

Asymmetric brane-worlds with induced gravity

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The Randall-Sundrum scenario, with a 1+3-dimensional brane in a 5-dimensional bulk spacetime, can be generalized in various ways. We consider the case where the Z_2 -symmetry at the brane is relaxed, and in addition the gravitational action is generalized to include an induced gravity term on the brane. We derive the complete set of equations governing the gravitational dynamics for a general brane and bulk, and identify how the asymmetry and the induced gravity act as effective source terms in the projected field equations on the brane. For a Friedmann brane in an anti de Sitter bulk, the solution of the Friedmann equation is given by the solution of a quartic equation. We find the perturbative solutions for small asymmetry, which has an effect at late times.

I. INTRODUCTION

Developments in particle physics suggest that our observable world may be a 1+3-dimensional “brane” surface embedded in a higher-dimensional “bulk” spacetime, with matter fields confined on the brane while gravity propagates in the bulk. An important family of such brane-world models is provided by the Randall-Sundrum (RS) scenario, with a single warped and non-compact extra dimension [1]. The effective Einstein equations on the brane in the general case were derived via the covariant Shiromizu-Maeda-Sasaki approach [2], together with equations governing the bulk degrees of freedom. A generalization of these equations for the case when the Z_2 -symmetry of the embedding is lifted and the bulk contains matter, was recently presented in [3]. Due to the asymmetry new terms appear in the effective Einstein equations, including a varying cosmological “constant”.

A correction arising from the quantum interaction of bulk gravitons and brane matter can be introduced into RS models by adding the so-called induced gravity contribution to the gravitational action:

$$S = \frac{1}{2\tilde{\kappa}^2} \int d^5x \sqrt{-\tilde{g}} \left[\tilde{R} - 2\tilde{\Lambda} \right] + \int_{\text{brane}} d^4x \sqrt{-g} \left[-\lambda + \frac{\gamma}{2\kappa^2} R \right]. \quad (1)$$

Tildes indicate the 5-dimensional counterparts of standard 4-dimensional quantities, λ is the brane tension and γ is the dimensionless induced-gravity parameter, with $\gamma = 0$ giving the standard RS model. The brane gravitational constant is given by [2]

$$\kappa^2 = \frac{\tilde{\kappa}^4 \lambda}{6}. \quad (2)$$

The effective Einstein equations (with Z_2 -symmetry) were derived for the general case in [4, 5], generalizing the earlier special cases, such as the Dvali-Gabadadze-Porrati models [6, 7], in which $\gamma = 1$ and Eq. (2) does not hold, since $\lambda = 0 = \tilde{\Lambda}$. The induced gravity models

contain two branches, labelled $\epsilon = \pm 1$, with very different properties. The $\epsilon = -1$ branch has an RS limit, including the case where induced gravity is a sub-dominant correction to the RS model [8, 9, 10].

In the present paper we lift the Z_2 -symmetry of the embedding, and derive the complete system of equations, including the effective Einstein equations on the brane. Some earlier results for this scenario in the particular case of Friedmann branes embedded in a vacuum bulk were presented in [9, 10].

In Section 2, by use of the Lanczos-Sen-Darmois-Israel junction conditions [11, 12, 13, 14], we derive the generalized effective Einstein equations in a form consistent with previous works. We show that, remarkably, the asymmetric contributions add *linearly* to the induced gravity corrections.

In Section 3 we apply the formalism to the case of Friedmann branes. We derive the generalized Friedmann and Raychaudhuri equations together with the constraints emerging from the Codazzi equation and other equations of the asymmetric case. Among the asymmetric features, we find a variable 4-dimensional cosmological “constant” Λ and contributions of the embedding to the dark radiation U . As in the Z_2 -symmetric induced gravity models, the non-linearity of the Friedmann equation in H^2 leads to two branches, $\epsilon = \pm 1$. The asymmetric RS model can be recovered in the $\epsilon = -1$ case, as the $\gamma \rightarrow 0$ limit.

Following the method of [3], in Section 4 we present an independent derivation of the Friedmann equation in the simplest case when the bulk is 5-dimensional anti de Sitter space-time (AdS), with *different* cosmological constants on the two sides of the brane. The comparison of the two forms of the Friedmann equation yields a quartic equation in H^2 , as in [9, 10]. We find a perturbative solution of the quartic equation for small asymmetry. Then, by use of the twice contracted Bianchi identity we identify both unknown functions Λ and U . Finally we analyze the high- and low-energy limits of the Friedmann equation.

II. EFFECTIVE EINSTEIN EQUATIONS

A. The general projection formalism

In this subsection we summarize the relevant results of [3].

The first and second fundamental forms of the brane are

$$g_{ab} = \tilde{g}_{ab} - n_a n_b, \quad (3)$$

$$K_{ab} = \nabla_a n_b = \frac{1}{2} \mathcal{L}_{\mathbf{n}} g_{ab}, \quad (4)$$

where n^a is the unit normal to the brane, with acceleration $\alpha^a = n^b \tilde{\nabla}_b n^a = g_b^a \alpha^b$. Following [3] we introduce the tensors

$$J_{ab} = K_{ac} K_b^c - \mathcal{L}_{\mathbf{n}} K_{ab} + \nabla_b \alpha_a - \alpha_b \alpha_a, \quad (5)$$

$$F_{ab} = K K_{ab} - K_{ac} K_b^c. \quad (6)$$

As shown in [3], the 5-dimensional Einstein equations $\tilde{G}_{ab} = -\tilde{\Lambda} \tilde{g}_{ab}$ are equivalent to the system comprised of the effective 4-dimensional Einstein equations, the Codazzi equation and the twice contracted Gauss equation:

$$G_{ab} = \frac{2}{3} \tilde{G}_{\{ab\}} + \frac{1}{2} n^c n^d \tilde{G}_{cd} g_{ab} + F_{ab} - \frac{1}{2} F g_{ab} - \mathcal{E}_{ab}, \quad (7)$$

$$\nabla_b K_a^b - \nabla_a K = g_a^b n^c \tilde{G}_{bc}, \quad (8)$$

$$R = \tilde{R} + F - 2J. \quad (9)$$

Here

$$Q_{\{ab\}} \equiv \left(g_a^c g_b^d - \frac{1}{4} g_{ab} g^{cd} \right) Q_{cd}, \quad (10)$$

denotes the projected trace-free part, and the ‘‘electric’’ part of the bulk Weyl tensor is

$$\mathcal{E}_{ab} \equiv \tilde{C}_{abcd} n^c n^d = J_{\{ab\}} - \frac{1}{3} \tilde{G}_{\{ab\}}. \quad (11)$$

The brane divides the bulk into two distinct regions. The jump and average of a quantity across the brane are $\Delta Q = Q^+ - Q^-$ and $\bar{Q} = \frac{1}{2}(Q^+ + Q^-)$. The junction conditions impose continuity of the first fundamental form, $\Delta g_{ab} = 0$, and the Lanczos equation [11]

$$\Delta K_{ab} = -\tilde{\kappa}^2 \left(\tau_{ab} - \frac{\tau}{3} g_{ab} \right), \quad (12)$$

which relates the jump of the second fundamental form to the total brane energy-momentum. (These conditions were first formulated in a coordinate-independent manner in [14].) Then [3]:

$$\Delta F_{ab} = -\tilde{\kappa}^2 \left[\bar{K} \left(\tau_{ab} - \frac{\tau}{3} g_{ab} \right) + \frac{\tau}{3} \bar{K}_{ab} - 2\bar{K}_{c(a} \tau_{b)}^c \right], \quad (13)$$

$$\bar{F}_{ab} = \bar{K}_{ab} \bar{K} - \bar{K}_{ac} \bar{K}_b^c + \delta F_{ab}, \quad (14)$$

$$\delta F_{ab} \equiv -\frac{\tilde{\kappa}^4}{4} \left(\tau_{ac} \tau_b^c - \frac{\tau}{3} \tau_{ab} \right). \quad (15)$$

The sums of the effective Einstein equation, Codacci equation and twice contracted Gauss equation, taken on the two sides of the brane, lead to

$$G_{ab} = -\frac{1}{2} \bar{\Lambda} g_{ab} + \tilde{\kappa}^2 \left(\bar{F}_{ab} - \frac{\bar{F}}{2} g_{ab} - \bar{\mathcal{E}}_{ab} \right), \quad (16)$$

$$\nabla_b \bar{K}_a^b = \nabla_a \bar{K}, \quad (17)$$

$$R = \frac{10}{3} \bar{\Lambda} + \bar{F} - 2\bar{J}. \quad (18)$$

The differences of the same equations give:

$$\Delta \mathcal{E}_{ab} = \Delta F_{\{ab\}}, \quad (19)$$

$$\tilde{\kappa}^2 \bar{K}_{ab} \tau^{ab} = -\Delta \bar{\Lambda}, \quad (20)$$

$$\nabla_b \tau_a^b = 0, \quad (21)$$

$$\tilde{\kappa}^2 \bar{K}_{ab} \tau^{ab} = \Delta J - \frac{5}{3} \Delta \bar{\Lambda}. \quad (22)$$

We have further decomposed the difference of the effective Einstein equations into tracefree and trace parts, in Eqs. (19) and (20).

B. Induced gravity equations

Up to this point, all results are unchanged compared to the asymmetric RS model, presented in [3] (here we have omitted the arbitrary bulk energy-momentum tensor in [3]). This is simply because induced gravity does not modify bulk physics, as seen in Eq. (1). However, Eq. (1) also shows that the induced gravity scenario adds a new term to the brane total energy-momentum [5]:

$$\tau_{ab} = -\lambda g_{ab} + T_{ab} - \frac{\gamma}{\kappa^2} G_{ab}, \quad (23)$$

where T_{ab} represents ordinary matter on the brane. Then it follows that

$$\delta F_{ab} - \frac{\delta F}{2} g_{ab} = \kappa^2 \left(-\frac{\lambda}{2} g_{ab} + T_{ab} \right) - \gamma G_{ab} + \frac{6}{\kappa^2 \lambda} \mathcal{S}[\kappa^2 T - \gamma G]_{ab}, \quad (24)$$

where the tensor functional $\mathcal{S}[Q]$ is defined by [5]

$$\mathcal{S}[Q]_{ab} = \frac{1}{4} \left[-Q_{ac} Q_b^c + \frac{Q}{3} Q_{ab} - \frac{1}{2} g_{ab} \left(-Q_{cd} Q^{cd} + \frac{Q^2}{3} \right) \right]. \quad (25)$$

The tensor $\mathcal{S}[T]_{ab}$ that is quadratic in T_{ab} , was introduced for the Z_2 -symmetric case in [2].

We define the brane cosmological ‘‘constant’’ [3]:

$$\Lambda = \Lambda_0 - \frac{\bar{L}}{4}, \quad (26)$$

where Λ_0 is a constant,

$$\Lambda_0 = \frac{\kappa^2 \lambda}{2} + \frac{\bar{\Lambda}}{2}, \quad (27)$$

while \bar{L} is a variable determined by the asymmetric embedding, being the trace of the following tensor:

$$\bar{L}_{ab} \equiv \bar{K}\bar{K}_{ab} - \bar{K}_{ac}\bar{K}_b^c - \frac{g_{ab}}{2} \left(\bar{K}^2 - \bar{K}_{cd}\bar{K}^{cd} \right). \quad (28)$$

Then the effective Einstein equations are

$$(1 + \gamma) G_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab} - \bar{\mathcal{E}}_{ab} + \frac{6}{\kappa^2 \lambda} \mathcal{S}[\kappa^2 T - \gamma G]_{ab} + \bar{L}_{\{ab\}}. \quad (29)$$

The asymmetric contributions [3] in Λ and in the source term $\bar{L}_{\{ab\}}$ add up *linearly* with the induced gravity contributions, derived first in [5]. Note that, as defined here, the cosmological ‘‘constant’’ is a *function* of the embedding of the brane.

Both the Codazzi equation (17) and Eq. (18), giving \bar{J} , are insensitive to brane physics, such as the choice of the matter energy-momentum tensor and the presence of the induced gravity correction. However the following equations do depend on these features. Using Eq. (23) in Eq. (19) and combining Eqs. (20) and (22), we obtain $\Delta \mathcal{E}_{ab}$ and ΔJ :

$$\begin{aligned} \frac{\kappa^2}{\bar{\kappa}^2} \Delta \mathcal{E}_{ab} &= -\bar{K} [\kappa^2 T_{\{ab\}} - \gamma G_{\{ab\}}] \\ &\quad - \frac{1}{3} \bar{K}_{\{ab\}} [\kappa^2 (2\lambda + T) - \gamma G] \\ &\quad + 2\bar{K}_{c\{a} [\kappa^2 T_{b\}^c - \gamma G_{b\}^c], \end{aligned} \quad (30)$$

$$\Delta J = \frac{2}{3} \Delta \tilde{\Lambda}. \quad (31)$$

These equations, together with Eq. (18), characterize the off-brane evolution of K_{ab} , and we will not deal with them further here.

In the asymmetric case there are also the additional constraints, Eqs. (20) and (21), to be imposed on the matter fields and geometrical properties of the brane:

$$\nabla_b T_a^b = 0, \quad (32)$$

$$-\lambda \bar{K} + \bar{K}^{ab} \left(T_{ab} - \frac{\gamma}{\kappa^2} G_{ab} \right) = -\frac{1}{\bar{\kappa}^2} \Delta \tilde{\Lambda}. \quad (33)$$

From the twice-contracted Bianchi identity in 4 dimensions, using Eq. (32), we find

$$\nabla^a (\bar{\mathcal{E}}_{ab} - \bar{L}_{\{ab\}}) = -\frac{6}{\kappa^2 \lambda} \nabla^a \mathcal{S}[\kappa^2 T - \gamma G]_{ab}. \quad (34)$$

The cosmological consequences of this equation will be exploited in the next section.

III. FRIEDMANN BRANE

For a Friedmann brane,

$$g_{ab} = -u_a u_b + a^2 h_{ab}, \quad (35)$$

$$T_{ab} = \rho u_a u_b + p a^2 h_{ab}, \quad (36)$$

$$G_{ab} = \frac{3(\dot{a}^2 + k)}{a^2} u_a u_b - (2a\ddot{a} + \dot{a}^2 + k) h_{ab}, \quad (37)$$

where u^a is the geodesic 4-velocity, a is the scale factor and h_{ab} is the 3-metric with constant curvature (and curvature index $k = 0, \pm 1$) of the maximally symmetric spatial slices, on which the energy density ρ and pressure p are constant. Then Eq. (25) implies

$$\mathcal{S}[\kappa^2 T - \gamma G]_{ab} = V u_a u_b + W a^2 h_{ab}, \quad (38)$$

where

$$\begin{aligned} V &= \left[\frac{\kappa^2 \rho}{\sqrt{12}} - \frac{\sqrt{3} \gamma (\dot{a}^2 + k)}{2a^2} \right]^2, \\ W &= \frac{\kappa^4 \rho (\rho + 2p)}{12} - \frac{\gamma \kappa^2 [(2\rho + 3p)(\dot{a}^2 + k) - 2\rho a \ddot{a}]}{6a^2} \\ &\quad + \frac{\gamma^2 (\dot{a}^2 + k)(\dot{a}^2 + k - 4a\ddot{a})}{4a^4}. \end{aligned} \quad (40)$$

We can introduce an effective ‘‘non-local’’ energy density U , which imprints bulk effects on the brane (generalizing the Z_2 -symmetric procedure [15]), via

$$-\bar{\mathcal{E}}_{ab} + \bar{L}_{\{ab\}} = \kappa^2 U \left(u_a u_b + \frac{a^2}{3} h_{ab} \right). \quad (41)$$

In the Z_2 -symmetric case, U arises from the Coulomb field of a bulk black hole, and is called ‘‘dark radiation’’; for an AdS bulk, $U = 0$. Asymmetry allows for a non-zero U in an AdS bulk. Then the effective Einstein equation (29) gives the generalized Friedmann equation,

$$\begin{aligned} \left[1 + \gamma \left(1 + \frac{\rho}{\lambda} \right) \right] \frac{(\dot{a}^2 + k)}{a^2} &= \frac{3\gamma^2}{2\lambda \kappa^2} \left(\frac{\dot{a}^2 + k}{a^2} \right)^2 \\ &\quad + \frac{1}{3} \left[\Lambda + \kappa^2 \rho \left(1 + \frac{\rho}{2\lambda} \right) + \kappa^2 U \right], \end{aligned} \quad (42)$$

and the generalized Raychaudhuri equation,

$$\begin{aligned} \left[1 + \gamma \left(1 + \frac{\rho}{\lambda} \right) \right] \frac{\ddot{a}}{a} - 3\gamma \left(\frac{\rho + p}{2\lambda} \right) \frac{(\dot{a}^2 + k)}{a^2} &= \\ \frac{1}{6} \left\{ 2\Lambda - \kappa^2 \left[3p \left(1 + \frac{\rho}{\lambda} \right) + \rho \left(1 + 2\frac{\rho}{\lambda} \right) \right] - 2\kappa^2 U \right\} \\ + \frac{3\gamma^2}{2\lambda \kappa^2} \left(\frac{\dot{a}^2 + k}{a^2} \right) \left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} \right). \end{aligned} \quad (43)$$

These have the same form as Eqs. (3.20,21) of [5]. All asymmetric features are absorbed in Λ and U , which are different from those of [5]. The conservation equation (32) has the usual form,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \quad (44)$$

and the other constraint (33) with no symmetric counterpart gives

$$\frac{1}{\tilde{\kappa}^2} \Delta \tilde{\Lambda} = \left[\lambda - p - \frac{\gamma}{\kappa^2} \left(2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} \right) \right] \bar{K} - \left[\rho + p + 2 \frac{\gamma}{\kappa^2} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} \right) \right] u_a u_b \bar{K}^{ab}. \quad (45)$$

In the RS model ($\gamma \rightarrow 0$ limit) this is just a constraint on the embedding and the perfect fluid. Induced gravity introduces \dot{a} and \ddot{a} contributions, transforming Eq. (45) into a dynamical equation, to be added to the system of the energy-balance equation (44) and the generalized Friedmann and Raychaudhuri equations (42) and (43). We will discuss the significance of this equation below.

The ‘‘non-local’’ conservation equation (34) reduces to

$$\dot{U} + 4 \frac{\dot{a}}{a} U = \frac{1}{4\kappa^2} \dot{\bar{L}}, \quad (46)$$

which shows how asymmetry can be a source for U even in the absence of a bulk black hole. For a Z_2 -symmetric embedding, we have $\bar{L} = 0$ and $U \propto a^{-4}$. An independent way to derive Eq. (46) is to take the time derivative of the generalized Friedmann equation, then eliminate $\dot{\rho}$ by Eq. (44) and eliminate the derivatives of a from the terms not containing γ by Eqs. (42) and (43).

The Lanczos equation becomes

$$\Delta K_{ab} = -\frac{\tilde{\kappa}^2}{3} \left\{ \left[2\rho + 3p - \lambda + \frac{3\gamma}{\kappa^2} \left(2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} \right) \right] u_a u_b + \left[\rho + \lambda - \frac{3\gamma}{\kappa^2} \left(\frac{\dot{a}^2 + k}{a^2} \right) \right] a^2 h_{ab} \right\}. \quad (47)$$

Up to this point the discussion has followed [3], the results of which can be recovered in the $\gamma \rightarrow 0$ limit. Since the Friedmann equation (42) is quadratic in $(\dot{a}^2 + k)$, we can take its square root, following [5], and obtain

$$H^2 + \frac{k}{a^2} = \frac{\kappa^2}{3\gamma} \left[\rho + \frac{(1+\gamma)}{\gamma} \lambda (1 + \epsilon \mathcal{A}) \right], \quad (48)$$

where

$$\mathcal{A}^2 = 1 + \frac{2\gamma}{(1+\gamma)^2 \lambda} \left[\rho - \frac{\gamma}{\kappa^2} (\Lambda + \kappa^2 U) \right], \quad (49)$$

and $\epsilon = \pm 1$ distinguishes the two branches. Although the above equations have the same form as those in [5], the function $\mathcal{A} = \mathcal{A}(\rho, U, \Lambda)$ depends on the asymmetric character of the embedding. Employing Eq. (48), the Raychaudhuri equation (43) can also be simplified,

$$\begin{aligned} -6\epsilon(1+\gamma)\mathcal{A} \frac{\ddot{a}}{a} &= \frac{\kappa^2}{\gamma} (\rho + 3p) + 2(\Lambda - \kappa^2 U) \\ -\kappa^2 \frac{(1+\gamma)^2}{\gamma^2} \lambda (1 + \epsilon \mathcal{A})^2 + \epsilon \kappa^2 \frac{(1+\gamma)}{\gamma} (\rho + 3p) \mathcal{A}. \end{aligned} \quad (50)$$

It is not immediate to obtain the RS limit from the above equations, since we need to go to second order in γ . In the limit of small induced gravity,

$$(1+\gamma)\mathcal{A} \rightarrow 1 + \gamma \left(1 + \frac{\rho}{\lambda} \right) - \frac{\gamma^2}{\lambda} \left[\rho \left(1 + \frac{\rho}{2\lambda} \right) + \frac{\Lambda}{\kappa^2} + U \right], \quad (51)$$

and to leading order we recover the Friedmann and Raychaudhuri equations of the asymmetric RS model in the $\epsilon = -1$ branch. The other branch does not allow this limit.

We can now eliminate the derivatives of a from Eq. (45), to obtain an algebraic relation among U , Λ , ρ , a and the embedding:

$$\begin{aligned} & \left(\bar{K} + u_a u_b \bar{K}^{ab} \right) \left[\frac{\kappa^2}{\gamma} (\rho + 3p) + 2(\Lambda - \kappa^2 U) \right. \\ & \left. - \kappa^2 \frac{(1+\gamma)^2}{\gamma^2} \lambda \right] - 3\epsilon \kappa^2 \frac{(1+\gamma)}{\gamma} \left[\frac{\Delta \tilde{\Lambda}}{\tilde{\kappa}^2} + \frac{\lambda}{\gamma} \bar{K} \right] \mathcal{A} \\ & + \kappa^2 \frac{(1+\gamma)^2}{\gamma^2} \lambda \left[-2\bar{K} + u_a u_b \bar{K}^{ab} \right] \mathcal{A}^2 = 0. \end{aligned} \quad (52)$$

To leading order, this constraint agrees with the corresponding constraint of the asymmetric RS model [3].

The Lanczos equation (47) becomes

$$\begin{aligned} \frac{3\epsilon}{\tilde{\kappa}^2} (1+\gamma) \mathcal{A} \Delta K_{ab} &= \left[\rho + 3p - \frac{(1+\gamma)^2}{\gamma} \lambda \right. \\ & \left. + \frac{2\gamma}{\kappa^2} (\Lambda - \kappa^2 U) - \epsilon \frac{(1+\gamma)}{\gamma} \lambda \mathcal{A} \right] u_a u_b \\ & + \frac{(1+\gamma)}{\gamma} \lambda \mathcal{A} [\epsilon + (1+\gamma) \mathcal{A}] a^2 h_{ab}. \end{aligned} \quad (53)$$

IV. ANTI DE SITTER BULK

Since the bulk equations are unchanged by the introduction of the induced gravity contribution on the brane, the static bulk solution admitting spatial sections with cosmological symmetry will be AdS. Then the off-brane evolution equations (30), (31) and (18) are trivially satisfied and $\bar{\mathcal{E}}_{ab} = 0$.

A. Determining the functions U and Λ

From the extrinsic curvature of a generic Friedmann brane [3] we find

$$h^{ab} \Delta K_{ab} = 6a \bar{B}, \quad (54)$$

where

$$B^\pm = - \left[\dot{a}^2 + k - \frac{\tilde{\Lambda}^\pm}{6} a^2 \right]^{1/2}. \quad (55)$$

(We have chosen the normal to the brane pointing towards the + region.) Then

$$12\bar{B}\Delta B + a^2\Delta\tilde{\Lambda} = 0, \quad (56)$$

and

$$\frac{\dot{a}^2 + k}{a^2} = \frac{\Lambda_0}{3} - \frac{\kappa^2\lambda}{6} + \left(\frac{\bar{B}}{a}\right)^2 + \left(\frac{\Delta B}{2a}\right)^2, \quad (57)$$

which is the Friedmann equation, written in terms of \bar{B} and ΔB . These can be determined from Eq. (56) and the h^{ab} -projection of the Lanczos equation (53):

$$H^2 + \frac{k}{a^2} = \frac{\Lambda_0}{3} + \frac{\kappa^2\lambda}{6} \left\{ \left[\frac{1 + \epsilon(1 + \gamma)\mathcal{A}}{\gamma} \right]^2 - 1 \right\} + \frac{\gamma^2(\Delta\tilde{\Lambda})^2}{96\kappa^2\lambda[1 + \epsilon(1 + \gamma)\mathcal{A}]^2}. \quad (58)$$

In the $\gamma \rightarrow 0$ limit, using Eq. (51), we recover the Friedmann equation of the asymmetric RS model with AdS bulk [3]. We have checked that the remaining constraints, Eqs. (52) and the Codazzi equation, are satisfied.

A comparison of the generic form of the Friedmann equation, Eq. (48), with the specific form for an AdS bulk, Eq. (58), gives an algebraic relation between Λ and U . We obtain a *quartic* polynomial

$$\sum_{i=0}^4 c_i x^i = 0, \quad (59)$$

in the variable

$$x = \Lambda - \Lambda_0 + \kappa^2 U, \quad (60)$$

with coefficients

$$\begin{aligned} c_0 &= -\frac{\gamma^2\lambda^2(\Delta\tilde{\Lambda})^4}{4\kappa^4}, \\ c_1 &= \frac{32\lambda^2(\lambda + \gamma\beta)(\Delta\tilde{\Lambda})^2}{\kappa^2}, \\ c_2 &= -1024\lambda^2 \left[\frac{\gamma^2(\Delta\tilde{\Lambda})^2}{32\kappa^4} + \beta^2 \right], \\ c_3 &= \frac{2048\gamma\lambda^2\beta}{\kappa^2}, \\ c_4 &= -\frac{1024\gamma^2\lambda^2}{\kappa^4}. \end{aligned} \quad (61)$$

Here we have introduced the notation

$$\beta = \rho + \lambda \left(1 + \frac{\gamma}{2} \right) - \frac{\gamma\Lambda_0}{\kappa^2}. \quad (62)$$

Equation (59) was obtained by eliminating \dot{a}^2 from the two forms of the Friedmann equation, Eqs. (48) and (58), then inserting \mathcal{A}^2 from Eq. (49), expressing \mathcal{A} from the resulting equation, taking its square and comparing again

with \mathcal{A}^2 as given by Eq. (49). With this procedure, the explicit dependence on ϵ is lost, and the two branches will re-emerge as roots of Eq. (59).

The functions U and $\Lambda - \Lambda_0$ obey Eq. (46), which becomes

$$\kappa^2 U = \frac{3(\rho + p)\dot{x}}{4\dot{\rho}}. \quad (63)$$

Once the solutions of Eq. (59) are found, U is given by Eq. (63). This is an alternative route to the lengthy procedure, described in [3], of comparing the two forms of the Raychaudhuri equation.

In the Z_2 -symmetric limit, with $\Delta\tilde{\Lambda} = 0$, the coefficients of the polynomial (59) reduce to

$$\begin{aligned} c_0 &= c_1 = 0, \quad c_2 = -1024\lambda^2\beta^2, \\ c_3 &= 2048\gamma\kappa^{-2}\lambda^2\beta, \quad c_4 = -1024\gamma^2\kappa^{-4}\lambda^2. \end{aligned} \quad (64)$$

The only solution is the double root $x = 0$. Then Eq. (63) is solved for $U = 0$ and we recover the result of [5], in the AdS case $\mathcal{E}_0 = 0$.

B. Perturbative solutions: small asymmetry

A small deviation from Z_2 -symmetry can be described via the dimensionless parameter

$$\alpha = \frac{(\Delta\tilde{\Lambda})^2}{6\kappa^4\lambda^2} \ll 1. \quad (65)$$

To leading order, the solution of Eq. (59) is

$$x = \frac{3\kappa^2\lambda^2}{32\beta^2} \left[\lambda + \gamma\beta - \epsilon\sqrt{\lambda(\lambda + 2\gamma\beta)} \right] \alpha. \quad (66)$$

From Eq. (63) we obtain

$$\begin{aligned} \kappa^2 U &= -\frac{9\kappa^2(\rho + p)\lambda^2}{128\beta^3(\lambda + 2\gamma\beta)} \left[(\lambda + 2\gamma\beta)(2\lambda + \gamma\beta) \right. \\ &\quad \left. - \epsilon(2\lambda + 3\gamma\beta)\sqrt{\lambda(\lambda + 2\gamma\beta)} \right] \alpha. \end{aligned} \quad (67)$$

Switching off the induced gravity in this perturbative solution gives

$$x = \frac{3(1 - \epsilon)\kappa^2\lambda^3}{32(\rho + \lambda)^2} \alpha, \quad (68)$$

$$\kappa^2 U = -\frac{9(1 - \epsilon)\kappa^2(\rho + p)\lambda^3}{64(\rho + \lambda)^3} \alpha. \quad (69)$$

The perturbative solution is in fact *exact* in the limit without induced gravity, as can be verified by direct inspection of Eqs. (59) and (63) in the $\gamma \rightarrow 0$ limit. For the $\epsilon = -1$ branch the results of the asymmetric RS model, presented in [3], are recovered.

Having obtained x and U , the explicit forms of the Friedmann and Raychaudhuri equations, Eqs. (48) and (50), can be written. For this, we remark that

$$\mathcal{A}^2 = 1 + \frac{2\gamma}{(1+\gamma)^2\kappa^2\lambda} (\kappa^2\rho - \gamma x - \gamma\Lambda_0). \quad (70)$$

In the Z_2 -symmetric limit this agrees with Eq. (3.26) of [5], for vanishing dark radiation term.

C. High energy limit

We now consider the perturbative small-asymmetry case at high energies ($\rho \gg \lambda$) in the early universe. We assume that there is no bare cosmological constant on the brane, i.e., $\Lambda_0 = 0$, and that the spatial sections are flat, $k = 0$. By Eqs. (62) and (66),

$$\beta \approx \rho, \quad x \approx \left(\frac{3}{32}\gamma\kappa^2\lambda\right) \frac{\lambda}{\rho} \alpha. \quad (71)$$

Then from Eq. (70),

$$\mathcal{A}^2 = \frac{2\gamma}{(1+\gamma)^2} \frac{\rho}{\lambda} \left[1 + O\left(\frac{\lambda}{\rho}, \alpha \frac{\lambda^2}{\rho^2}\right)\right], \quad (72)$$

so that to lowest order, there is no effect from asymmetry. The Friedmann equation (48) is the same as in the symmetric case:

$$H^2 = \frac{\kappa^2\rho}{3\gamma} \left[1 + \epsilon \sqrt{\frac{2\lambda}{\gamma\rho}} \left\{1 + O\left(\sqrt{\frac{\lambda}{\rho}}, \alpha \frac{\lambda^2}{\rho^2}\right)\right\}\right]. \quad (73)$$

D. Low energy limit

In the low-energy, late-universe regime, induced gravity models with $\epsilon = 1$ are interesting because they can produce acceleration even without dark energy [7, 8]. Asymmetric RS models can also produce acceleration (see [3] and references therein). We consider $\epsilon = \pm 1$ models in the perturbative small-asymmetry case, with $\Lambda_0 = 0 = k$. We assume that

$$\frac{\rho}{\lambda} \ll \alpha \ll 1, \quad (74)$$

which is readily achieved by choosing the brane tension high enough relative to the very low energy scale of the late universe.

By Eqs. (62) and (66),

$$\beta \approx \lambda \left(1 + \frac{\gamma}{2}\right), \quad (75)$$

$$x \approx \frac{3}{16}\kappa^2\lambda \left(\frac{1-\epsilon+\gamma}{2+\gamma}\right)^2 \alpha. \quad (76)$$

Then from Eqs. (70) and (74),

$$\mathcal{A} \approx 1 - \frac{3}{16} \left[\frac{\gamma(1-\epsilon+\gamma)}{(1+\gamma)(2+\gamma)}\right]^2 \alpha. \quad (77)$$

To lowest order, the first correction is from asymmetry, in contrast to the high-energy case. The Friedmann equation (48) becomes

$$H^2 = \frac{2\kappa^2(1+\gamma)\lambda}{3\gamma^2} \left[1 - \frac{3\gamma^4}{32(1+\gamma)^2(2+\gamma)^2} \alpha\right], \quad (78)$$

for $\epsilon = 1$, and

$$H^2 = \frac{\kappa^2\lambda}{16(1+\gamma)} \alpha, \quad (79)$$

for $\epsilon = -1$.

It follows that for $\epsilon = 1$, the late-time acceleration from induced gravity is slightly reduced by asymmetry, whereas for the $\epsilon = -1$ branch (with an RS limit), the asymmetry introduces a small acceleration.

V. CONCLUDING REMARKS

Combining the method developed in [3] with that of [5], we have developed a general formalism describing gravitational dynamics on the asymmetrically embedded brane with induced gravity. The decomposition of the bulk Einstein equations yielded the tensorial, vectorial and scalar projections. They are equivalent to the projected Einstein, the Codazzi and the twice contracted Gauss equations. In the derivation of the final set of equations the generalized junction conditions were applied. The resulting effective Einstein equation contains all terms from the asymmetric RS model, to which the induced gravity contributions add linearly.

For Friedmann branes, the generalized Friedmann and Raychaudhuri equations were derived, together with the relevant constraint equations. They contain two undetermined functions U and $\Lambda - \Lambda_0$. These depend on the bulk geometry. As an application of the formalism, we discussed in detail the AdS bulk with different cosmological constants on the two sides of the brane. Determining the functions U and $\Lambda - \Lambda_0$ reduces in this case to solving a quartic polynomial. This feature of the model was already remarked on and exploited in [9], where a perturbative solution in terms of a small induced gravity contribution was analyzed. Similarly, in [10] a quartic equation in $H^2 + k/a^2$ was presented. Our method in determining the Friedmann equation also yielded a quartic equation. We have found and discussed those solutions of the quartic equation for which the deviation from Z_2 -symmetry is small. In the high-energy, early-universe regime, the asymmetry does not affect the Friedmann equation to lowest order. In the low-energy, late-universe regime, the asymmetry does have an effect. In the $\epsilon = -1$ branch, it contributes a late-time acceleration, while in the $\epsilon = 1$ branch is slightly reduces the late-time acceleration arising from induced gravity.

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