Non-Gaussianities in two-field inflation

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Abstract

We study the bispectrum of the curvature perturbation on uniform energy density hypersurfaces in models of inflation with two scalar fields evolving simultaneously. In the case of a separable potential, it is possible to compute the curvature perturbation up to second order in the perturbations, generated on large scales due to the presence of non-adiabatic perturbations, by employing the δN-formalism, in the slow-roll approximation. In this case, we provide an analytic formula for the nonlinear parameter $f_{\rm NL}$. We apply this formula to double inflation with two massive fields, showing that it does not generate significant non-Gaussianity; the nonlinear parameter at the end of inflation is slow-roll suppressed. Finally, we develop a numerical method for generic two-field models of inflation, which allows us to go beyond the slow-roll approximation and confirms our analytic results for double inflation.

1   Introduction

A key prediction of a period of inflation in the very early universe, is the generation of a spectrum of primordial perturbations. Such perturbations naturally arise from the zero point vacuum fluctuations in quantum fields, which are stretched to arbitrarily large scales during inflation. The distribution of the primordial density perturbations thus provides an important test of any inflation model. In particular, slow-roll models of inflation generically predict an almost scale-invariant, almost Gaussian distribution of primordial density perturbations [1, 2].

Typically one calculates the power spectrum of density perturbations and the relative amplitude of gravitational waves. However, there is limited information in the power spectrum over the restricted range of scales where the primordial power spectrum can be reliably inferred from observations. As a result, there has been growing interest in calculating the distribution of primordial perturbations from inflation, considering not only the power spectrum but also the bispectrum and other measures of possible deviations from purely Gaussian distribution as a possible discriminant between different models [3, 4]. For instance, it is known that the size of the bispectrum during single-field, slow-roll inflation is related to the spectral tilt of the power spectrum and is thus constrained to be small [5]. On the other hand, inflationary models with higher derivative operators, such as ghost inflation or inflation based on the Dirac-Born-Infeld action, can produce higher non-Gaussianity [6, 7, 8, 9], possibly detectable. Furthermore, non-adiabatic perturbations produced during *multi-field* inflation can certainly generate detectable non-Gaussianity in the density field *after* inflation, in models such as the curvaton scenario [10] (see also [11]) or reheating [12, 13, 14, 15].

What is less clear is whether nonlinear large-scale evolution of perturbations *during* inflation is capable of producing significant non-Gaussianity. Such a study requires a consistent treatment of nonlinear perturbations, not just in the matter fields but also in the gravitational field. Recently a number of authors have developed gauge-invariant descriptions of the nonlinear curvature perturbation on large scales [16, 17, 18, 19, 20, 21].

Rigopoulos *et al.* [22] have used numerical simulations of the stochastic dynamics on large scales and found significant non-Gaussianity even in simple two-field models. It is this question which we wish to address in this paper by presenting analytical and numerical estimates of the non-Gaussianity in two-field inflation models. We use the δN-formalism, as advocated by Lyth and Rodriguez [20], to calculate the evolution of the curvature perturbation after Hubble exit. We find that in the case of a separable potential, it is possible to construct an analytical formula to express the bispectrum of the

curvature perturbation in terms of the potentials and slow-roll parameters of the two fields. Furthermore, we also use numerical solutions to go beyond the slow-roll approximation after Hubble exit, confirming our analytic results. In the case of double inflation with two massive fields, we find no significant non-Gaussianity produced, which confirms a previous discussion by Alabidi and Lyth in Ref. [23].

Note that the multi-field scenario considered here is different from that considered in [11, 24], where only one field is contributing to the energy density during inflation.

In Sec. 2 we review the two- and three-point statistics of field perturbations during slow-roll inflation, and how in the δN-formalism these can be related to the two- and three-point statistics of the curvature perturbation on uniform density hypersurfaces in arbitrary multi-field models. In Sec. 3 we show how one can actually calculate the evolution of the primordial power spectrum and bispectrum using the slow-roll approximation in simple two-field models with a separable potential, during inflation. In Sec. 4 we discuss the issue of how the end of inflation can affect the curvature perturbation and the non-Gaussianity. In Sec. 5 we present both an analytical and numerical study of non-Gaussianity in the double quadratic inflation model studied by Rigopoulos *et al.* [22], going beyond the slow-roll approximation. We present our conclusions in Sec. 6.

2   Non-Gaussian perturbations in multi-field inflation

Here we review some of the results concerning perturbations and non-Gaussianities in multi-field inflationary models. For simplicity, we will consider a model of inflation driven by a set of minimally coupled scalar fields with canonical kinetic terms and potential W, described by the action

$$S = -\frac{m_{\mathrm{P}}^2}{2} \int d^4x \sqrt{-g} \left[\frac{1}{2} \sum_I \partial^\mu \varphi_I \partial_\mu \varphi_I + W(\varphi_1, \varphi_2, \ldots) \right], \qquad (1)$$

where $m_{\mathrm{P}}^2 \equiv 8\pi G$ is the reduced squared Planck mass.

We consider scalar perturbations in a quasi-homogeneous flat Friedmann-Lemaître-Robertson-Walker spacetime with scale factor $a(t)$ and perturbed metric

$$ds^2 = -(1 + 2A)dt^2 + 2B_{,i}dtdx^i + a(t)^2 \left[(1 - 2\psi)\delta_{ij} + 2E_{,ij} \right] dx^i dx^j. \qquad (2)$$

Primordial cosmological perturbations are usually expressed in terms of the curvature perturbation on uniform energy density hypersurfaces, denoted

by ζ, defined in [25, 26] in linear perturbation theory, and generalized at higher order in the perturbations in [16, 17, 18]. (See also [19, 27, 28] for comparison between different variables at second order.) This quantity is widely used, especially because it is conserved, on large scales, for adiabatic perturbations [29]. Here we want to compute the three-point correlation function of ζ and hence we need to define ζ only up to *second order* in the perturbations,

$$\zeta \equiv -\psi - \psi^2 - \frac{H}{\dot{\rho}}\delta\rho + \frac{1}{\dot{\rho}}\dot{\psi}\delta\rho + \frac{H}{\dot{\rho}^2}\delta\rho\dot{\delta\rho} + \frac{1}{2\dot{\rho}}\left(\frac{H}{\dot{\rho}}\right)^{\cdot}\delta\rho^2, \tag{3}$$

where ρ is the background energy density, $\delta\rho$ its perturbation, and $H \equiv \dot{a}/a$ is the Hubble rate, while a dot denotes the derivative with respect to cosmic time t.

According to the so called δN-formalism [30, 31, 32], ζ, evaluated at some time $t = t_c$, is equivalent, *on large scales*, to the perturbation of the number of e-foldings $\mathcal{N}(t_c, t_*, \mathbf{x})$ from an initial *flat* hypersurface at $t = t_*$, to a final *comoving* – or, equivalently, *uniform density* – hypersurface at $t = t_c$. Hence, one has, on large scales,

$$\zeta(t_c, \mathbf{x}) \simeq \delta N(t_c, t_*, \mathbf{x}) \equiv \mathcal{N}(t_c, t_*, \mathbf{x}) - N(t_c, t_*), \tag{4}$$

where

$$N(t_c, t_*) \equiv \int_*^c H dt \tag{5}$$

is the unperturbed value of $\mathcal{N}$. Equation (4) simply follows from the definition of $\mathcal{N}$ as the volume expansion rate of the $t = \mathrm{const}$ hypersurface, integrated along the integral curve of the unit vector orthogonal to the $t = \mathrm{const}$ hypersurface [31]. This definition is not restricted to linear theory but holds also at second or higher order in the cosmological perturbations [32, 33, 18, 21].

One can then take t_* as the time, during inflation, when the relevant perturbation scales exited the Hubble radius, $k = aH$, and t_c as some time $> t_*$ during or after inflation. Then the number of e-foldings $\mathcal{N}$ can be viewed as a function of the field configuration $\varphi_I(t_*, \mathbf{x})$ on the flat hypersurface at $t = t_*$ and of the time t_c. If one splits the scalar fields in a background value and a perturbation, $\varphi_I(t, \mathbf{x}) \equiv \varphi_I(t) + \delta\varphi_I(t, \mathbf{x})$, $\delta N(t_c, t_*, \mathbf{x})$ can be expanded in series of the initial field perturbations $\delta\varphi_I(t_*, \mathbf{x})$. Retaining only terms up to second order, one obtains

$$\delta N(t_c, t_*, \mathbf{x}) = \sum_I N_{,I}\delta\varphi_*^I + \frac{1}{2}\sum_{IJ} N_{,IJ}\delta\varphi_*^I\delta\varphi_*^J, \tag{6}$$

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where

$$N_{,I} \equiv \frac{\partial N}{\partial \varphi_*^I}, \qquad N_{,IJ} \equiv \frac{\partial^2 N}{\partial \varphi_*^I \partial \varphi_*^J}, \tag{7}$$

are the first and second derivatives of the *unperturbed* number of e-foldings $N(t_c, t_*)$, with respect to the unperturbed values of the fields at Hubble crossing. Note that in general $\mathcal{N}$ depends on the fields, $\varphi_I(t)$, and their first time derivatives, $\dot{\varphi}_I(t)$. However, if slow-roll conditions

$$3H\dot{\varphi}_I \simeq -W_{,I}, \quad \text{at} \ \ t = t_* \tag{8}$$

are satisfied at Hubble exit, then $\mathcal{N}$ depends only on the field values [31].

In [20] it was shown that Eq. (4) with (6) can be used to compute ζ up to second order in the perturbations in multi-field models of inflation. In particular, the δN-formalism is *equivalent* to integrate the evolution of ζ on super-Hubble scales from Hubble exit until t_c.

2.1 Two-point statistics

The power spectrum of the curvature perturbation ζ, $\mathcal{P}_\zeta$, is defined as

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle \equiv (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \mathcal{P}_\zeta(k_1). \tag{9}$$

Let us now take the scalar field perturbations $\delta\varphi_I$ as uncorrelated stochastic variables at early times, with scale invariant spectrum of massless scalar fields in de-Sitter space. Their two-point correlation functions hence satisfy

$$\langle \delta\varphi_{\mathbf{k}_1}^I \delta\varphi_{\mathbf{k}_2}^J \rangle = (2\pi)^3 \delta_{IJ} \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \mathcal{P}_*(k_1), \qquad \mathcal{P}_*(k) \equiv \frac{H_*^2}{4\pi^2}, \tag{10}$$

where H_* is evaluated at Hubble exit, $k = aH$. From Eqs. (9) and (10), making use of Eqs. (4) and (6), one obtains

$$\mathcal{P}_\zeta = \sum_I N_{,I}^2 \mathcal{P}_*. \tag{11}$$

Another interesting observable that can be derived from the two-point statistics is the scalar spectral index of $\mathcal{P}_\zeta$, defined as

$$n_\zeta - 1 \equiv \frac{d\ln \mathcal{P}_\zeta}{d\ln k}. \tag{12}$$

For multi-field inflation, the expression for the scalar spectral index has been given, for instance, in [31, 1]. At lowest order in slow-roll one has [2]

$$\frac{d\ln \mathcal{P}_\zeta}{d\ln k} \simeq \frac{d\ln \mathcal{P}_\zeta}{dN} = \frac{1}{H} \frac{d\ln \mathcal{P}_\zeta}{dt}. \tag{13}$$

The first equality follows from taking k at Hubble crossing, $k = aH \propto H \exp N$, while the second is a consequence of the definition of N, Eq. (5). Making use of the expression for the scalar power spectrum $\mathcal{P}_\zeta$ given in (11), and of

$$\frac{d}{dt} = \sum_I \dot{\varphi}_I \frac{\partial}{\partial \varphi_I}, \tag{14}$$

one thus obtains

$$n_\zeta - 1 = -2\epsilon + \frac{2}{H} \frac{\sum_{IJ} \dot{\varphi}_J N_{,JI} N_{,I}}{\sum_K N_{,K}^2}, \tag{15}$$

where ϵ is the well known slow-roll parameter defined as

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \tag{16}$$

This expression can be shown to be equivalent to that given in [31]. Indeed, the particular combination of second derivatives, $\dot{\varphi}_J N_{,IJ}$, that appears in the spectral index (15) can be eliminated in favor of second derivatives of the potential using the slow-roll approximation to give [1]

$$n_\zeta - 1 = -2\epsilon - \frac{2}{m_{\rm P}^2 \sum_K N_{,K}^2} + \frac{2m_{\rm P}^2 \sum_{IJ} V_{,JI} N_{,I} N_{,J}}{V \sum_K N_{,K}^2}. \tag{17}$$

2.2 Three-point statistics

Let us now discuss the three-point statistics of the curvature perturbations. The bispectrum of the curvature perturbation ζ is defined as

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle \equiv (2\pi)^3 \delta^{(3)}(\sum_i \mathbf{k}_i) B_\zeta(k_1, k_2, k_3). \tag{18}$$

Observational limits are usually put on the so called *nonlinear parameter* $f_{\rm NL}$. We use here the definition of k-dependent $f_{\rm NL}$ given in [5],[1]

$$-\frac{6}{5} f_{\rm NL} \equiv \frac{\Pi_i k_i^3}{\sum_i k_i^3} \frac{B_\zeta}{4\pi^4 \mathcal{P}_\zeta^2}. \tag{19}$$

One can compute the bispectrum using Eqs. (4) and (6). For the three-point correlation function of ζ, this yields

$$\begin{aligned}
\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle &= \sum_{IJK} N_{,I} N_{,J} N_{,K} \langle \delta\varphi^I_{\mathbf{k}_1} \delta\varphi^J_{\mathbf{k}_2} \delta\varphi^K_{\mathbf{k}_3} \rangle + \\
&\quad \frac{1}{2} \sum_{IJKL} N_{,I} N_{,J} N_{,KL} \langle \delta\varphi^I_{\mathbf{k}_1} \delta\varphi^J_{\mathbf{k}_2} (\delta\varphi^K \star \delta\varphi^L)_{\mathbf{k}_3} \rangle + \text{perms},
\end{aligned} \tag{20}$$

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[1]Note that this choice of sign for $f_{\rm NL}$ differs from that used by [51, 52]. The minimal detectable $f_{\rm NL}$ with an ideal CMB experiment is slightly larger than unity [53, 51].

where a star denotes the convolution and we have neglected correlation functions higher than the four-point (see [35, 34]).

The first line represents the contribution from the three-point correlation functions of the fields. For purely Gaussian fields, this vanishes. However, Seery and Lidsey [35] have found that during slow-roll inflation the field three-point correlation functions do not vanish and read

$$\langle \delta\varphi^I_{\mathbf{k}_1} \delta\varphi^J_{\mathbf{k}_2} \delta\varphi^K_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^{(3)}(\sum_i \mathbf{k}_i) \frac{4\pi^4}{\Pi_i k_i^3} \mathcal{P}_*^2 \sum_{\text{perms}} \frac{\dot{\varphi}_I \delta_{JK}}{4Hm_{\text{P}}^2} \mathcal{M}, \qquad (21)$$

with

$$\mathcal{M} = \mathcal{M}(k_1, k_2, k_3) \equiv -k_1 k_2^2 - 4\frac{k_2^2 k_3^2}{k_t} + \frac{1}{2}k_1^3 + \frac{k_2^2 k_3^2}{k_t^2}(k_2 - k_3) , \qquad (22)$$

where $k_t = k_1 + k_2 + k_3$, and it is assumed that all the k_i are of the same order of magnitude so that they cross the Hubble radius approximately at the same time. The function $\mathcal{M}$, which has been written in a slightly different form from that in Ref. [35], parameterizes the k-dependence of the three-point functions. The sum is over all simultaneous rearrangements of the indices I, J, and K, and the momenta k_1, k_2, and k_3 in $\mathcal{M}$, such that the relative position of the k_i is respected [35].

The sum over the permutations of $\mathcal{M}$ over all the k_i reads

$$\begin{aligned} \mathcal{F}(k_1, k_2, k_3) &\equiv \sum_{\text{perms}} \mathcal{M}(k_1, k_2, k_3) \\ &= -2 \left(\frac{1}{2} \sum_{i\neq j} k_i k_j^2 + 4\frac{\sum_{i>j} k_i^2 k_j^2}{k_t} - \frac{1}{2} \sum_i k_i^3 \right), \qquad (23) \end{aligned}$$

and one can sum over the permutations and evaluate the first line of Eq. (20) using Eq. (21) and

$$\sum_{IJK} N_{,I} N_{,J} N_{,K} \sum_{\text{perms}} \dot{\varphi}_I \delta_{JK} \mathcal{M}(k_1, k_2, k_3) = -H \sum_I N_{,I}^2 \mathcal{F}(k_1, k_2, k_3), \qquad (24)$$

where we have used $\sum_I N_{,I} \dot{\varphi}_I = -H$.

To evaluate the second line of Eq. (20), as in [35] we assume that the connected part of the four-point correlation functions is negligible and we make use of Wick's theorem to reduce the four-point functions to products of two-point functions. Hence, we can finally rewrite the bispectrum using its definition (18) and Eq. (20). This yields, after few manipulations,

$$B(k_1, k_2, k_3) = 4\pi^4 \mathcal{P}_\zeta^2 \frac{\sum_i k_i^3}{\Pi_i k_i^3} \left(\frac{-1}{4m_{\text{P}}^2 \sum_K N_{,K}^2} \frac{\mathcal{F}}{\sum_i k_i^3} + \frac{\sum_{IJ} N_{,I} N_{,J} N_{,IJ}}{(\sum_K N_{,K}^2)^2} \right). \qquad (25)$$

To derive this expression we have used Eq. (11) to replace $\mathcal{P}_*^2$ with $\mathcal{P}_\zeta^2$.

One can relate the bispectrum to the nonlinear parameter f_{NL} by using Eq. (19),

$$-\frac{6}{5}f_{\mathrm{NL}} = \frac{\mathcal{P}_*}{2m_{\mathrm{P}}^2\mathcal{P}_\zeta}(1+f) + \frac{\sum_{IJ}N_{,I}N_{,J}N_{,IJ}}{(\sum_K N_{,K}^2)^2}, \tag{26}$$

where

$$f = f(k_1,k_2,k_3) \equiv -1 - \frac{\mathcal{F}}{2\sum_i k_i^3} \tag{27}$$

is a function of the shape of the momentum triangle with the range of values $0 \le f \le \frac{5}{6}$ [5], and we have used Eq. (11) in the first term on the right hand side of Eq. (26). The lower bound on f is obtained in the geometrical limit when one of the three k_i's is much smaller than the other two, e.g., $k_1 \ll k_2 \approx k_3$, in which case f_{NL} becomes independent of k [5, 36], while the upper bound is obtained in the equilateral triangle configuration, $k_1 \approx k_2 \approx k_3$.

The first term on the right hand side of Eq. (26) is momentum dependent and comes from the three-point correlation functions of the fields, Eq. (21), computed by quantizing the perturbations *inside* the Hubble radius during inflation [35, 37]. Notice that $\mathcal{P}_*$ can be related to the amplitude of gravitational waves. Introducing the ratio between tensor to scalar modes,

$$r \equiv \frac{8\mathcal{P}_*}{m_{\mathrm{P}}^2\mathcal{P}_\zeta}, \tag{28}$$

this term can be written as

$$-\frac{6}{5}f_{\mathrm{NL}}^{(3)} \equiv \frac{r}{16}(1+f), \tag{29}$$

and is constrained by observations to be small, $r/16 \ll 1$. This expression simplifies the bound on this term found in Ref. [37].

The second term [20],

$$-\frac{6}{5}f_{\mathrm{NL}}^{(4)} \equiv \frac{\sum_{IJ}N_{,I}N_{,J}N_{,IJ}}{(\sum_K N_{,K}^2)^2}, \tag{30}$$

is momentum independent and *local* in real space, because it is due to the evolution of nonlinearities *outside* the Hubble radius during inflation.

The nonlinear parameter f_{NL} can only be larger than unity if the momentum independent term $f_{\mathrm{NL}}^{(4)}$ is large. We will devote the next section to compute this quantity in two-field models.

3 Two-field inflation with separable potential

We will consider now two scalar fields $\varphi_1 \equiv \phi$ and $\varphi_2 \equiv \chi$, described by the action

$$S = -\frac{m_{\mathrm{P}}^2}{2} \int d^4x \sqrt{-g} \left[\frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} \partial^\mu \chi \partial_\mu \chi + W(\phi, \chi) \right] . \tag{31}$$

We assume that the potential of the fields is separable into the sum of two potentials each of which is dependent on only one of the two fields,

$$W(\phi, \chi) = U(\phi) + V(\chi) . \tag{32}$$

3.1 Slow-roll dynamics of background fields

The Klein-Gordon equations for the background fields read

$$\ddot{\phi} + 3H\dot{\phi} + U' = 0, \tag{33}$$
$$\ddot{\chi} + 3H\dot{\chi} + V' = 0, \tag{34}$$

where

$$U' \equiv \frac{dU}{d\phi}, \qquad V' \equiv \frac{dV}{d\chi}. \tag{35}$$

The unperturbed Friedmann equations read

$$H^2 = \frac{1}{3m_{\mathrm{P}}^2} \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 + W \right), \tag{36}$$

$$\dot{H} = -\frac{1}{2m_{\mathrm{P}}^2} \left(\dot{\phi}^2 + \dot{\chi}^2 \right). \tag{37}$$

Inflation takes place when $\epsilon = -\dot{H}/H^2 < 1$. Here it is assumed that the slow-roll conditions are satisfied for both fields during inflation. In this case, the Klein-Gordon and the first Friedmann equations reduce to

$$3H\dot{\phi} \simeq -U', \qquad 3H\dot{\chi} \simeq -V', \qquad 3m_{\mathrm{P}}^2 H^2 \simeq W. \tag{38}$$

It is hence convenient to extend the definition of slow-roll parameter ϵ for one field and define [38]

$$\epsilon^\phi \equiv \frac{m_{\mathrm{P}}^2}{2} \left(\frac{U'}{W} \right)^2, \qquad \epsilon^\chi \equiv \frac{m_{\mathrm{P}}^2}{2} \left(\frac{V'}{W} \right)^2, \tag{39}$$

with

$$\epsilon = \epsilon^\phi + \epsilon^\chi. \tag{40}$$

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One can also define the two slow-roll parameters

$$\eta^\phi \equiv m_{\mathrm{P}}^2 \frac{U''}{W}, \qquad \eta^\chi \equiv m_{\mathrm{P}}^2 \frac{V''}{W}. \tag{41}$$

Note that throughout this paper, for simplicity, we will assume $U' \geq 0$ and $V' \geq 0$ so that we may eliminate first derivatives of the potential in favor of the slow-roll parameters $\sqrt{\epsilon^\phi}$ and $\sqrt{\epsilon^\chi}$.

We will use the slow-roll equations (38) to write the number of e-foldings (5) during inflation as [30, 1]

$$N = -\frac{1}{m_{\mathrm{P}}^2} \int_*^c \frac{U}{U'} d\phi - \frac{1}{m_{\mathrm{P}}^2} \int_*^c \frac{V}{V'} d\chi. \tag{42}$$

There is a crucial difference between single field inflation and inflation with many fields [38]. In single field inflation the slow-roll solution forms a one-dimensional phase space. This means that once the inflationary attractor has been reached, there is a *unique* trajectory. In particular, the end of inflation takes place at a fixed value of the inflaton field which in turn corresponds to a fixed energy density. However, if two fields are present, the phase-space becomes two-dimensional and there is an infinite number of possible classical trajectories in field space. The values of the two fields at the end of inflation will in general depend on the choice of trajectory. Interestingly, for the potential (32), under the slow-roll conditions (38), there exists a dimensionless integral of motion C, using which one can label each slow-roll classical trajectory,

$$C \equiv -m_{\mathrm{P}}^2 \int \frac{d\phi}{U'} + m_{\mathrm{P}}^2 \int \frac{d\chi}{V'}. \tag{43}$$

The number of e-foldings N in Eq. (42) characterizes the evolution *along* a given trajectory, while the quantity C allows us to parameterize motion *off* the classical trajectory, and it will turn out to be very useful in computing the curvature perturbation during inflation. In this respect, the study presented here is very similar to the one discussed in Ref. [38], in the case of a potential that is a separable product rather than a separable sum.

3.2   Perturbed expansion during slow-roll

In order to calculate ζ, given in Eq. (4), we need to calculate the perturbed expansion due to the field quantum fluctuations on an initial spatially *flat* slice up to a final *comoving* or *uniform density* hypersurface at t_c. To calculate the power spectrum of ζ (11) we require only the first derivative of the

expansion with respect to the initial field values. But to calculate the scalar spectral index (15) or the three-point correlation function (20) we need to compute the perturbation in the number of e-foldings expanded up to *second order* in the initial field perturbations. We will thus proceed as follows. We will first compute the first derivative of $N(t_c, t_*)$ with respect to the fields, i.e., $N_{,\phi}$ and $N_{,\chi}$, by differentiating $N(t_c, t_*)$ in Eq. (42) in $d\phi_*$ and $d\chi_*$. With these two first derivatives we will be able to compute the second derivatives $N_{,IJ}$.

We first note that each of the integrals in Eq. (42) depends upon both ϕ_* and χ_*. For instance, the value of the integral over ϕ depends upon χ_* through the dependence of the limit ϕ_c upon $C(\phi_*, \chi_*)$, the integral of motion (43) labelling the classical trajectory. In other words one has

$$
\begin{aligned}
dN = {} & \frac{1}{m_{\mathrm{P}}^2} \left[\left(\frac{U}{U'} \right)_* - \frac{\partial \chi_c}{\partial \phi_*} \left(\frac{V}{V'} \right)_c - \frac{\partial \phi_c}{\partial \phi_*} \left(\frac{U}{U'} \right)_c \right] d\phi_* \\
& + \frac{1}{m_{\mathrm{P}}^2} \left[\left(\frac{V}{V'} \right)_* - \frac{\partial \chi_c}{\partial \chi_*} \left(\frac{V}{V'} \right)_c - \frac{\partial \phi_c}{\partial \chi_*} \left(\frac{U}{U'} \right)_c \right] d\chi_*.
\end{aligned}
\tag{44}
$$

In order to compute the derivatives of the number of e-foldings with respect the initial fields ϕ_* and χ_*, one needs to compute $\partial \chi_c / \partial \chi_*$, $\partial \chi_c / \partial \phi_*$, etc. Since the value of ϕ_c and χ_c on a given classical trajectory is a function of the conserved quantity C, one has

$$
\begin{aligned}
d\phi_c &= \frac{d\phi_c}{dC} \left(\frac{\partial C}{\partial \phi_*} d\phi_* + \frac{\partial C}{\partial \chi_*} d\chi_* \right), \\
d\chi_c &= \frac{d\chi_c}{dC} \left(\frac{\partial C}{\partial \phi_*} d\phi_* + \frac{\partial C}{\partial \chi_*} d\chi_* \right).
\end{aligned}
\tag{45}
$$

Making use of Eq. (43), one can easily compute

$$
\frac{\partial C}{\partial \phi_*} = -\frac{m_{\mathrm{P}}^2}{U'_*}, \qquad \frac{\partial C}{\partial \chi_*} = \frac{m_{\mathrm{P}}^2}{V'_*}.
\tag{46}
$$

We now require that $t = t_c$ coincides with a $\rho = \mathrm{const}$ hypersurface, which during slow-roll is given by

$$
U(\phi_c) + V(\chi_c) = \mathrm{const}.
\tag{47}
$$

Differentiating this condition yields

$$
\frac{d\phi_c}{dC} U'_c + \frac{d\chi_c}{dC} V'_c = 0.
\tag{48}
$$

Furthermore, differentiating Eq. (43) with respect to the trajectory C and using Eq. (48), one finally obtains

$$m_{\mathrm{P}}^2 \frac{d\phi_c}{dC} = -\left[U_c'\left(\frac{1}{V_c'^2} + \frac{1}{U_c'^2}\right)\right]^{-1},$$
$$m_{\mathrm{P}}^2 \frac{d\chi_c}{dC} = \left[V_c'\left(\frac{1}{V_c'^2} + \frac{1}{U_c'^2}\right)\right]^{-1}. \tag{49}$$

Thus, substituting Eqs. (46) and (49) in Eq. (45), one finds

$$\frac{\partial \phi_c}{\partial \phi_*} = \frac{W_c}{W_*} \frac{\epsilon_c^\chi}{\epsilon_c} \left(\frac{\epsilon_c^\phi}{\epsilon_*^\phi}\right)^{1/2}, \qquad \frac{\partial \phi_c}{\partial \chi_*} = -\frac{W_c}{W_*} \frac{\epsilon_c^\chi}{\epsilon_c} \left(\frac{\epsilon_c^\phi}{\epsilon_*^\chi}\right)^{1/2},$$
$$\frac{\partial \chi_c}{\partial \phi_*} = -\frac{W_c}{W_*} \frac{\epsilon_c^\phi}{\epsilon_c} \left(\frac{\epsilon_c^\chi}{\epsilon_*^\phi}\right)^{1/2}, \qquad \frac{\partial \chi_c}{\partial \chi_*} = \frac{W_c}{W_*} \frac{\epsilon_c^\phi}{\epsilon_c} \left(\frac{\epsilon_c^\chi}{\epsilon_*^\chi}\right)^{1/2}, \tag{50}$$

where ϵ is the overall slow-roll parameter given in Eq. (16) and we have written the derivatives of the potentials that appear in Eqs. (46) and (49) in terms of the slow-roll parameters defined in Eq. (39).

One can now evaluate the derivatives of the number of e-foldings with respect to the initial fields ϕ_* and χ_*, using in Eq. (44) the expressions (50). For the first derivative this yields

$$m_{\mathrm{P}} \frac{\partial N}{\partial \phi_*} = \frac{1}{\sqrt{2\epsilon_*^\phi}} \frac{U_* + Z_c}{W_*}, \tag{51}$$

$$m_{\mathrm{P}} \frac{\partial N}{\partial \chi_*} = \frac{1}{\sqrt{2\epsilon_*^\chi}} \frac{V_* - Z_c}{W_*}, \tag{52}$$

where

$$Z_c = (V_c \epsilon_c^\phi - U_c \epsilon_c^\chi)/\epsilon_c. \tag{53}$$

To compute the second derivatives we differentiate Eqs. (51) and (52) with respect to ϕ_* and χ_*. This yields

$$m_{\mathrm{P}}^2 \frac{\partial^2 N}{\partial \phi_*^2} = 1 - \frac{\eta_*^\phi}{2\epsilon_*^\phi} \frac{U_* + Z_c}{W_*} + \frac{m_{\mathrm{P}}}{W_* \sqrt{2\epsilon_*^\phi}} \frac{\partial Z_c}{\partial \phi_*}, \tag{54}$$

$$m_{\mathrm{P}}^2 \frac{\partial^2 N}{\partial \chi_*^2} = 1 - \frac{\eta_*^\chi}{2\epsilon_*^\chi} \frac{V_* - Z_c}{W_*} - \frac{m_{\mathrm{P}}}{W_* \sqrt{2\epsilon_*^\chi}} \frac{\partial Z_c}{\partial \chi_*}, \tag{55}$$

$$m_{\mathrm{P}}^2 \frac{\partial^2 N}{\partial \phi_* \partial \chi_*} = \frac{m_{\mathrm{P}}}{W_* \sqrt{2\epsilon_*^\phi}} \frac{\partial Z_c}{\partial \chi_*} = -\frac{m_{\mathrm{P}}}{W_* \sqrt{2\epsilon_*^\chi}} \frac{\partial Z_c}{\partial \phi_*}. \tag{56}$$

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In single field inflation, perturbations on large scales are adiabatic and hence ζ remains constant during inflation [29]. In two-field models, however, the large-scale perturbations are not in general adiabatic and ζ evolves during inflation. In the previous expressions, Eqs. (51), (52) and (54–56), the function Z_c takes account of this evolution by expressing how the local expansion depends upon the local field values on the uniform density hypersurface during inflation.

3.3 Curvature perturbation and non-Gaussianity during inflation

We now compute the curvature perturbation and the non-Gaussianity on a comoving or uniform energy density hypersurface during slow-roll inflation, as given in Eq. (4). From the first derivatives of N, Eqs. (51) and (52), one can derive an expression for the power spectrum of the curvature perturbation (11). It can be simplified by introducing two dimensionless variables, related to the values of the potentials and first slow-roll parameters at t_* and t_c,

$$u \equiv \frac{U_* + Z_c}{W_*}, \qquad v \equiv \frac{V_* - Z_c}{W_*}. \tag{57}$$

The power spectrum then reads

$$\mathcal{P}_\zeta = \frac{W_*}{24\pi^2 m_\mathrm{P}^4} \left(\frac{u^2}{\epsilon_*^\phi} + \frac{v^2}{\epsilon_*^\chi} \right) . \tag{58}$$

Note that this can never be less than the power spectrum derived by considering only adiabatic perturbations restricted to the background trajectory (at fixed C). In multi-field inflation there is an additional contribution to the power spectrum due to non-adiabatic perturbations at Hubble exit. Thus one has $\mathcal{P}_\zeta \geq \mathcal{P}_{\zeta|_C}$, where $\mathcal{P}_{\zeta|_C}$ can be obtained by taking the limit $t_c \to t_*$ in Eq. (58), since for adiabatic perturbations the curvature perturbation does not change after Hubble crossing. In this limit

$$u \to \frac{\epsilon_*^\phi}{\epsilon_*}, \qquad v \to \frac{\epsilon_*^\chi}{\epsilon_*}, \qquad \text{for } t_c \to t_*, \tag{59}$$

and one finds

$$\mathcal{P}_{\zeta|_C} = \frac{W_*}{24\pi^2 \epsilon_* m_\mathrm{P}^4} . \tag{60}$$

To compute the scalar spectral index n_ζ with Eq. (15) and the nonlinear parameter f_NL with Eq. (26) one needs the second derivatives $N_{,IJ}$, given in

Eqs. (54–56). In the expression for the scalar spectral index the derivatives of Z_c with respect to the fields contained in these equations mutually cancel. Thus, by using Eq. (15), one finds

$$n_\zeta - 1 = -2\epsilon_* - 4\frac{u\left(1 - \frac{\eta_*^\phi}{2\epsilon_*^\phi}u\right) + v\left(1 - \frac{\eta_*^\chi}{2\epsilon_*^\chi}v\right)}{\frac{u^2}{\epsilon_*^\phi} + \frac{v^2}{\epsilon_*^\chi}}. \tag{61}$$

Note that in the limit $t_c \to t_*$, using Eq. (59), one finds the spectral index for purely adiabatic perturbations,

$$n_{\zeta|_C} - 1 = -6\epsilon_* + 2\eta_*^{\sigma\sigma}, \tag{62}$$

where we have defined [39]

$$\eta^{\sigma\sigma} \equiv (\epsilon^\phi \eta^\phi + \epsilon^\chi \eta^\chi)/\epsilon. \tag{63}$$

This represents the effective mass of adiabatic fluctuations *tangent* to the background trajectory.

The momentum dependent nonlinear parameter, $f_{\rm NL}^{(3)}$, defined in Eq. (29), takes a very simple form, derived from Eq. (60),

$$-\frac{6}{5}f_{\rm NL}^{(3)} = \epsilon_* \frac{\mathcal{P}_{\zeta|_C}}{\mathcal{P}_\zeta}(1 + f) \leq \epsilon_*(1 + f), \tag{64}$$

where f is the momentum dependence, defined in Eq. (27).

In order to give an expression for $f_{\rm NL}^{(4)}$, one needs $\partial Z_c/\partial\phi_*$ and $\partial Z_c/\partial\chi_*$, which can be computed by employing Eq. (50). One finds

$$\sqrt{\epsilon_*^\phi}\frac{\partial Z_c}{\partial\phi_*} = -\sqrt{\epsilon_*^\chi}\frac{\partial Z_c}{\partial\chi_*} = \frac{\sqrt{2}}{m_{\rm P}}W_*\mathcal{A}, \tag{65}$$

where we have defined

$$\mathcal{A} \equiv -\frac{W_c^2}{W_*^2}\frac{\epsilon_c^\phi \epsilon_c^\chi}{\epsilon_c}\left(1 - \frac{\eta_c^{ss}}{\epsilon_c}\right), \tag{66}$$

with [39]

$$\eta^{ss} \equiv (\epsilon^\chi \eta^\phi + \epsilon^\phi \eta^\chi)/\epsilon. \tag{67}$$

This represents the effective mass of isocurvature fluctuations orthogonal to the background trajectory.

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We are thus able to give an analytic expression for $f_{\rm NL}^{(4)}$ defined in Eq. (30). One finds

$$-\frac{6}{5}f_{\rm NL}^{(4)} \;=\; 2\frac{\frac{u^2}{\epsilon_*^\phi}\left(1-\frac{\eta_*^\phi}{2\epsilon_*^\phi}u\right)+\frac{v^2}{\epsilon_*^\chi}\left(1-\frac{\eta_*^\chi}{2\epsilon_*^\chi}v\right)+\left(\frac{u}{\epsilon_*^\phi}-\frac{v}{\epsilon_*^\chi}\right)^2\mathcal{A}}{\left(\frac{u^2}{\epsilon_*^\phi}+\frac{v^2}{\epsilon_*^\chi}\right)^2}. \qquad (68)$$

This is the exact expression for the amplitude of the nonlinear parameter for the curvature perturbation ζ during slow-roll inflation in an arbitrary two-field inflation model with separable potential, and represents one of the main results of this paper.

As for the power spectrum and spectral index, we can take the limit of $t_c \to t_*$, which gives the nonlinear parameter for purely adiabatic perturbations in the two-field case. Using Eq. (59) in Eqs. (64) and (68), yields

$$-\frac{6}{5}f_{\rm NL|C} = \epsilon_*(1+f)+2\epsilon_*-\eta_*^{\sigma\sigma} \qquad (69)$$

which coincides with the single field case result [5, 36]. Note that, using Eqs. (62) and (28) for the adiabatic case, this can be expressed as a consistency relation between observables [5],

$$-\frac{6}{5}f_{\rm NL|C} = -\frac{1}{2}(n_{\zeta|C}-1)+\frac{r_{|C}}{8}f, \qquad (70)$$

and is thus constrained to be small.

Although the general expression (68) for two fields, which allows for non-adiabatic perturbations, is rather involved, we can qualitatively discuss the order of magnitude we expect for $f_{\rm NL}^{(4)}$. Since $\max(u,v)\leq 1$ and $u+v=1$, the denominator in Eq. (68) is of order ε_*^{-2}, where we use ε to denote generic first-order slow-roll parameters, ϵ or η. The first two terms in the numerator of Eq. (68) are of order ε_*^{-1}, so their contribution to $f_{\rm NL}^{(4)}$ is of order ε_*. Only the third term, which is of order $\mathcal{A}\varepsilon_*^{-2}$, leads to a contribution that is not automatically slow-roll suppressed.

Although η^{ss} may become larger than unity during inflation, the prefactor in front of the parenthesis in Eq. (66) can be very small or vanishing, suppressing the contribution of $\mathcal{A}$ to the nonlinear parameter. If this is not the case, since $\mathcal{A}$ does not appear in the expression for the spectral index n_ζ, Eq. (61), for $\eta_c^{ss}/\epsilon_c \gg 1$ one may have models with large $f_{\rm NL}$ and quasi-scale-invariant spectral index, corresponding to large deviations from the consistency relation for purely adiabatic perturbations, Eq. (70). We leave to future work the investigation of models where $\mathcal{A}$, and thus $f_{\rm NL}$, are large.

4 Perturbations after inflation

So far we have considered only the curvature perturbation during inflation. In order to relate our calculations to observables we need to calculate ζ *after* inflation when the universe is radiation dominated, on a uniform energy density hypersurface for $t = t_c > t_e$, where t_e denotes the end of inflation. In this case t_c defines a uniform radiation density hypersurface.

In single field inflation large-scale perturbations are adiabatic and thus the curvature perturbation on uniform density hypersurfaces remains constant both *during* and *after* inflation, independent of the detailed physics occurring at the end of inflation. In two-field models we need to take account of how the local expansion depends upon the local field values both during inflation and at the end of inflation in order to calculate ζ some time after inflation.

If one of the two fields – for instance ϕ – has stabilized before the end of inflation, so that the end of inflation is dominated by a single field χ, then Z_c becomes constant, and the power spectrum, the scalar spectral index and the non-linear parameters after the end of inflation are simply given by Eqs. (58), (61), (64) and (68) with $Z_c = U_c =$ const. Note that in this case $\mathcal{A} = 0$ at the end of inflation and the nonlinear parameter $f_{\mathrm{NL}}^{(4)}$ will be small.

If both fields are "active" at the end of inflation, and inflation takes place on a hypersurface $q(\phi_e, \chi_e) =$ const at $t = t_e$, undefined – i.e., not necessarily a uniform energy density, one can separate $\delta N(t_c, t_*)$ into the sum of two pieces, the first from $t = t_*$ to the end of inflation $t = t_e$, the second from $t = t_e$ to a uniform density hypersurface after inflation $t = t_c$,

$$\delta N(t_c, t_*) = \delta N(t_e, t_*) + \delta N(t_c, t_e). \tag{71}$$

The first piece, $\delta N(t_e, t_*)$, can be computed by a similar calculation to that of Sec. 3.2. It is possible to show that δN expanded at second order is found by replacing Z_c in Eqs. (51), (52) and (54–56) by

$$Z_e = \left(V_e \frac{\partial q}{\partial \phi_e} \sqrt{\epsilon_e^\phi} - U_e \frac{\partial q}{\partial \chi_e} \sqrt{\epsilon_e^\chi} \right) \left(\frac{\partial q}{\partial \phi_e} \sqrt{\epsilon_e^\phi} + \frac{\partial q}{\partial \chi_e} \sqrt{\epsilon_e^\chi} \right)^{-1}. \tag{72}$$

Then one needs to compute $\delta N(t_c, t_e)$.

In scenarios such as the curvaton or modulated reheating scenarios, it is assumed that the end of inflation hypersurface is effectively unperturbed, $\delta N(t_e, t_*) \simeq 0$, and that the primordial density perturbation originates from isocurvature field perturbations, δs, during inflation, which introduce a perturbation $\delta N(t_c, t_e) \propto \delta s$ only *after* or at the *end* of inflation.

An alternative possibility is to assume that inflation ends due to a sudden instability triggered by some function of the fields reaching a critical value

[40, 41, 42]. This is what happens in hybrid inflation [43, 44] where the false vacuum state is destabilized when the inflaton field, ϕ, reaches a critical value, $\phi = \phi_e$. If we assume instantaneous reheating of the universe one has [33]

$$
\begin{aligned}
\delta N(t_c, t_e, \mathbf{x}) &= \mathcal{N}\left(\rho(t_c, \mathbf{x})\right) - \mathcal{N}\left(\rho(t_e, \mathbf{x})\right) \\
&= -\left[\frac{dN}{d\rho}\delta\rho + \frac{1}{2}\frac{d^2N}{d\rho^2}\delta\rho^2\right]_e \\
&= -H_e\left[\frac{\delta\rho}{\dot{\rho}} + \frac{1}{2}\left(\frac{\dot{\rho}}{2\rho} - \frac{\ddot{\rho}}{\dot{\rho}}\right)\frac{\delta\rho^2}{\dot{\rho}^2}\right]_e.
\end{aligned}
\tag{73}
$$

We assume that the energy density is conserved at the end of inflation, so that the background value of the energy density in this expression is $\rho(t_e) = U(\phi_e) + V(\chi_e)$. Furthermore, if the equation of state becomes radiation-like, so that $\dot{\rho} = -4H\rho$, we then have

$$
\delta N(t_c, t_e, \mathbf{x}) = \frac{1}{4}\left[\frac{\delta\rho}{\rho} - \frac{1}{2}\frac{\delta\rho^2}{\rho^2}\right]_e.
\tag{74}
$$

Note that the numerical coefficient that appears in front of the brackets in this expression is dependent upon the equation of state after inflation has ended and would be $1/3(1 + w)$ for an equation of state $P = w\rho$.

In a hybrid-type limit, where the false vacuum dominates the self-interaction energy of the slowly-rolling fields, we can take the effective potential to be almost completely flat (consistent with the slow-roll approximation) so that $[\delta\rho/\rho]_e$ is negligible and $\delta N(t_c, t_e) \ll \delta N(t_e, t_*)$. The primordial curvature perturbation ζ is then given by the curvature of the end of inflation hypersurface, $\zeta(t_c) = \delta N(t_e, t_*)$.

5   Double quadratic inflation

To be more specific and give a quantitative estimate of the non-Gaussianity, it is instructive to consider the case of massive fields with potential given by Eq. (32) with [45, 46]

$$
U = \frac{1}{2}m_\phi^2\phi^2, \qquad V = \frac{1}{2}m_\chi^2\chi^2.
\tag{75}
$$

These scalar fields thus have no explicit interaction, but can interact gravitationally during inflation.

5.1 Slow-roll analysis

We can apply our earlier analysis to calculate the curvature perturbation during slow-roll inflation in this model.

The integrals (42) and (43) yield

$$m_{\rm P}^2 N(t_c, t_*) = \frac{1}{4}\left(\phi_*^2 + \chi_*^2\right) - \frac{1}{4}\left(\phi_c^2 + \chi_c^2\right), \tag{76}$$

$$m_{\rm P}^{-2} C(t_c, t_*) = m_\phi^{-2} \ln\left(\frac{\phi_*}{\phi_c}\right) - m_\chi^{-2} \ln\left(\frac{\chi_*}{\chi_c}\right), \tag{77}$$

where, as for N, we have fixed the limits of integration in the definition of C in Eq. (43) to run from t_* to t_c. In the standard treatment, one can parameterize the slow-roll trajectories of the scalar fields in polar coordinates [45]

$$\chi = 2m_{\rm P}\sqrt{N}\sin\theta, \qquad \phi = 2m_{\rm P}\sqrt{N}\cos\theta. \tag{78}$$

The advantage of doing so is that the angular variable θ can be related to the number of e-foldings by the expression [45, 46, 47]

$$N = N_0 \frac{(\sin\theta)^{2/(R^2-1)}}{(\cos\theta)^{2R^2/(R^2-1)}}, \tag{79}$$

where we have defined the ratio between the masses of the fields as

$$R = m_\chi/m_\phi. \tag{80}$$

Thus, Eq. (79) implies that an inflationary model is completely specified by giving R and the values of the fields ϕ and χ – or equivalently N and θ – at some given time. Once specified an initial condition, Eq. (79) can be used to evolve the background fields or equivalently the slow-roll parameters. Then one can use Eqs. (58), (61), (64) and (68), to evaluate completely *analytically* the power spectrum $\mathcal{P}_\zeta$, the scalar spectral index n_ζ and the nonlinear parameter $f_{\rm NL}$ for double inflation.

Before studying in detail these equations for a specific case, one can estimate the amount of non-Gaussianity generally produced *during* and *after* inflation. *During* inflation, as explained in Sec. 3.3, $f_{\rm NL}$ can be large only if, in Eq. (68), the contribution proportional to the quantity $\mathcal{A}$ defined in Eq. (66) is large. In double inflation this contribution can become temporarily large, at most of order unity, at the turn of the trajectory in field space. However, *after* inflation, the fields have settled to their minima, which implies $Z_c = 0$ and also $\mathcal{A} = 0$. In this case the dependence on the field values

at $t = t_c$ in Eq. (57) disappears and we can simply rewrite Eqs. (58), (61), (64) and (68) by using the potential (75) and Eq. (76) with $\phi_c = \chi_c = 0$, as

$$\mathcal{P}_\zeta \ = \ \frac{H_*^2}{4\pi^2 m_\mathrm{P}^2} N(t_c, t_*), \tag{81}$$

$$n_\zeta - 1 \ = \ -2\epsilon_* - \frac{1}{N(t_c, t_*)}, \tag{82}$$

$$-\frac{6}{5} f_\mathrm{NL} \ = \ \frac{1}{2N(t_c, t_*)} \left[2 + f(k_1, k_2, k_3) \right], \tag{83}$$

where $N(t_c, t_*)$ is the number of e-foldings between Hubble exit and the moment when $\phi_c = \chi_c = 0$, which roughly coincides with the end of inflation. These results coincide with the estimate of Ref. [23]. We conclude that double inflation *cannot* produce large non-Gaussianity – i.e., $|f_\mathrm{NL}| \ll 1$ – these being suppressed by the slow-roll conditions on the fields *at Hubble exit*. We now turn to a more detailed analysis of Eqs. (58), (61), (64) and (68), and compare with a numerical results for the evolution on large scales without assuming slow-roll.

5.2   Numerical analysis

The δN-formalism assumes that the fields are in slow-roll at Hubble exit, Eq. (8) (see [48] for a development of the δN-formalism applicable to more general situations). Furthermore, in deriving Eqs. (58), (61), (64), (68) and the background evolution relation (79), we have assumed slow-roll all along the inflationary evolution, from t_* to t_c.

While still assuming that slow-roll conditions are satisfied at Hubble exit, one can go *beyond* the slow-roll approximation by *numerically* solving the full background evolution equations for the fields, Eqs. (33) and (34), together with the Friedmann equation (36), and computing the evolution of the scale factor a. One can then evaluate the expansion $N = \ln a$ as a function of the field initial values (ϕ_*, χ_*), up to some final time t_c which may be some time *during* or *after* slow-roll inflation. Then, one can numerically calculate the first and second partial derivatives of $N(t_c, t_*)$ with respect to ϕ_* and χ_* by the *finite-difference method* [49], i.e., by computing N for different values close to (ϕ_*, χ_*). Indeed, this provides an efficient method to compute the non-Gaussianity numerically for *any* two-field model, also when the field potential is not separable.

We study, as an example, the specific case considered by Rigopoulos *et al.* in [22]: mass ratio $R = 1/9$ and initial condition $\phi = \chi = 13m_\mathrm{P}$. We assume that inflation ends on a hypersurface $\epsilon_e = 1$ and that Hubble exit
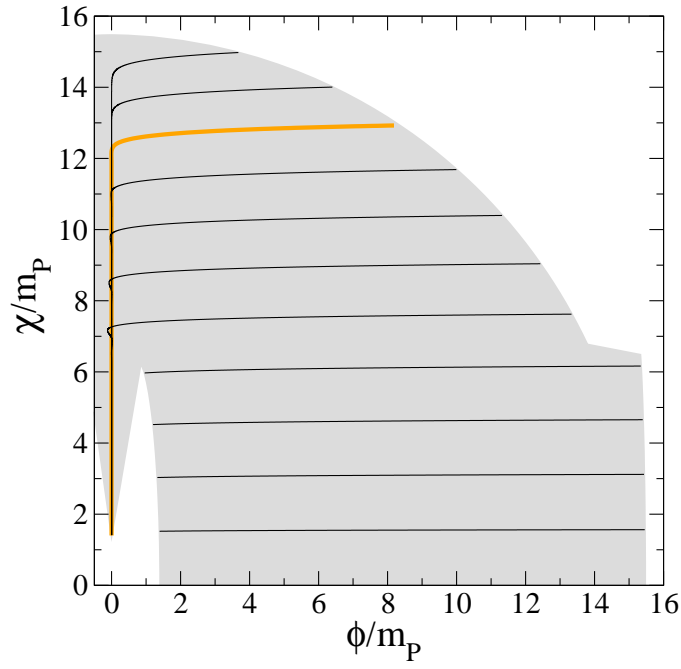
Figure 1: Examples of trajectories in field space are shown from Hubble exit, starting 60 e-foldings before the end of inflation at $\epsilon_e = 1$, for $R = 1/9$. The thick (orange) line represents the trajectory starting from $\phi = \chi = 13m_P$, and shown from Hubble exit, ($\phi_* = 8.2, \chi_* = 12.9$), to the end of inflation, ($\phi_e = 0.0, \chi_e = 1.4$). The grey shading represents the space of all possible trajectories.
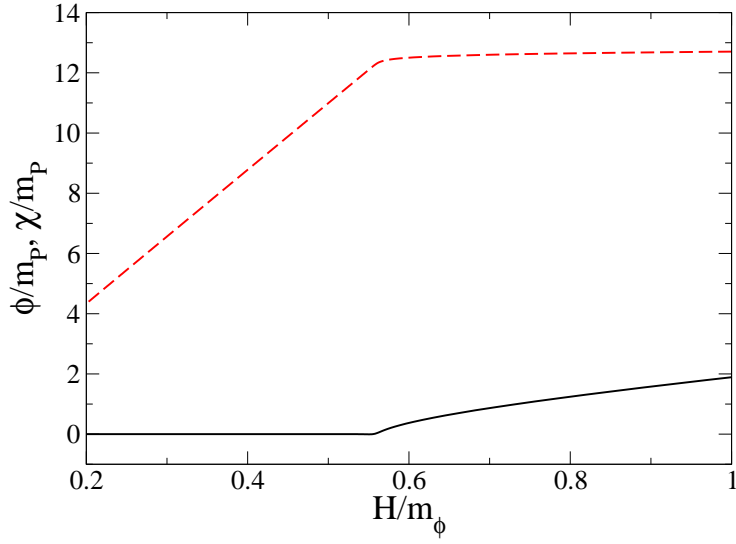
Figure 2: The values of the fields ϕ (solid line) and χ (dashed red line) during the inflationary trajectory of Fig. 1, are shown as a function of the Hubble rate H, from $N \simeq 42$ to $N \simeq 4$ e-foldings from the end of inflation. Note that time increases from right to left.
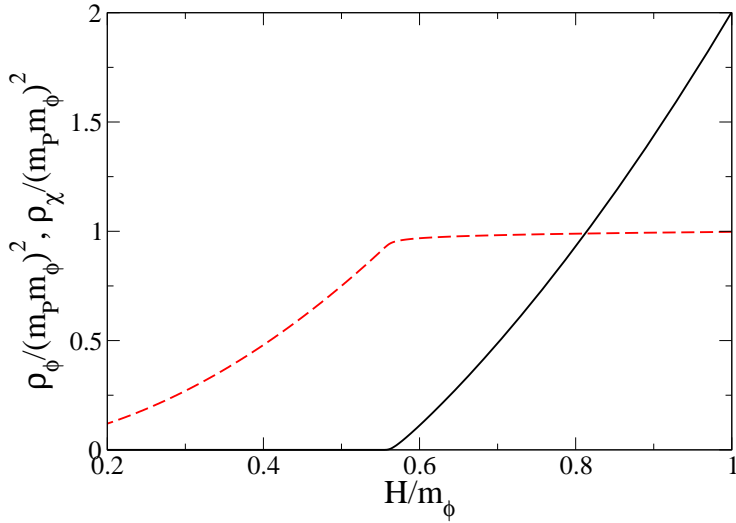

Figure 3: The energy densities of the fields ρ_ϕ (solid line) and ρ_χ (dashed red line) normalized to $(m_P m_\phi)^2$, are shown as for Fig. 2.
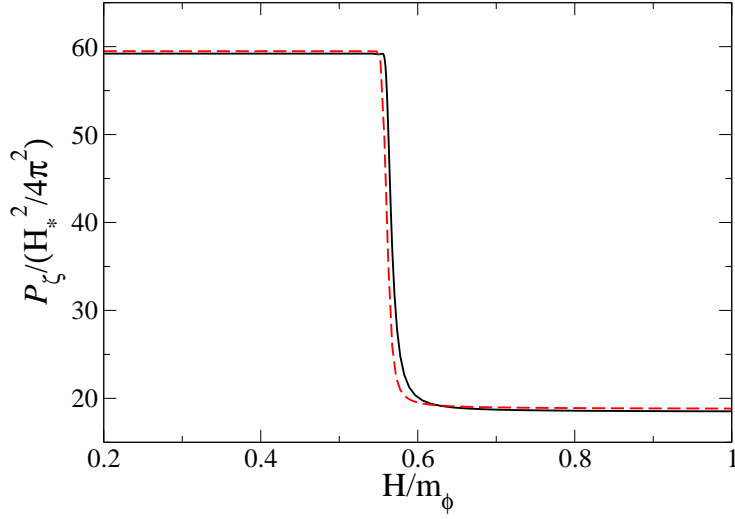
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Figure 4: The power spectrum $\mathcal{P}_\zeta$ of the large-scale uniform density pertur-
bation ζ during the inflationary trajectory of Fig. 1, is shown as a function
of the Hubble rate H, from $N \simeq 42$ to $N \simeq 4$ e-foldings from the end of
inflation ($m_{\mathrm{P}} = 1$). The solid and the dashed lines represent the analytic
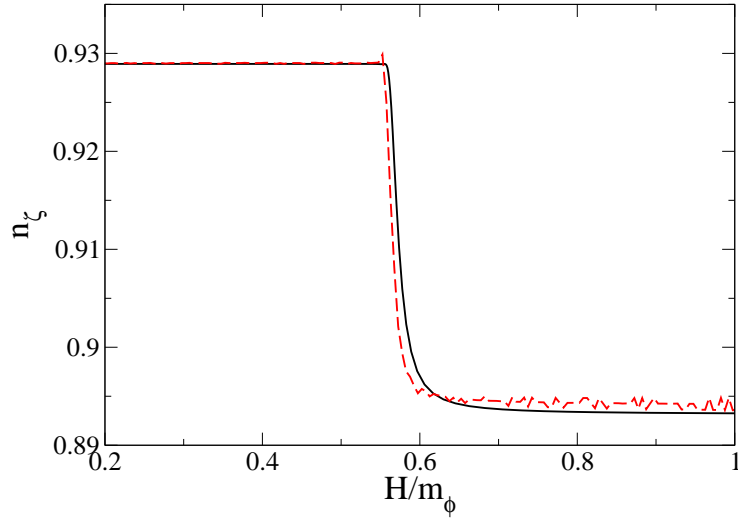and numerical calculations, respectively.

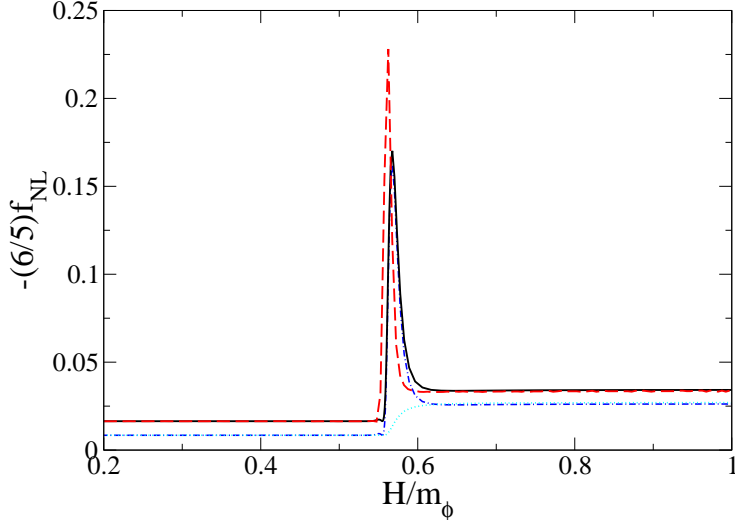Figure 5: The spectral index n_ζ is shown as for Fig. 4.

Figure 6: The nonlinear parameter f_{NL}, in the limit where $f(k_1, k_2, k_3) = 0$ ($k_1 \ll k_2 \approx k_3$) is shown as for Fig. 4. The (turquoise) dotted line represents $-(6/5)f_{\mathrm{NL}}^{(3)}/(1+f)$ while the (blue) dashed-dotted line represents $-(6/5)f_{\mathrm{NL}}^{(4)}$, both computed analytically.

takes place at $t = t_*$, corresponding to 60 e-foldings before the end of inflation. One finds that $(\phi_e = 0.0, \chi_e = 1.4)$, corresponding to $\theta_e = \pi/2$, and that $(\phi_* = 8.2, \chi_* = 12.9)$, corresponding to $\theta_* = 0.32\pi$.

The trajectory in field space is shown in Fig. 1. One can see that assuming slow-roll from t_* to t_e is well justified. Indeed, the *turn* of trajectory in field space, corresponding to the moment when ϕ exits slow-roll, takes place near $\phi = 0$, when the energy density of ϕ has already become negligible. This can also be checked in Figs. 2 and 3, where the values of the fields and their energy densities are shown, respectively, at the turn of the trajectory in field space, from $N \simeq 42$ to $N \simeq 4$ e-foldings from the end of inflation.

Now we compute $\mathcal{P}_\zeta$, n_ζ and $f_{\mathrm{NL}} = f_{\mathrm{NL}}^{(3)} + f_{\mathrm{NL}}^{(4)}$ analytically, using Eqs. (58), (61), (64) and (68), and numerically, using the procedure explained above. In Figs. 4–6 we compare the analytic and numerical calculation of the power spectrum $\mathcal{P}_\zeta$, the scalar spectral index n_ζ, the nonlinear parameter f_{NL}, as a function of the Hubble rate during inflation. The two calculations agree remarkably well, within an accuracy that depends on the numerical precision.

As expected, when the heavy field ϕ dominates the total energy density, for $H/m_\phi \gtrsim 0.8$, the uniform density curvature perturbation ζ remains constant. It starts growing when H drops to $0.55 \lesssim H/m_\phi \lesssim 0.8$, corresponding to both the light and heavy fields contributing to the total energy density.

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During this phase, the large scale total entropy perturbation is non-vanishing, and sources the evolution of ζ [38]. After few oscillations, the heavy field energy density is red-shifted and the light field dominates the universe while ζ becomes constant in time again. Consequently, the power spectrum and the spectral index grow monotonically, following the growth of ζ.

The evolution of the nonlinear parameter $-f_{\rm NL}^{(4)}$, however, is non-monotonic. It grows sharply during the intermediate phase $0.55 \lesssim H/m_\phi \lesssim 0.8$, corresponding to the heavy field leaving slow-roll, but it then decreases. This growth corresponds to the growth of $\mathcal{A}$ defined in Eq. (66).

One can compute the value of $f_{\rm NL}$ at the end of inflation, for $\epsilon_e \simeq 1$, in the limit where $f(k_1, k_2, k_3) = 0$ ($k_1 \ll k_2 \approx k_3$). One finds $-(6/5)f_{\rm NL}^{(4)} = 0.008$ and $-(6/5)f_{\rm NL}^{(3)} = 0.008$ which coincide with the analytical result of Eq. (83), $-(6/5)f_{\rm NL} = 0.016 \simeq 1/60$. We conclude that $f_{\rm NL}$ in this model is much smaller that unity.

Note that, as shown in Fig. 4, the curvature perturbation becomes constant after the energy density of the more massive field (ϕ in our case) becomes negligible. The isocurvature perturbations in the massive field are suppressed at the end of inflation and hence the nonlinear parameter that we have calculated is indeed the primordial $f_{\rm NL}$ constrained by observations, unless we have some curvaton-type mechanism which alters the large-scale curvature perturbation after inflation.

6   Conclusion

In this work we have studied the non-Gaussianity, in terms of the bispectrum, generated by models of inflation with two fields evolving during inflation. We have first reviewed the use of the δN-formalism to compute the curvature perturbation on uniform density hypersurfaces during inflation, up to second order in the perturbations, when the slow-roll conditions are satisfied by all fields.

We have then specialized to the two-field case, and assumed that the potential is separable into the sum of two potentials, each of which is dependent on only one of the two fields. In this case the number of e-foldings can be analytically expanded up to second order in the initial field fluctuations and we have derived an analytic formula, Eq. (68), to compute the nonlinear parameter $f_{\rm NL}$ in terms of the values of the potential and slow-roll parameters of the fields. Using this formula, one can identify the conditions for the model to generate large non-Gaussianity. In particular, we have shown that a large $f_{\rm NL}$ after inflation requires a large η^{ss} – defined in Eq. (67) – at the end of inflation, although a large η^{ss} does not necessarily imply large

non-Gaussianity.

As a specific model, we have considered double inflation with two massive scalar fields. In this case, the background evolution of the fields can be computed analytically and we have been able to compute the evolution of the nonlinear parameter during inflation. As shown in Sec. 5.1, one expects the non-Gaussianity generated in double inflation to be slow-roll suppressed, $|f_{\mathrm{NL}}| \ll 1$, as in the single field case. Since our analytic formula relies on the slow-roll conditions of all fields from Hubble exit until the end of inflation, we have extended our analysis by developing a numerical method, based on the δN-formalism, to compute the curvature perturbation up to second order in the perturbations, that allows us to relax the slow-roll assumption after Hubble exit. This method, which can be extended to any two-field model, confirms our analytic results.

Our results agree with a previous discussion of non-Gaussianity in a double quadratic potential by Alabidi and Lyth [23], based on an estimate of δN given in [1] using the integral (42). In Ref. [1] it is argued that N in Eq. (42) is dominated by the field values at Hubble exit and hence one can neglect the dependence of N upon the field values at the final time, t_c. This is not in general true when evaluating the perturbation ζ during inflation. Including the dependence of ϕ_c and χ_c upon the field values at Hubble exit considerably complicates our analysis, but allows to follow perturbations during inflation. In our numerical example we see that the non-linearity parameter does evolve during inflation, especially as the trajectory turns the corner in field space, reflecting its dependence upon the final field values. However this does not alter the general conclusion that the non-linearity parameter remains small at the end of inflation.

Our results disagree with those of Rigopoulos *et al.* [22] who found that double quadratic inflation, in particular the model studied in Sec. 5.2, can generate large non-Gaussianity. The formalism used by Rigopoulos *et al.* differs from the δN-formalism in two aspects: First, it takes into account the sub-Hubble evolution of the field perturbations via stochastic terms. Second, it integrates *explicitly* the coupled evolution of the adiabatic and entropy super-Hubble perturbations. However, we have explicitly calculated in this model the non-Gaussianity due to the three-point function for the field perturbations at Hubble exit, using the result of Seery and Lidsey [8], and shown that it is small, as previously argued by Lyth and Zaballa [37]. Furthermore, the δN-formalism, on super-Hubble scales, is equivalent to integrating the evolution of ζ. Thus, the contribution to f_{NL} coming from the super-Hubble evolution should be equally taken into account by both formalisms and we are unable to explain the discrepancy between the results.

We conclude, on the basis of our analysis, that nonlinear evolution on

super-Hubble scales during double quadratic inflation does not appear to be capable of generating a detectable level of non-Gaussianity in the bispectrum. However, the possibility of producing a large non-Gaussianity is left open in models of two-field inflation where the mass of the isocurvature field orthogonal to the classical trajectory becomes larger than the Hubble rate and both the first slow-roll parameters of the two fields do not vanish at the end of inflation.

Note added: After completing this work we have learnt [50] that our conclusion that f_{NL} is small for the specific double quadratic inflation model investigated in Sec. 5 is in qualitative agreement with an improved numerical calculation by the authors of [22].

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