}$$

The uniform-density and comoving curvature perturbations differ by an amount proportional to the comoving density perturbation

$$\delta\rho_c \equiv \delta\rho - 3h\delta q = -\frac{\rho'}{h}(\zeta + \mathcal{R}) \tag{29}$$

which can be related to the Newtonian potential by the Einstein constraint equations as

$$\frac{\kappa^2}{2}a^2\delta\rho_c = \partial^2\Psi \,, \tag{30}$$

and thus vanishes on large scales if Ψ remains finite.

IV. JUNCTION CONDITIONS

We will choose coordinates on either side of the matching surface such that the matching surface occurs on a constant-η hypersurface, i.e., $\eta = \eta_-$ on one side and $\eta = \eta_+$ on the other. This fixes the temporal gauge at the collision on either side of the matching. Of course, from the perturbation in this specific gauge we can construct gauge-invariant combinations such as Φ and ζ.

Following the treatment of matching across infinitesimal shells in general relativity [22] we will require continuity of the first fundamental form on the matching surface (the induced metric):

$$[q_{ij}]_-^+ = 0 \,. \tag{31}$$

Note that the extrinsic curvature is the Lie derivative of the induced metric normal to the constant-η hypersurface. The matching condition (31) follows from assuming that the extrinsic curvature remains finite in the limit that the thickness of the matching surface (the duration of the transition) tends to zero. This is gauranteed in Einstein gravity if the usual energy conditions are obeyed [26], but this will be an additional assumption if we allow the energy conditions to be violated in the transition. If, for example, geometry itself is only an emergent phenomena, as sometimes claimed by string theorists, our classical matching conditions may not be sufficient to model a truly stringy transition. In that case we would require string theory to supply an alternative prescription for matching fundamental degrees of freedom on one side of the transition to the other.

The jump in the extrinsic curvature is due to the localised source (6)
(see also [17])

$$[K_{ij}]_-^+ = \kappa^2 \left(S_{ij} - \frac{3}{2}q_{ij}S \right) \,. \tag{32}$$

Although we will consider only linearised perturbations about an FRW metric to describe the evolution approaching or leaving the matching surface, we could in principle describe non-linear evolution through the bounce. The junction conditions (31) and (32) are obtained from the full non-linear equations of general relativity across an interval $-\epsilon < \Delta\eta < +\epsilon$ in the "thin-shell" (sudden transition) limit where $\epsilon \to 0$ and the extrinisic curvature remains finite, but where the integral of the energy momentum tensor remains finite. This is a singular hypersurface as the Ricci tensor must diverge in this limit in order for the integral, S_{ij}, to remain finite.

Equations (11) and (31) require, to zeroth-order, continuity of the scale factor

$$[a]_-^+ = 0 \,, \tag{33}$$

and, to first-order in the perturbations,

$$[-\psi\delta_{ij} + E_{,ij}]_-^+ = 0 \,. \tag{34}$$

We can split this into a trace and tracefree part which requires that ψ and E are both continuous across the transition separately. In practice E is dependent on the spatial gauge, so the physical matching condition is that ψ, which describes the perturbation in the intrinsic curvature of the spatial hypersurface, is continuous across the surface:

$$[\psi]_-^+ = 0 \,. \tag{35}$$

Equations (12), (32) and (6) require, to zeroth-order, that the discontinuity in the Hubble expansion is due to the background energy-momentum source

$$\left[\frac{h}{a}\right]_-^+ = -\frac{\kappa^2}{2}P_{s0}\,, \tag{36}$$

and, to first-order,

$$\left[\frac{1}{3}\delta\theta\right]_-^+ = \left[-\frac{1}{a}(\psi' + h\phi) + \frac{1}{3a}\partial^2\sigma\right]_-^+ = -\frac{\kappa^2}{2}\delta P_s\,, \tag{37}$$

$$[\sigma]_-^+ = \kappa^2\delta\Pi_s\,, \tag{38}$$

where we have equated the trace and tracefree parts. Note that for any energy-momentum tensor describing (one or many) scalar fields we have no anistropic stress at first-order and the shear σ must therefore be continuous.

In order to interpret the junction condition for the trace part of the extrinsic curvature it is useful to remember that this is related to the energy density via the Einstein constraint equations. Using the constraint equations (15) and (16) we can re-interpret the junction conditions (36) and (37) as describing a sudden change in the energy density across the matching surface.

V.   SINGLE SCALAR FIELD COSMOLOGY

In order to make this formalism more concrete we will consider the simplest case of a single scalar field. If we demand that the transition is triggered by a local physical quantity, this is equivalent to demanding that Σ is a function of the local values of φ and $\dot\varphi$, where to first-order we have

$$\varphi = \varphi_0 + \delta\varphi\,, \tag{39}$$

$$\dot\varphi = \frac{1}{a}\left((1 - \phi)\varphi'_0 + \delta\varphi'\right)\,. \tag{40}$$

Note that the perturbation of the local proper time derivative of the field includes a term due to the perturbation of the lapse function, ϕ.

The condition $\Sigma_- = 0$ at the bounce then imposes the constraints

$$\Sigma_1\delta\varphi_- + \frac{1}{a}\Sigma_2(\delta\varphi'_- - \varphi'_{0-}\phi_-) = 0\,, \tag{41}$$

so there is a linear relation between the first-order perturbations of the field and its derivative at the transition, fixed by the ratio Σ_1/Σ_2.

Similarly we assume that the surface stress is a function of the local field and its proper time derivative so that

$$P_s(\varphi, \dot\varphi) = P_{s0} + P_{s1}\delta\varphi_- + \frac{1}{a}P_{s2}(\delta\varphi'_- - \varphi'_0\phi_-)\,, \tag{42}$$

$$\delta\Pi_s(\varphi, \dot\varphi) = \Pi_{s1}\delta\varphi_- + \frac{1}{a}\Pi_{s2}(\delta\varphi'_- - \varphi'_0\phi_-)\,, \tag{43}$$

Using the constraint (41) we can write

$$\delta P_s = \left(P_{s1} - \frac{\Sigma_1}{\Sigma_2}P_{s2}\right)\delta\varphi_-\,, \tag{44}$$

$$\delta\Pi_s = \left(\Pi_{s1} - \frac{\Sigma_1}{\Sigma_2}\Pi_{s2}\right)\delta\varphi_-\,. \tag{45}$$

A.   Non-adiabatic perturbations

A local rotation of the perturbations in phase-space is a useful technique to understand the evolution of large-scale perturbations in cosmology by identifying adiabatic and entropy perturbations [28]. We define adiabatic perturbations

to be perturbations along the zeroth order (background) trajectory in phase space for all variables. Thus to first order we have

$$\frac{\delta x}{\dot{x}} = \frac{\delta y}{\dot{y}} \; \forall \; x, y \, . \tag{46}$$

Any perturbation orthogonal to the background trajectory represents a relative entropy (or isocurvature) perturbation

$$S_{xy} = \frac{h}{a} \left(\frac{\delta x}{\dot{x}} - \frac{\delta y}{\dot{y}} \right) \, , \tag{47}$$

where we include the local Hubble rate h/a to make S_{xy} dimensionless.

For a single scalar field described by the two-dimensional phase-space $(\varphi, \dot{\varphi})$ we can define the intrinsic entropy perturbation of the single field

$$S_\varphi = \frac{h}{a} \left(\frac{\dot{\delta\varphi}}{\ddot{\varphi}_0} - \frac{\delta\varphi}{\dot{\varphi}_0} \right) \, , \tag{48}$$

where remember that a dot denotes differentiation with respect to proper time. Thus we can re-write this in terms of conformal time derivatives as

$$S_\varphi = h \left(\frac{\delta\varphi' - \varphi'\phi}{\varphi'' - h\varphi'} - \frac{\delta\varphi}{\varphi'} \right) \, . \tag{49}$$

The comoving density perturbation for a single scalar field is

$$\delta\rho_c = \frac{1}{a^2} \left(\varphi'(\delta\varphi' - \varphi'\phi) - (\varphi'' - h\varphi')\delta\varphi \right) \, , \tag{50}$$

and hence is related to the relative entropy (48)

$$\delta\rho_c = \frac{\varphi'(\varphi'' - h\varphi')}{a^2 h} S_\varphi \, . \tag{51}$$

This is in turn related to the Newtonian potential via the Einstein constraint equation (30), and hence for a single scalar field we have

$$\begin{aligned} S_\varphi &= \frac{2h}{\kappa^2 \varphi'(\varphi'' - h\varphi')} \partial^2 \Psi \, , \\ &= -\frac{4}{9h^2(1+w)(1+c_s^2)} \partial^2 \Psi \, . \end{aligned} \tag{52}$$

where we have written $w = P/\rho$ and $c_s^2 = \dot{P}/\dot{\rho}$ for the scalar field energy and pressure.

We can re-write Eq. (41) as a constraint relating the field fluctuations on the transition surface to the instrinsic entropy fluctuation

$$\left(\Sigma_1 + \frac{\varphi'' - h\varphi'}{a\varphi'} \Sigma_2 \right) \delta\varphi_- = -\Sigma_2 \left(\frac{\varphi'' - h\varphi'}{ah} \right) S_\varphi \, . \tag{53}$$

where we have used Eq. (48) to eliminate $\delta\varphi' - \varphi'_0\phi$.

We will write this as

$$\delta\varphi_- = -s_\Sigma \frac{\varphi'}{h} S_{\varphi-} \, , \tag{54}$$

where

$$s_\Sigma \equiv \frac{\Sigma_2 (\varphi'' - h\varphi')}{\Sigma_1 a\varphi' + \Sigma_2 (\varphi'' - h\varphi')} \, . \tag{55}$$

The dimensionless ratio s_Σ defines the rate of change of Σ due to the change of φ versus that due to the change $\dot{\varphi}$. We see that in the absence of intrinsic entropy fluctuations ($S_{\varphi-} = 0$) there can be no field perturbations at the matching surface, $\delta\varphi_- = 0$. [33]

From Eqs. (54), (44) and (45) we have

$$\delta P_s = -(s_P - s_\Sigma)\frac{P'_s}{h_-}S_{\varphi-},$$

(56)

$$\delta \Pi_s = -(s_\Pi - s_\Sigma)\frac{\Pi'_s}{h_-}S_{\varphi-},$$

(57)

where we have defined

$$s_P \equiv \frac{P_{s2}(\varphi'' - h\varphi')}{P_{s1}a\varphi' + P_{s2}(\varphi'' - h\varphi')},$$

(58)

$$s_\Pi \equiv \frac{\Pi_{s2}(\varphi'' - h\varphi')}{\Pi_{s1}a\varphi' + \Pi_{s2}(\varphi'' - h\varphi')}.$$

(59)

Thus we see from Eqs. (44) and (45) that we require a non-zero intrinsic entropy perturbation for the surface stress tensor to be perturbed at the transition. For purely adiabatic incoming perturbations the matching surface stress tensor is necessarily unperturbed.

B. Incoming perturbations on matching surface

We can now express the metric perturbations ψ_-, $\delta\theta_-$ and σ_- on the matching surface in terms of the incoming field perturbations $\delta\varphi_-$ and the gauge-invariant curvature perturbations $\mathcal{R}_-$ and Ψ_-.

From the definition of the comoving curvature perturbation, $\mathcal{R}$ in Eq.(24), we can write the curvature perturbation on the matching surface as

$$\psi_- = \mathcal{R}_- - \frac{h_-}{\varphi'}\delta\varphi_-.$$

(60)

On the other hand, the shear perturbation on the matching surface can be written, using Eq. (22), in terms of ψ_- and the Newtonian potential

$$\sigma_- = \frac{1}{h_-}(\Psi_- - \psi_-),$$

(61)

and hence

$$\sigma_- = \frac{1}{h}\left(\Psi_- - \mathcal{R}_- + \frac{h_-}{\varphi'}\delta\varphi_-\right).$$

(62)

Finally, the perturbed expansion $\delta\theta$ is related to the field perturbation via the momentum constraint (17).

$$a\delta\theta_- = -\frac{3\kappa^2}{2}\varphi'\delta\varphi_- + \partial^2\sigma_-,$$

(63)

and hence

$$a\delta\theta_- = -\frac{3\kappa^2}{2}\varphi'\delta\varphi_- + \frac{1}{h_-}\partial^2\left(\Psi_- - \mathcal{R}_- + \frac{h_-}{\varphi'}\delta\varphi_-\right).$$

(64)

Equations (60), (62) and (64) give the geometrical perturbations on the incoming side of the matching surface in terms of the field fluctuations $\delta\varphi_-$ and the gauge-invariant metric perturbations $\mathcal{R}_-$ and Ψ_-. However we have seen that the field perturbations on the matching surface $\delta\varphi_-$ can be related to the gauge-invariant entropy perturbations S_- for a single fluid through Eq. (54). Thus we have

$$\psi_- = \mathcal{R}_- + s_\Sigma S_{\varphi-},$$

(65)

$$a\delta\theta_- = \frac{9(1+w_-)h_-}{2}s_\Sigma S_{\varphi-} + \frac{1}{h_-}\partial^2(\Psi_- - \mathcal{R}_- - s_\Sigma S_{\varphi-}),$$

(66)

$$\sigma_- = \frac{1}{h_-}(\Psi_- - \mathcal{R}_- - s_\Sigma S_{\varphi-}).$$

(67)

We note that in the absence of entropy perturbations we have

$$\psi_- = \mathcal{R}_- , \tag{68}$$

$$a\delta\theta_- = \frac{1}{h_-}\partial^2 \left(\Psi_- - \mathcal{R}_-\right) , \tag{69}$$

$$\sigma_- = \frac{1}{h_-} \left(\Psi_- - \mathcal{R}_-\right) . \tag{70}$$

Although we have written the three metric perturbations on the matching surface ψ_-, $\delta\theta_-$ and σ_- in terms of the three gauge-invariant variable $\mathcal{R}_-$, Ψ_- and $S_{\varphi-}$, two of the gauge-invariant variables, Ψ_- and $S_{\varphi-}$, are related by the constraint Eq. (52). This requires the entropy perturbation to be much smaller than Ψ on large (typically super-Hubble) scales.

C. Outgoing perturbations

We will now reconstruct the outgoing perturbations after the transition assuming a fluid dominated cosmology with fixed equation of state $w_+ \equiv (P/\rho)_+$.

The outgoing comoving curvature perturbation is written using Eq. (27) as

$$\mathcal{R}_+ = \psi_+ + \frac{2}{9(1 + w_+)h_+} \left(-a\delta\theta_+ + \partial^2\sigma_+\right) . \tag{71}$$

and thus using the junction conditions (35), (37) and (38) we have

$$\mathcal{R}_+ = \psi_- + \frac{2}{9(1 + w_+)h_+} \left(-a\left[\delta\theta_- - \frac{3}{2}\kappa^2\delta P_s\right] + \partial^2\left[\sigma_- + \kappa^2\delta\Pi_s\right]\right) . \tag{72}$$

We can write this in terms of the incoming gauge-invariant perturbations with a single scalar field, using Eqs. (56), (57), (65) and (66), as

$$\mathcal{R}_+ = \mathcal{R}_- + \left[\left(1 - \frac{(1 + w_-)h_-}{(1 + w_+)h_+}\right) s_\Sigma \right.$$
$$\left. - \frac{2\kappa^2}{9(1 + w_+)h_+} \left(\frac{3a}{2}\frac{P_s'}{h_-}(s_P - s_\Sigma) + \frac{\Pi_s'}{h_-}(s_\Pi - s_\Sigma)\partial^2\right)\right] S_{\varphi-,} . \tag{73}$$

The outgoing gauge-invariant perturbation Ψ_+ is given from Eq. (22) as

$$\Psi_+ \equiv \psi_+ + h_+\sigma_+ . \tag{74}$$

The junction conditions (35) and (38) then give

$$\Psi_+ = \psi_- + h_+ \left(\sigma_- + \kappa^2\delta\Pi_s\right) . \tag{75}$$

Again we can write this in terms of the incoming gauge-invariant perturbations with a single scalar field, using Eqs. (57), (65) and (67), as

$$\Psi_+ = \frac{h_+}{h_-}\Psi_- - \frac{h_+ - h_-}{h_-} \left(\mathcal{R}_- + s_\Sigma S_{\varphi-}\right) - \frac{h_+}{h_-}\kappa^2\Pi_s'(s_\Pi - s_\Sigma)S_{\varphi-} . \tag{76}$$

For the special case of purely adiabatic field perturbations we have the simple result

$$\mathcal{R}_+ = \mathcal{R}_- , \tag{77}$$

$$\Psi_+ = \frac{h_+}{h_-}\Psi_- + \left(1 - \frac{h_+}{h_-}\right)\mathcal{R}_- . \tag{78}$$

Thus the comoving curvature perturbation is constant for adiabatic perturbations across an arbitrary transition surface, $\Sigma(\varphi, \dot\varphi) = 0$, even if there is an abrupt change in the background expansion rate h, or if the localised stress-tensor on the transition surface is an arbitrary function of the scalar field or its proper time derivative. In contrast,

the Newtonian potential does change across the transition, unless there is no change in the background expansion rate, i.e., $h_+ = h_-$, or the special case of vanishing comoving shear, in which case $\Psi = \mathcal{R}$.

The adiabatic limit is a very restrictive case, but it is one that is a good approximation in cases where there is a unique attractor in phase-space and the scalar field perturbations are in a squeezed state, as is the case during slow-roll inflation, or an ekpyrotic-type collapse.

In practice the intrinsic entropy perturbation of the scalar field is related to the divergence of the Newtonian potential through Eq. (52). Thus we can write the general transfer matrix through a sudden transition as

$$
\begin{pmatrix} \mathcal{R}_+ \\ \Psi_+ \end{pmatrix} = \begin{pmatrix} T_{RR} & T_{R\Psi} \\ T_{\Psi R} & T_{\Psi\Psi} \end{pmatrix} \begin{pmatrix} \mathcal{R}_- \\ \Psi_- \end{pmatrix} .
\tag{79}
$$

where we can split the transfer matrix into terms of order k^0, k^2 and k^4:

$$
T_{ij} = \sum_{n=0,1,2} k^{2n} T_{ij}^{(2n)} .
\tag{80}
$$

On large scales we expect the outgoing perturbations to be determined by the k^0-coefficients:

$$
T_{RR}^{(0)} = 1 ,
\tag{81}
$$

$$
T_{R\Psi}^{(0)} = 0 ,
\tag{82}
$$

$$
T_{\Psi R}^{(0)} = \frac{h_- - h_+}{h_-} ,
\tag{83}
$$

$$
T_{\Psi\Psi}^{(0)} = \frac{h_+}{h_-} .
\tag{84}
$$

In particular we see that, even if we allow the surface energy-momentum tensor to be an arbitrary function of the local field and its proper time derivative, the comoving curvature perturbation is conserved in this large-scale limit.

On finite scales the full transfer matrix includes the additional non-zero coefficients:

$$
T_{R\Psi}^{(2)} = \frac{4}{9h_-^2(1+w_-)(1+c_{s-}^2)} \left[\left(1 - \frac{(1+w_-)h_-}{(1+w_+)h_+} \right) s_\Sigma - \frac{\kappa^2 a P_s'}{3(1+w_+)h_+h_-}(s_P - s_\Sigma) \right] ,
\tag{85}
$$

$$
T_{R\Psi}^{(4)} = \frac{4}{9h_-^2(1+w_-)(1+c_{s-}^2)} \left[\frac{2\kappa^2}{9(1+w_+)h_+} \left(\frac{\Pi_s'}{h_-}(s_\Pi - s_\Sigma) \right) \right] ,
\tag{86}
$$

$$
T_{\Psi\Psi}^{(2)} = -\frac{4}{9h_-^2(1+w_-)(1+c_{s-}^2)} \left[\frac{h_+ - h_-}{h_-} s_\Sigma + \frac{h_+}{h_-}\kappa^2 \Pi_s'(s_\Pi - s_\Sigma) \right] .
\tag{87}
$$

All the other coefficients in Eq. (80) are zero. If all the functions Σ, P_s and Π_s are functions only of the local value of the scalar field, φ, and not its time derivative, then s_Σ, s_P and s_Π all vanish and only the k^0 coefficients $T_{ij}^{(0)}$ are non-zero.

VI. CONCLUSIONS

In this paper we have presented the general cosmological matching conditions for linear scalar metric perturbations at a sudden transition across a spacelike hypersurface, allowing for a perturbed surface stress tensor leading to a finite change in the extrinsic curvature. This describes the behaviour of perturbations on scales much larger than the characteristic length associated with causal propagation during a cosmological transition such as the end of inflation, or a bounce at the end of a pre big bang-type model.

Durrer and Vernizzi [17] emphasized that an inhomogeneous surface stress at the transition might lead to a change in the comoving curvature perturbation, $\mathcal{R}$. This could be of great importance in the ekpyrotic or cyclic scenarios where the comoving curvature perturbation during the pre big bang phase does not have a scale-invariant spectrum [5], even though the Newtonian potential, Ψ, does.

We studied the case of a sudden transition from a universe dominated by a single scalar field to a radiation-dominated era, giving expressions for the outgoing gauge-invariant perturbations Ψ and $\mathcal{R}$ in terms of the incoming perturbations. We found that if the matching surface (both the transition time and the surface stress tensor) can be characterised by functions of the local scalar field and its proper time derivatives then any change in the comoving curvature perturbation on large scales must be due to intrinsic entropy pertubations in the scalar field. However

during slow-roll inflation or an ekpyrotic collapse the scalar field is driven into a squeezed state on super-Hubble scales and entropy perturbations are heavily suppressed. In particular the intrinsic entropy perturbation of a scalar field is proportional to the divergence of the Newtonian potential, $\partial^2\Psi$, and has a steep blue spectrum even when the Newtonian potential is almost scale invariant.

If we simplify the analysis by considering only adiabatic perturbations then the surface stress tensor must be unperturbed, and we obtain

$$[\mathcal{R}]^+_- = 0 \,, \tag{88}$$

$$[\Psi]^+_- = \left(\frac{H_+ - H_-}{H_-}\right)(\Psi_- - \mathcal{R}_-) \,. \tag{89}$$

If the background expansion is continuous, $[H]^+_- = 0$, then we recover the standard result that both $\mathcal{R}$ and Ψ are both continuous [14, 15, 16]. A sudden transition in the Hubble expansion can lead to a sudden jump in the Newtonian potential Ψ, but the comoving curvature perturbation remains constant. This is in accordance with the general rule that there exists a conserved curvature perturbation for adiabatic perturbations on large scales so long as local energy conservation holds [31, 32].

Our results re-inforce previous claims (see, e.g., [5]) based on the evolution of cosmological perturbations in general relativity, and recent studies [19, 20, 21] which seek to go beyond general relativity. We emphasise that our conclusions apply to any model of a sudden transition between two epochs described by general relativity, even if general relativity breaks down during the transition, so long as physics is still local and the transition is described in terms of local moduli or matter variables.

Acknowledgements

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