Scalar perturbations from brane-world inflation

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We investigate the scalar metric perturbations about a de Sitter brane universe in a 5-dimensional anti de Sitter bulk. We compare the master-variable formalism, describing metric perturbations in a 5-dimensional longitudinal gauge, with results in a Gaussian normal gauge. For a vacuum brane (with constant brane tension) there is a continuum of normalizable Kaluza-Klein modes, with $m > \frac{3}{2}H$, which remain in the vacuum state. A light radion mode, with $m = \sqrt{2}H$, satisfies the boundary conditions for two branes but is not normalizable in the single-brane case. When matter is introduced (as a test field) on the brane, this mode, together with the zero-mode and an infinite ladder of discrete tachyonic modes, become normalizable. However, the boundary condition requires the self-consistent 4-dimensional evolution of scalar field perturbations on the brane and the dangerous growing modes are not excited. These normalizable discrete modes introduce corrections at first-order to the scalar field perturbations computed in a slow-roll expansion. On super-Hubble scales, the correction is smaller than slow-roll corrections to the de Sitter background. However on small scales the corrections can become significant.

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I. INTRODUCTION

If gravity propagates in extra spatial dimensions, while Standard Model fields are confined to the 3 observed spatial dimensions, then the observable universe could be described by a 3-brane in a $4 + d$-dimensional bulk spacetime (see [1, 2, 3] for reviews on the subject). At low energies, the extra-dimensional effects should be small in order to recover the successes of 4-dimensional general relativity, but at high energies these effects could be dominant. If the early universe included a period of inflation, then the extra-dimensional gravitational effects could introduce significant changes to the dynamics and generation of primordial perturbations at high energy. Any imprints left on the perturbation spectra will be constrained by increasingly high precision observations of the cosmic microwave background, providing in principle constraints on extra-dimensional theories.

The study of general cosmological perturbations in brane-worlds is complicated because the motion of the brane in the higher-dimensional bulk makes it impossible, in general, to separate the evolution of different Kaluza-Klein modes. It is only possible to fix the brane location and obtain a separable wave equation for perturbations in the special case of a de Sitter (or Minkowski or anti-de Sitter) brane in 5-dimensional anti-de Sitter (AdS) spacetime [4, 5]. This is useful as it provides a zeroth-order approximation for slow-roll inflation on the brane. In this case the behaviour of tensor [6] or vector [7] metric perturbations has been described, while neglecting matter sources on the brane. In particular one can estimate high-energy corrections to the spectrum of gravitational waves produced from vacuum fluctuations in the bulk spacetime during inflation [6, 8, 9, 10].

The amplitude of scalar perturbations on the brane due to inflaton field fluctuations has also been estimated in the extreme slow-roll limit where the coupling of field fluctuations to bulk metric perturbations is neglected [11]. Energy conservation on the brane is sufficient to ensure that there exists a scalar curvature perturbation for matter that is conserved for adiabatic density perturbations in a large scale limit [12, 13]. In this limit, this approach by-passes the need to study bulk scalar metric perturbations coupled to matter on the brane. But we do need to understand the bulk metric perturbations in order to go beyond the zeroth-order slow-roll approximation (for partial attempts see Refs. [14, 15, 16]) and to distinguish 5-dimensional effects from a modified 4-dimensional theory [17]. (Note that the bulk metric perturbations are also needed in order to compute the Sachs-Wolfe effect, since the large-scale curvature perturbation does not determine the brane metric perturbations [13, 18].)

In this paper we investigate bulk scalar metric perturbations about a de Sitter brane. We first consider scalar metric perturbations in the absence of matter perturbations. In this case there are no normalizable light modes (with effective 4-dimensional mass $m^2 < \frac{9}{4}H^2$) for a single brane [9], but in the presence of a second brane there is a normalizable "radion" mode [19]. We discuss how this discrete mode appears either as a displacement ("bending") of the brane or as a bulk metric perturbation in different gauges. The radion is massless for two Minkowski branes [20] but appears as an "instability" for two de Sitter branes [19, 21, 22, 23], although the effect of the radion "instability" on the brane becomes small on large scales [24].

We then go on to consider the bulk metric perturbations excited by scalar field fluctuations on a single de Sitter brane as a first step towards calculating the effect of first-order slow-roll corrections on the exact de Sitter solutions. We show that scalar field perturbations can excite an infinite ladder of apparently tachyonic, normalizable bulk modes. However the boundary condition requires the self-consistent 4-dimensional evolution of scalar field perturbations on the brane and the dangerous growing modes, that one might expect to find for tachyonic modes, are not allowed. We comment on the possible effect of metric backreaction upon the scalar fluctuations during inflation.

We present our conclusions in Section VI.

II. RANDALL-SUNDRUM COSMOLOGY

The Randall-Sundrum (RS) model [25] provides the basis for a simple realization of the brane-world idea in cosmology [26, 27, 28]. The background bulk is 5D AdS spacetime with a negative cosmological constant Λ_5 and the brane has Friedman-Robertson-Walker geometry. For a general brane and bulk geometry, the 5D field equations are

$$ {}^{(5)}G_{AB} + \Lambda_5 \, {}^{(5)}g_{AB} = 0 \,. \tag{1} $$

One can define an energy scale μ corresponding to the curvature scale of the bulk, via $\Lambda_5 = -6\mu^2$.

The induced metric on the brane is

$$ g_{AB} = {}^{(5)}g_{AB} - n_A n_B \,, \tag{2} $$

where n^A is the unit vector normal to the brane. The 4D matter fields determine the brane trajectory in the bulk via the junction conditions, by producing the jump in the extrinsic curvature at the brane. Without loss of generality, the surface energy-momentum on the brane can be split into two parts, $T_{\mu\nu} - \lambda g_{\mu\nu}$, where $T_{\mu\nu}$ is the matter energy-momentum tensor and λ is a constant brane tension. The junction condition with Z_2-symmetry is then [26, 27]

$$ K^{\mu\ +}_{\nu} - K^{\mu\ -}_{\nu} = 2K^{\mu\ +}_{\nu} = -\kappa_5^2 \left[T^{\mu}_{\nu} - \frac{1}{3}\delta^{\mu}_{\nu} \left(T - \lambda\right) \right] , \tag{3} $$

where the extrinsic curvature of the brane is $K_{\mu\nu} = g_{\mu}{}^{C} g_{\nu}{}^{D} \left[{}^{(5)}\nabla_C n_D \right]$ and κ_5^2 is the 5-dimensional coupling of matter to gravity. The effective Einstein equations for the induced metric on the brane are then [27]

$$ G_{\mu\nu} = \kappa_4^2 T_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu} \,, \tag{4} $$

where the effective 4D coupling of matter to gravity on the brane at low energies is given by $\kappa_4^2 = \mu\kappa_5^2$ and we have chosen the arbitrary constant $\lambda = 6\mu/\kappa_5^2$. As well as the high-energy corrections due to the tensor $\Pi_{\mu\nu}$ which is quadratic in the energy-momentum tensor $T_{\mu\nu}$, the effective Einstein equations include a non-local contribution $E_{\mu\nu}$ from the projection of the 5D Weyl tensor.

In order to study inhomogeneous bulk metric perturbations, we choose a specific form for the unperturbed 5D spacetime that accommodates any spatially flat FRW cosmological solution on the brane at $y = 0$,

$$ ds^2 = -n^2(t,y)dt^2 + a^2(t,y)d\vec{x}^{\,2} + b^2(t,y)dy^2 \,. \tag{5} $$

The scale factor on the brane is $a_o(t) = a(t,0)$. The junction conditions (3) for this background metric yield

$$ \frac{a'^{+} - a'^{-}}{a} = -\frac{\kappa_5^2}{3}\left(\lambda + \rho\right) , \qquad \frac{n'^{+} - n'^{-}}{n} = -\frac{\kappa_5^2}{3}\left(\lambda - 3P - 2\rho\right) , \tag{6} $$

where ρ and P are respectively the energy density and pressure associated with the homogeneous brane energy-momentum tensor $T_{\mu\nu}$.

Allowing arbitrary first-order scalar metric perturbations then gives the metric [29, 30]

$$ g_{AB} = \begin{bmatrix} -n^2(1+2A) & a^2 B_{,i} & nA_y \\ a^2 B_{,j} & a^2\left\{(1+2\mathcal{R})\delta_{ij} + 2E_{,ij}\right\} & a^2 B_{y,i} \\ nA_y & a^2 B_{y,i} & b^2(1+2A_{yy}) \end{bmatrix} . \tag{7} $$

Note that we are using the common cosmological notation of scalar perturbations to denote scalars with respect to 3-space slices at fixed t and y.

The perturbed energy-momentum tensor for matter on the brane, with background energy density ρ and pressure P, can be written as

$$T^\mu_\nu = \begin{bmatrix} -(\rho+\delta\rho) & \delta q_{,j} \\ -a^{-2}\left\{\delta q^{,i} - (\rho+P)B^{,i}\right\} & (P+\delta P)\delta^i_j + \delta\pi^i_j \end{bmatrix}, \tag{8}$$

where $\delta\pi^i_j = \delta\pi^{,i}{}_{,j} - \frac{1}{3}\delta^i_j\,\delta\pi^{,k}{}_{,k}$ is the tracefree anisotropic stress perturbation. Substituting in the junction conditions, this requires (see e.g. [31, 32]),

$$\delta K^0_0 = \frac{\kappa_5^2}{6}\left(2\delta\rho + 3\delta P\right), \tag{9}$$

$$\delta K^0_i = -\frac{\kappa_5^2}{2}\delta q_{,i}, \tag{10}$$

$$\delta K^i_j = -\frac{\kappa_5^2}{6}(\delta\rho - \vec{\nabla}^2\delta\pi)\delta^i_j - \frac{\kappa_5^2}{2}\delta\pi^{,i}{}_{,j}. \tag{11}$$

The components of the perturbed extrinsic curvature in an arbitrary gauge are given by [33]

$$\delta K^0_0 = \frac{1}{b}\left[A' - \frac{n'}{n}A_{yy} + \frac{1}{n}\dot{A}_y + \frac{b^2}{n^2}\left\{\ddot{\xi} + \left(2\frac{\dot{b}}{b} - \frac{\dot{n}}{n}\right)\xi\right\} + \left\{\left(\frac{n'}{n}\right)' - \frac{n'}{n}\frac{b'}{b}\right\}\xi\right], \tag{12}$$

$$\delta K^i_j = \frac{1}{b}\left[\mathcal{R}' - \frac{a'}{a}A_{yy} + \frac{1}{n^2}\frac{\dot{a}}{a}(nA_y + b^2\dot{\xi}) + \left\{\left(\frac{a'}{a}\right)' - \frac{a'}{a}\frac{b'}{b}\right\}\xi\right]\delta^i_j + \frac{1}{b}\left[E' - B_y - \frac{b^2}{a^2}\xi\right]^{,i}_{,j}, \tag{13}$$

$$\delta K^0_i = -n^{-2}\left[\frac{1}{2}\frac{a^2}{b}\left(B' - \dot{B}_y - \frac{n}{a^2}A_y\right) + b\left\{-\dot{\xi} + \left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)\xi\right\}\right]_{,i}, \tag{14}$$

where a dot denotes a derivative with respect to time t and a prime a derivative with respect to y. Here we took into account the fact that the position of the brane is generally displaced from $y=0$ in a general gauge. The brane bending scalar $\xi(t,\vec{x})$ describes the perturbed position of the brane.

III. MASTER VARIABLE AND ALTERNATIVE BULK GAUGES

A. 5D longitudinal gauge

To eliminate any gauge dependence on the choice of 3-space coordinates we can work with the spatially gauge-invariant combinations

$$\sigma_t = -B + \dot{E}, \quad \sigma_y = -B_y + E', \tag{15}$$

which are subject only to temporal and bulk gauge transformations. The bulk and temporal gauges are fully determined by setting $\sigma_t = \sigma_y = 0$, which we refer to as the 5D longitudinal gauge [29, 34] to avoid possible confusion with quantities in the 4D longitudinal gauge on the brane.

We can define the remaining metric perturbations in the 5D longitudinal gauge as

$$\tilde{A} = A - \frac{1}{n}\left(\frac{a^2}{n^2}\sigma_t\right)^{\cdot} + \frac{n'}{n}\frac{a^2}{b^2}\sigma_y, \tag{16}$$

$$\tilde{\mathcal{R}} = \mathcal{R} - \frac{\dot{a}}{a}\frac{a^2}{n^2}\sigma_t + \frac{a'}{a}\frac{a^2}{b^2}\sigma_y, \tag{17}$$

$$\tilde{A}_y = A_y + n\left(\frac{a^2}{n^2}\sigma_t\right)' + \frac{b^2}{n}\left(\frac{a^2}{b^2}\sigma_y\right)^{\cdot}, \tag{18}$$

$$\tilde{A}_{yy} = A_{yy} - \frac{\dot{b}}{b}\frac{a^2}{n^2}\sigma_t + \frac{1}{b}\left(\frac{a^2}{b}\sigma_y\right)'. \tag{19}$$

These are equivalent to the gauge-invariant bulk perturbations originally introduced in covariant form in [35, 36] and in a coordinate-based approach in [34]. The spatial trace part of the 5D Einstein equations simplifies in the 5D longitudinal gauge to

$$\tilde{A} + \tilde{\mathcal{R}} + \tilde{A}_{yy} = 0 \,. \tag{20}$$

Mukohyama [35] (see also [36]) was the first to show that the perturbed 5D Einstein equations, in the absence of bulk matter perturbations, ${}^{(5)}\delta G^A_B = 0$, are solved in an AdS background if the metric perturbations are derived from a "master variable", Ω:

$$\tilde{A} = -\frac{1}{6a}\left\{\frac{1}{b^2}\left[2\Omega'' - \left(2\frac{b'}{b} + \frac{n'}{n}\right)\Omega'\right] + \frac{1}{n^2}\left[\ddot{\Omega} - \left(2\frac{\dot{b}}{b} + \frac{\dot{n}}{n}\right)\dot{\Omega}\right] - \mu^2\Omega\right\}, \tag{21}$$

$$\tilde{A}_y = \frac{1}{na}\left(\dot{\Omega}' - \frac{n'}{n}\dot{\Omega} - \frac{\dot{b}}{b}\Omega'\right), \tag{22}$$

$$\tilde{A}_{yy} = \frac{1}{6a}\left\{\frac{1}{b^2}\left[\Omega'' - \left(2\frac{n'}{n} + \frac{b'}{b}\right)\Omega'\right] + \frac{1}{n^2}\left[2\ddot{\Omega} - \left(2\frac{\dot{n}}{n} + \frac{\dot{b}}{b}\right)\dot{\Omega}\right] + \mu^2\Omega\right\}, \tag{23}$$

$$\tilde{\mathcal{R}} = \frac{1}{6a}\left\{\frac{1}{b^2}\left[\Omega'' + \left(\frac{n'}{n} - \frac{b'}{b}\right)\Omega'\right] + \frac{1}{n^2}\left[-\ddot{\Omega} + \left(\frac{\dot{n}}{n} - \frac{\dot{b}}{b}\right)\dot{\Omega}\right] - 2\mu^2\Omega\right\}. \tag{24}$$

The remaining perturbed 5D Einstein equations then yield a single wave equation governing the evolution of the master variable Ω in the bulk:

$$-\left(\frac{b}{na^3}\dot{\Omega}\right)^{\cdot} + \left(\frac{n}{ba^3}\Omega'\right)' + \left(\mu^2 - \frac{k^2}{a^2}\right)\frac{nb}{a^3}\Omega = 0 \,, \tag{25}$$

where k is the comoving wave-number along the brane, $\vec{\nabla}^2 \to -k^2$. Note that this is not the standard form for a 5-dimensional wave-equation for a canonical scalar field. It can be re-written in a standard form by defining $\omega \equiv a^{-3}\Omega$ but we shall work with the original variable introduced in [35].

The advantage of the master variable approach is that the 5D field Ω describes all the degrees of freedom of the bulk scalar metric perturbations. In particular the perturbed brane location ξ is directly related to the anisotropic stress by boundary conditions at the brane [see Eqs. (11) and (13)]. Hence any radion mode describing the perturbation in the relative distance between two branes must be encoded in the bulk metric perturbations.

B. Gaussian normal gauge

An alternative choice of gauge commonly used is a Gaussian normal (GN) gauge, where the bulk y coordinate measures the proper distance in the bulk, including first-order metric perturbations [29]. This requires the metric perturbations B_y, A_y and A_{yy} to vanish, but leaves a residual gauge freedom to pick the 4D gauge on any given constant-y hypersurface (analogous to the residual gauge freedom on a spatial hypersurface in the 4D synchronous gauge).

At this stage, one must make a distinction between a *general Gaussian Normal gauge* defined as above and the particular *brane Gaussian Normal gauge* in which the above conditions are supplemented by the requirement that the brane lies at $y = 0$.

A technique used to study the propagation of gravitational waves in a vacuum spacetime is to work in a (general) GN gauge in which the perturbations are transverse and tracefree in the background spacetime. The transverse and tracefree condition in a Gaussian normal gauge actually over constrains the problem except for the special case of a maximally symmetric 4D (anti-)de Sitter brane [29].

When the 5D perturbations of the metric $h_{AB} \equiv \delta^{(5)}g_{AB}$ are transverse, ${}^{(5)}\nabla^C\, h_{AC} = 0$, and tracefree, ${}^{(5)}g^{AB}\, h_{AB} = 0$, the perturbed Einstein equations can be written as a wave equation,

$${}^{(5)}\Box\, h_{AB} = 2\, {}^{(5)}R_{CADB}\, h^{CD} \,, \tag{26}$$

where ${}^{(5)}\Box = {}^{(5)}\nabla_C\, {}^{(5)}\nabla^C$. The background Riemann tensor in AdS$_5$ is given by

$${}^{(5)}R_{ABCD} = \frac{\Lambda_5}{6}\left[{}^{(5)}g_{AC}\, {}^{(5)}g_{BD} - {}^{(5)}g_{AD}\, {}^{(5)}g_{BC}\right]. \tag{27}$$

Therefore to linear order in the metric perturbations, and enforcing the transverse and traceless conditions, the field equations in the absence of matter are given by

$$^{(5)}\Box\, h_{AB} = -\frac{1}{3}\Lambda_5\, h_{AB}\,. \tag{28}$$

The tracefree condition, in a GN gauge, requires

$$A + 3\mathcal{R} - k^2 E = 0\,. \tag{29}$$

The transverse condition in general gives rise to four constraint equations, which can be written, using Eq. (29), as

$$-k^2 B + 2\left(\dot{A} + 4\frac{\dot{a}}{a}A\right) \;=\; 0\,, \tag{30}$$

$$\left\{\frac{a^2}{n^2}\left[\left(\frac{\dot{n}}{n} - 5\frac{\dot{a}}{a}\right)B - \dot{B}\right] - 2A - 4\mathcal{R}\right\}_{,i} \;=\; 0\,, \tag{31}$$

$$2\left(\frac{a'}{a} - \frac{n'}{n}\right)A \;=\; 0\,. \tag{32}$$

Unless $(a/n)' = 0$, i.e., unless a and n have the same y-dependence, the five constraint equations require that the four GN scalar metric perturbations are all identically zero. Using the background field equation ${}^{(5)}G^4_0 = 0$ [see Eq. (1)], this implies that it is only possible to use the transverse and tracefree GN gauge for a separable bulk metric, which corresponds to a Minkowski or (anti-)de Sitter brane given in Eq. (34).

Thus, only in the special case of a (anti-)de Sitter or Minkowski brane, the wave equation (28) gives an evolution equation for the scalar metric perturbation:

$$\frac{1}{n^2}\left\{\ddot{A} - \left(\frac{\dot{n}}{n} - 7\frac{\dot{a}}{a}\right)\dot{A}\right\} + \left[8\left(\frac{\dot{a}}{an}\right)^2 + 2\left(\frac{a'}{a}\right)^2 - 2\mu^2\right]A + \frac{k^2}{a^2}A = A'' + 4\frac{a'}{a}A'\,, \tag{33}$$

and the remaining scalar metric perturbations can be deduced from the constraint equations (29)–(31). The GN gauge choice necessarily eliminates the radion mode from the metric perturbations. In this gauge the radion in a two-brane system must be described instead as a relative perturbation of the coordinate position of the branes.

IV. BULK GRAVITONS WITH A DE SITTER BRANE

A. Separable background

In order to solve for the y-dependence of the bulk gravitons and to study the time-dependence of the perturbations on the brane, we will consider the special case of a de Sitter brane (with constant Hubble rate H, energy density ρ and pressure $-\rho$) in an AdS bulk, which gives a separable form for the bulk metric [4],

$$ds^2 = N^2(y)\left[-dt^2 + a_o^2(t)d\vec{x}^{\,2}\right] + dy^2\,, \tag{34}$$

$$a_o(t) = \exp Ht\,, \quad N(y) = \frac{H}{\mu}\sinh\mu(y_{\rm h} - |y|)\,, \tag{35}$$

where $y = \pm y_{\rm h}$ are Cauchy horizons, with

$$y_{\rm h} = \frac{1}{\mu}\coth^{-1}\left(1 + \frac{\rho}{\lambda}\right)\,. \tag{36}$$

Any constant-y hypersurface corresponds to an exponentially expanding de Sitter slice for $\rho > 0$, giving a dS_4 slicing of AdS_5. The original RS solution [25] with Minkowski spacetime on the brane (M_4 slicing of AdS_5) is recovered in the limit $\rho/\lambda \to 0$, when $N \to \exp(-\mu|y|)$ and $y_{\rm h} \to \infty$. At very high energies, $\rho \gg \lambda$, deviations from the RS solution will be significant. The junction conditions in Eq. (6) require that $p = -\rho =$ constant. This will be a good approximation to a potential-dominated scalar field rolling slowly down a sufficiently flat potential [11].

It is often useful to work in terms of the conformal bulk-coordinate $z = \int dy/N(y)$:

$$z = \mathrm{sgn}(y)H^{-1}\ln\left[\coth\tfrac{1}{2}\mu(y_{\rm h} - |y|)\right]\,. \tag{37}$$

The Cauchy horizon is now at $|z|=\infty$, and the brane is located at $z=\pm z_{\rm b}$, with

$$z_{\rm b}=\frac{1}{H}\sinh^{-1}\frac{H}{\mu}\,. \tag{38}$$

The line element, Eq. (34), becomes

$$ds^2 \;=\; N^2(z)\left[-dt^2+dz^2+{\rm e}^{2Ht}d\vec{x}^{\,2}\right]\,, \tag{39}$$

$$N(z) \;=\; \frac{H}{\mu\sinh H|z|}\,. \tag{40}$$

In the RS limit, $\rho\to 0$ and $H\to 0$, so that $N\to[1+\mu(|z|-z_{\rm b})]^{-1}$.

B. Master variable

In the dS$_4$ slicing of AdS$_5$, the master variable wave equation (25) reduces to

$$\frac{1}{N^2}\left(-\ddot{\Omega}+3H\dot{\Omega}\right)+\Omega''-2\frac{N'}{N}\Omega'=\left(\frac{k^2}{a_o^2N^2}-\mu^2\right)\Omega\,. \tag{41}$$

The solutions can be separated into eigenmodes of the time-dependent equation on the brane and the bulk mode equation, $\Omega(t,y;\vec{x})=\int d^3\vec{k}\,dm\,\alpha_m(t)u_m(y)e^{i\vec{k}\cdot\vec{x}}$, where

$$\ddot{\alpha}_m-3H\dot{\alpha}_m+\left[m^2+\frac{k^2}{a_o^2}\right]\alpha_m \;=\; 0\,, \tag{42}$$

$$u_m''-2\frac{N'}{N}u_m'+\left[\frac{m^2}{N^2}+\mu^2\right]u_m \;=\; 0\,. \tag{43}$$

Note that the Hubble damping term $-3H\dot{\alpha}_m$ has the "wrong sign", i.e., this is not the standard wave equation for a scalar field in 4D. We recover the RS solutions in the limit $H\to 0$, $\rho\to 0$, in which case $\varphi_m=\exp(\pm i\omega t)$, with $\omega^2=k^2+m^2$, and u_m can be given in terms of Bessel functions [25].

If we write $\alpha_m=a_o^2\varphi_m$ and use conformal time $\eta=-1/(a_oH)$, Eq. (42) can be rewritten as

$$\frac{d^2\varphi_m}{d\eta^2}+\left[k^2-\frac{2-(m^2/H^2)}{\eta^2}\right]\varphi_m=0\,. \tag{44}$$

This is the same form of the time-dependent mode equation commonly given for a massive scalar field in 4D de Sitter spacetime. The general solution is

$$\varphi_m(\eta;\vec{k})=\sqrt{-k\eta}\,Z_\nu(-k\eta)\,,\quad \nu^2=\frac{9}{4}-\frac{m^2}{H^2}\,, \tag{45}$$

where Z_ν is a linear combination of Bessel functions of order ν. The solutions oscillate at early-times/small-scales for all m, with an approximately constant amplitude while they remain within the Hubble radius ($k\gg a_oH$). 'Heavy modes', with $m>\frac{3}{2}H$, continue to oscillate as they are stretched to super-Hubble scales, but their amplitude rapidly decays away, $|\varphi_m^2|\propto a_o^{-3}$. But for 'light modes' with $m<\frac{3}{2}H$, the perturbations become over-damped at late-times/large-scales ($k\ll a_oH$), and decay more slowly: $|\varphi_m^2|\propto a_o^{2\nu-3}$.

Defining $\Psi_m\equiv N^{-3/2}u_m$, it is possible to rewrite the off-brane equation (43) in Schrödinger-like form,

$$\frac{d^2\Psi_m}{dz^2}-V\Psi_m=-m^2\Psi_m\,, \tag{46}$$

where

$$V(z)=-\frac{1}{4}\mu^2N^2(z)+\frac{9}{4}H^2=-\frac{H^2}{4\sinh^2(Hz)}+\frac{9}{4}H^2\,. \tag{47}$$

For $z\to\infty$ we have $V\to\frac{9}{4}H^2$ and we have a continuum of massive modes above the mass gap [5] $m^2>\frac{9}{4}H^2$ which become oscillating plane waves as $z\to\infty$. The time-evolution of the mode functions for these heavy modes, Eq. (45), shows that they remain underdamped at late times, i.e., the continuum of massive modes is not excited by de Sitter inflation on the brane, as has previously been noted for vector [7] and tensor [6] modes.

The general solution to the Schrödinger equation (46) is (for $y \geq 0$) [37]

$$\Psi_m = [\sinh \mu(y_h - y)]^{-1/2} \, W_{\nu-1/2}(\coth \mu(y_h - y)) = (\sinh Hz)^{1/2} \, W_{\nu-1/2}(\cosh Hz), \tag{48}$$

where W_α is a linear combination of Legendre polynomials of order α. In a single-brane model, the normalization of the solution is determined by the condition

$$\int_{z_b}^{\infty} |\Psi_m|^2 dz = \int_0^{y_h} N^{-4} |u_m|^2 dy < \infty . \tag{49}$$

The general solution will be non-normalizable for light massive modes with $m^2 \leq \frac{9}{4}H^2$ (which diverge as $z \to \infty$). Modes with $m^2 \leq \frac{9}{4}H^2$ are only normalizable if the boundary conditions at $z = z_b$ allow us to kill the divergent part of the solution at $z \to \infty$. There are no such modes for a single vacuum de Sitter brane [9], but there is one normalizable light mode when a second de Sitter brane is present [19].

1. *Boundary conditions at the brane*

In terms of the master variable, Ω, in the AdS background with dS brane, the boundary conditions (9)–(11) require

$$\left(\ddot{\Omega}' - \frac{N'}{N}\ddot{\Omega}\right) + 2H\left(\dot{\Omega}' - \frac{N'}{N}\dot{\Omega}\right) = \kappa_5^2 a_o \delta P \, , \tag{50}$$

$$\dot{\Omega}' - \frac{N'}{N}\dot{\Omega} = \kappa_5^2 a_o \delta q \, , \tag{51}$$

$$-3H\left(\dot{\Omega}' - \frac{N'}{N}\dot{\Omega}\right) - \frac{k^2}{a^2}\left(\Omega' - \frac{N'}{N}\Omega\right) = \kappa_5^2 a_o \delta\rho \, . \tag{52}$$

For a vacuum brane, $\delta T_\mu^\nu = 0$, these reduce to a single boundary condition on the master variable,

$$\Omega' = \frac{N'}{N}\Omega \, . \tag{53}$$

2. *Radion mode*

The vacuum boundary condition (53) is trivially satisfied for any z by the mode

$$u_{\rm r} \propto N \, , \quad m^2 = m_{\rm r}^2 = 2H^2 \, , \tag{54}$$

which is a solution of the bulk mode equation (43). The Schrödinger wave function $\Psi_{\rm r} \propto N^{-1/2}$ diverges as $z \to \infty$ (where $N \to 0$) so this mode is non-normalizable in the single-brane model. However in a stationary two-brane model, where the second brane is at any fixed $z_2 > z_b$, this mode is normalizable [19] and automatically obeys the boundary condition (53) for any z_2.

We identify this mode as the "radion", which exists in the two-brane model but is non-normalizable for a single brane. The time dependence of this mode is given by Eq. (45) with $\nu = \frac{1}{2}$,

$$\varphi_{\rm r}(\eta, k) = \sqrt{-k\eta} \, Z_{1/2}(-k\eta) \, , \tag{55}$$

and hence the master variable on the brane on large scales or at late times, for which $k/a_o H \ll 1$, is

$$\Omega_{\rm r} \approx C_1 a_o^2 + C_2 a_o \, . \tag{56}$$

The physical effect of the radion on the brane-world can be interpreted as an effective energy-momentum perturbation. The perturbed 5D Weyl tensor is felt on the brane through its projection $\delta E_{\mu\nu}$, which has an effective energy and momentum density on the brane, given in terms of the master variable by [30]

$$\kappa_4^2 \delta\rho_E = \frac{k^4}{3a_o^5}\Omega\,, \tag{57}$$

$$\kappa_4^2 \delta q_E = \frac{k^2}{3a_o^2}\left(\frac{\Omega}{a_o}\right)^{\cdot}\,, \tag{58}$$

$$\kappa_4^2 \delta\pi_E = \frac{1}{2a_o^3}\left(\ddot{\Omega} - H\dot{\Omega} + \frac{k^2}{3a_o^2}\Omega\right)\,. \tag{59}$$

The time dependence of the radion mode on large scales gives the physical effect, from Eqs. (57)–(59), as

$$\kappa_4^2 \delta\rho_{E\,\mathrm{r}} \approx H^4\left[\frac{C_1}{3}a_o\left(\frac{k}{a_oH}\right)^4 + \frac{C_2}{3}\left(\frac{k}{a_oH}\right)^4\right]\,, \tag{60}$$

$$\kappa_4^2 \delta q_{E\,\mathrm{r}} \approx H^3\left[\frac{C_1}{3}a_o\left(\frac{k}{a_oH}\right)^2 + \frac{C_2}{9}\left(\frac{k}{a_oH}\right)^4\right] \tag{61}$$

$$\kappa_4^2 \delta\pi_{E\,\mathrm{r}} \approx H^2\left[\frac{C_1}{a_o} + \frac{C_2}{45a_o^2}\left(\frac{k}{a_oH}\right)^4\right]\,. \tag{62}$$

In order to derive the contribution from the decaying mode, we expanded the solution for Ω_r as

$$\Omega_\mathrm{r} = C_1 a_o^2 + C_2 a_0\left[1 - \frac{1}{6}\left(\frac{k}{a_oH}\right)^2 + \frac{1}{120}\left(\frac{k}{a_oH}\right)^4\right]\,. \tag{63}$$

The decaying mode corresponds to dark radiation with isotropic pressure $\delta P_E = \frac{1}{3}\delta\rho_E$ [38], but the dominant mode is supported by a non-negligible momentum density on large scales, driven by the anisotropic pressure exerted by the radion field [19].

C. Gaussian normal gauge

In the 5D longitudinal gauge, the radion is encoded as a discrete mode in the bulk metric perturbations. In the GN coordinates, the radion must be described instead as a relative perturbation of the coordinate position, i.e. the brane-bending scalar ξ. In this subsection, we show that these two descriptions are equivalent.

In the dS_4 slicing of AdS_5, Eq. (33) reduces to

$$-\frac{1}{N^2}\left(\ddot{A} + 7H\dot{A} + 10H^2 A + \frac{k^2}{a_o^2}A\right) + A'' + 4\frac{N'}{N}A' = 0. \tag{64}$$

This wave equation can be separated via $A(t,y;\vec{x}) = \int d^3\vec{k}\, dm\, f_m(t) g_m(y) e^{i\vec{k}.\vec{x}}$, as

$$\ddot{f}_m + 7H\dot{f}_m + 10H^2 f_m + \left[m^2 + \frac{k^2}{a^2}f_m\right] = 0, \tag{65}$$

$$g_m'' + 4\frac{N'}{N}g_m' + \frac{m^2}{N^2}g_m = 0. \tag{66}$$

Defining $\Phi_m \equiv N^{3/2}g_m$, we can rewrite the off-brane equation in the same Schrödinger-like form as Eq. (46), but with potential

$$V(z) = \frac{15}{4}\frac{H^2}{\sinh^2(Hz)} + \frac{9}{4}H^2. \tag{67}$$

This is the same effective potential as for tensor perturbations [6]. Again, modes with $0 < m^2 \leq \frac{9}{4}H^2$ are not normalizable in a single-brane model unless the boundary conditions at $z = z_\mathrm{b}$ kill off the divergent part of the solution at $z \to \infty$.

1. Boundary conditions

In the (general) GN gauge, the boundary conditions for metric perturbations, Eqs. (9)–(14), for the vacuum brane include a contribution from the brane-bending scalar ξ:

$$
\begin{aligned}
A' &= -\ddot{\xi} + H^2\xi, && (68)\\
\mathcal{R}' &= -H\dot{\xi} + H^2\xi, && (69)\\
E' &= \frac{1}{a_o^2}\xi. && (70)
\end{aligned}
$$

The evolution equation for ξ can be derived from the y-derivative the traceless condition (29),

$$A' + 3\mathcal{R}' - k^2E' = 0, \tag{71}$$

which yields

$$\ddot{\xi} + 3H\dot{\xi} - 4H^2\xi + \frac{k^2}{a_o^2}\xi = 0. \tag{72}$$

We note that the brane-bending has a tachyonic effective mass, $m^2 = -4H^2$, for a de Sitter brane [19, 21].

2. Radion mode

It is possible to find a particular solution for A supported by the brane-bending scalar [39]

$$A(t,y) = F(y)\left[-\ddot{\xi}(t) + H^2\xi(t)\right], \tag{73}$$

where F obeys

$$F'' + 4\frac{N'}{N}F' + \frac{2H^2}{N^2}F = 0. \tag{74}$$

Comparing this bulk equation with Eq. (66), we see that the radion supports a discrete bulk mode with $m^2 = 2H^2$, and $F(y)$ is given by

$$F(y) = D_1 \coth\mu(y_h - |y|) + D_2\left[1 + \coth^2\mu(y_h - |y|)\right], \tag{75}$$

where D_1 and D_2 are integration constants. The boundary condition (68) requires $F'(0) = 1$ and hence gives one relation between D_1 and D_2. From the constraint equations we get the solutions for other metric perturbations:

$$
\begin{aligned}
B &= \frac{2F(y)}{a_o^2}\left[\dot{\xi} - H\xi\right], && (76)\\
\mathcal{R} &= F(y)H\left[-\dot{\xi} + H\xi\right], && (77)\\
E &= \frac{F(y)}{a_o^2}\xi\,. && (78)
\end{aligned}
$$

This mode is not normalizable in a single-brane model. In a static two-brane model, it becomes normalizable, and we need to consider the bending of the second brane, ξ_2. Then we replace ξ by $\xi - \xi_2$ in the final result and $\xi - \xi_2$ satisfies the same 4D wave equation as ξ. As expected, the radion in a two-brane system is described as a relative perturbation of the coordinate position of the branes $\xi - \xi_2$. The radion supports a discrete bulk mode with $m^2 = 2H^2$.

3. Projected Weyl tensor

The equivalence of the two descriptions of the radion can be shown by evaluating the projected Weyl tensor. The effective energy-momentum tensor of the projected Weyl tensor is simply related to normal derivatives of the GN metric perturbations [29],

$$\kappa_4^2\delta\rho_E = -\left(A'' + 2\frac{N'}{N}A'\right), \quad (79)$$

$$\kappa_4^2\delta q_E = -\frac{a_o^2}{2}\left(B'' + 2\frac{N'}{N}B'\right), \quad (80)$$

$$\kappa_4^2\delta\pi_E = E'' + 2\frac{N'}{N}E'. \quad (81)$$

The solution for ξ on large scales is

$$\xi = c_1 a_o \left[1 + \frac{1}{6}\left(\frac{k}{a_o H}\right)^2 + \frac{1}{24}\left(\frac{k}{a_o H}\right)^4\right] + \frac{c_2}{a_o^4}. \quad (82)$$

Then, from the solutions for the metric perturbations Eqs. (73), (76)–(78), we can evaluate the projected Weyl tensor,

$$\kappa_4^2\delta\rho_E = \frac{2\mu^2H^2}{\sinh^4\mu y_h}D_2\left[\frac{c_1}{3}a_o\left(\frac{k}{a_oH}\right)^4 + 15\frac{c_2}{a_o^4}\right], \quad (83)$$

$$\kappa_4^2\delta q_E = \frac{2\mu^2H}{\sinh^4\mu y_h}D_2\left[\frac{c_1}{3}a_o\left(\frac{k}{a_oH}\right)^2 + 5\frac{c_2}{a_o^4}\right], \quad (84)$$

$$\kappa_4^2\delta\pi_E = \frac{2\mu^2}{\sinh^4\mu y_h}D_2\left[\frac{c_1}{a_o} + \frac{c_2}{a_o^6}\right]. \quad (85)$$

These agree with the results obtained using the master variable, Eqs. (60)–(62).

V. SCALAR FIELD ON THE BRANE

The simplest dynamical model of inflation on the brane involves a scalar field confined to the brane, which obeys the standard 4D wave equation on the brane:

$$\Box\phi = \frac{dV}{d\phi}. \quad (86)$$

In the original computation [11] of the spectrum of scalar perturbations generated by such a slow-roll brane inflation scenario, it has been assumed that, since the scalar field in this scenario is intrinsically 4D, the usual formula for the quantum fluctuations of a 4D scalar field should apply, giving $\delta\phi \sim H/(2\pi)$ at Hubble crossing. This should be valid for linear perturbations of a massless scalar field in de Sitter spacetime where the perturbations in the energy-momentum tensor are only second-order in the field fluctuations. On scales much larger than the Hubble radius at the end of slow-roll inflation, one can then calculate the curvature perturbation on uniform-density hypersurfaces which should be conserved, so long as energy is conserved on the brane, for adiabatic perturbations [12, 13]. The only difference from the standard inflationary calculation of field fluctuations then comes from the fact that the background Hubble rate H is governed by the modified Friedmann equation [26].

However, one would like to check whether 5D effects could spoil this reasoning. In particular the inflaton perturbations are linked with the 5D metric perturbations at first-order in a slow-roll expansion via the junction conditions. In this section we investigate the nature of the scalar metric perturbations that are produced by field fluctuations on the brane.

A. Bulk scalar modes

In previous sections, we have seen that in a single-brane model, there are no light modes for a vacuum brane, and that in a two-brane model the only light mode (with $m^2 < \frac{9}{4}H^2$) is the radion mode. In this section we include matter perturbations on the brane and show, using the master variable to describe bulk metric perturbations, that the matter perturbations support an infinite ladder of normalizable modes.

For the matter, we consider a 4D inflaton scalar field ϕ with potential $V(\phi)$. Scalar field perturbations have vanishing anisotropic stress at linear order, $\delta\pi = 0$ in Eq. (11), and hence the boundary condition at the brane for the off-diagonal part of the extrinsic curvature in Eq. (14) requires the brane position to be unperturbed ($\xi = 0$) in the 5D longitudinal gauge. In this case the 5D longitudinal gauge coincides with the 4D longitudinal gauge on the brane. The remaining boundary conditions for the master variable Ω can then be written in the general form [40]

$$a_o\kappa_5^2\delta\rho = -\frac{k^2}{a^2}\left(\Omega' - \frac{a'}{a}\Omega\right) - 3\frac{\dot{a}}{a}\left(\dot{\Omega}' - \frac{n'}{n}\dot{\Omega}\right), \tag{87}$$

$$a_o\kappa_5^2\delta q = -\left(\dot{\Omega}' - \frac{n'}{n}\dot{\Omega}\right), \tag{88}$$

$$\begin{aligned} a_o\kappa_5^2\delta P &= \ddot{\Omega}' - \frac{a'}{a}\ddot{\Omega} + 2\frac{\dot{a}}{a}\left(\dot{\Omega}' - \frac{n'}{n}\dot{\Omega}\right) + \left\{4\frac{\dot{a}}{a}\left(\frac{a'}{a} - \frac{n'}{n}\right) + 2\left(\frac{\dot{a}}{a}\right)' - \left(\frac{\dot{n}}{n}\right)'\right\}\dot{\Omega} \\ &\quad - \frac{2}{3}\left(\frac{a'}{a} - \frac{n'}{n}\right)\frac{k^2}{a^2}\Omega + \mu^2\left(\frac{a'}{a} - \frac{n'}{n}\right)\Omega - \left(\frac{a'}{a} - \frac{n'}{n}\right)\left(2\frac{a'}{a} - \frac{n'}{n}\right)\Omega', \end{aligned} \tag{89}$$

where the brane matter perturbations on the left-hand sides will be expressed in terms of the scalar field perturbation $\delta\phi$ and of the induced metric perturbations in the 4D longitudinal gauge.

For $dV/d\phi \neq 0$, the brane is no longer strictly de Sitter, but in order to make the problem tractable, we impose two approximations. The first is to assume zeroth-order slow-roll for the background, which means that in practice we consider the background as a strict de Sitter brane configuration. The second simplification is to ignore the brane metric perturbation contributions to $\delta\rho$, δP. In the standard 4D calculation, this latter approximation is known to be valid in the *slow-roll limit* in the 4D longitudinal gauge. In the present case, this can be justified only in retrospect once we have done the simplified calculation.

With these two approximations, the junction conditions (87)–(89) reduce to

$$\left(\ddot{\Omega}' - \frac{N'}{N}\ddot{\Omega}\right) + 2H\left(\dot{\Omega}' - \frac{N'}{N}\dot{\Omega}\right) = \kappa_5^2 a_o\left[\dot{\phi}\dot{\delta\phi} - V'(\phi)\delta\phi\right], \tag{90}$$

$$\dot{\Omega}' - \frac{N'}{N}\dot{\Omega} = \kappa_5^2 a_o\dot{\phi}\delta\phi, \tag{91}$$

$$-3H\left(\dot{\Omega}' - \frac{N'}{N}\dot{\Omega}\right) - \frac{k^2}{a_o^2}\left(\Omega' - \frac{N'}{N}\Omega\right) = \kappa_5^2 a_o\left[\dot{\phi}\dot{\delta\phi} + V'(\phi)\delta\phi\right], \tag{92}$$

where the contributions from the induced metric perturbations on the right-hand sides are neglected.

Defining

$$\mathcal{F}(t) = \Omega' - \frac{N'}{N}\Omega, \tag{93}$$

at the brane, and combining the junction conditions, we get a single evolution equation,

$$\ddot{\mathcal{F}} - \left(H + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\mathcal{F}} + \frac{k^2}{a_o^2}\mathcal{F} = 0. \tag{94}$$

This gives the boundary condition for the time dependence of the master variable Ω. From Eq. (91) the scalar field fluctuation $\delta\phi$ is given in terms of $\mathcal{F}$ by

$$\kappa_5^2\delta\phi = \frac{\dot{\mathcal{F}}}{a_o\dot{\phi}}. \tag{95}$$

Then it can be verified that Eq. (94) is consistent with the equation of motion for $\delta\phi$,

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \frac{k^2}{a_o^2}\delta\phi + V''(\phi)\delta\phi = 0, \tag{96}$$

for an arbitrary $V(\phi)$, to lowest order (i.e., neglecting the metric perturbations).

Assuming that ϕ is slow-rolling, so that $|\ddot{\phi}/\dot{\phi}| \ll H$ in Eq. (94), the solution for $\mathcal{F}$ is

$$\mathcal{F}(\eta) = C_1\frac{\cos(-k\eta)}{-k\eta} + C_2\frac{\sin(-k\eta)}{-k\eta}. \tag{97}$$

This should be compared with the time evolution of Ω given by each of the mode functions α_m in Eq. (45). One might expect that the boundary condition can be satisfied by summing up mode functions only with positive m^2. However, it turns out that this is not possible, and negative-m^2 modes are unavoidable. We use the formulas for summation of Bessel functions,

$$\sum_{\ell=0}^{\infty}(-1)^\ell\left(2\ell+\frac{3}{2}\right)z^{-3/2}J_{2\ell+3/2}(z) = \sqrt{\frac{1}{2\pi}}\,\frac{\sin z}{z}, \tag{98}$$

$$\sum_{\ell=0}^{\infty}(-1)^\ell\left(2\ell+\frac{1}{2}\right)z^{-3/2}J_{2\ell+1/2}(z) = \sqrt{\frac{1}{2\pi}}\,\frac{\cos z}{z}\,. \tag{99}$$

These show that an infinite sum of mode functions

$$\alpha_m=(-k\eta)^{-3/2}J_\nu(-k\eta)\,,\quad \text{where } \nu^2=\frac{9}{4}-\frac{m^2}{H^2}\,, \tag{100}$$

can satisfy the boundary condition imposed on $\mathcal{F}$, where the spectrum of KK modes is given by

$$\frac{m^2}{H^2} = -2(2\ell-1)(\ell+1)\quad \text{for } C_1, \tag{101}$$

$$\frac{m^2}{H^2} = -2\ell(2\ell+3)\quad \text{for } C_2. \tag{102}$$

These modes include an infinite ladder of tachyonic modes with $m^2<0$. However, the boundary condition requires us to include only the decaying solution for these tachyonic modes. The dangerous growing mode solution is excluded once the junction condition is imposed. Thus there is no instability.

We should choose the solution in the y-direction so that the metric perturbations remain small as $y\to y_h$ and the mode is normalizable for a single brane. Unlike the case of the radion mode for a vacuum brane, the test scalar field on the brane allows us to choose only the normalizable modes. The solution for Ω in the bulk is (for $y\geq 0$)

$$\begin{aligned}\Omega(\eta,y) \;=\;& C_1\sqrt{2\pi}\sum_{\ell=0}^{\infty}(-1)^\ell\left(2\ell+\frac{1}{2}\right)\frac{\sinh\mu(y_h-y)\,Q_{2\ell}(\coth\mu(y_h-y))}{\mu\,Q^1_{2\ell}(\coth\mu y_h)}\,(-k\eta)^{-3/2}J_{2\ell+1/2}(-k\eta)\\ &+C_2\sqrt{2\pi}\sum_{\ell=0}^{\infty}(-1)^\ell\left(2\ell+\frac{3}{2}\right)\frac{\sinh\mu(y_h-y)\,Q_{2\ell+1}(\coth\mu(y_h-y))}{\mu\,Q^1_{2\ell+1}(\coth\mu y_h)}\,(-k\eta)^{-3/2}J_{2\ell+3/2}(-k\eta)\,,\end{aligned} \tag{103}$$

where Q_α is a Legendre polynomial of the second kind and Q^α_β is an associated Legendre function of the second kind. Using the asymptotic behavior of $Q_n(z)$,

$$Q_n(z)\to\sqrt{\pi}\,\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}\,(2z)^{-n-1}\quad (\text{for}\quad z\to\infty), \tag{104}$$

we can check that this solution is normalizable [see Eq. (49)]. Thus if there is a matter perturbation on the brane, normalizable discrete modes are supported in a single-brane model. In the following discussion, we concentrate on a single-brane model.

On large scales, $k\eta\to 0$, the Bessel function behaves as $J_\nu(-k\eta)\propto a_o^{-\nu}$, so that the mode with smallest ν, i.e. the mode with $m^2=2H^2$, gives the dominant contribution in the C_1 mode. Thus on large scales, the solution for Ω is given by the $m^2=2H^2$ mode,

$$\Omega_{m^2=2H^2}=C_1 a_o\,\mu(y_h-y)\sinh\mu(y_h-y). \tag{105}$$

Then we can determine the solution for metric perturbations,

$$\mathcal{R}=-A=\frac{1}{2}C_1\mu^2\cosh\mu y_h. \tag{106}$$

The scalar field perturbation is also given in terms of C_1 from Eq. (95),

$$\kappa_5^2\delta\phi=-\frac{H}{\dot\phi}C_1\mu\sinh\mu y_h\,. \tag{107}$$

The relation between scalar field and metric perturbations,

$$\mathcal{R} = -\frac{\dot{\phi}}{2H}\kappa_{4,\mathrm{eff}}^2\delta\phi \quad \text{where } \kappa_{4,\mathrm{eff}}^2 = \kappa_5^2\mu\left[1+\left(\frac{H}{\mu}\right)^2\right]^{1/2}, \tag{108}$$

is the same as the standard 4D result, except for the high-energy correction of the 4D Newton constant.

B. Metric backreaction

We now investigate the corrections to the evolution of scalar field fluctuations which come from the effect of the metric perturbations that the field fluctuations themselves induce on the brane.

We expand the scalar field perturbations in terms of slow-roll parameters,

$$\delta\phi = \delta\phi_0 + \delta\phi_1 + \ldots, \tag{109}$$

where the zeroth-order order solution corresponds to the solution to Eq. (96). The first-order equation can be derived from the scalar field equation of motion (86),

$$\ddot{\delta\phi}_1 + 3H\dot{\delta\phi}_1 + \frac{k^2}{a_o^2}\delta\phi_1 = -V''\delta\phi_0 - 3\dot{\phi}\dot{\mathcal{R}} + \dot{\phi}\dot{A} - 2V'A, \tag{110}$$

where we have included the scalar metric perturbations in the 4D longitudinal gauge induced by the zeroth-order field fluctuations, $\delta\phi_0$.

The perturbed effective Einstein equations (4) on the brane are given by

$$\frac{\kappa_{4,\mathrm{eff}}^2}{2}\left(\dot{\phi}\dot{\delta\phi}_0 + V'\delta\phi_0\right) + \frac{\kappa_4^2}{2}\delta\rho_E = 3H\dot{\mathcal{R}} - 3H^2A + \frac{k^2}{a_o^2}\mathcal{R}, \tag{111}$$

$$\frac{\kappa_{4,\mathrm{eff}}^2}{2}\left(\dot{\phi}\dot{\delta\phi}_0 - V'\delta\phi_0\right) + \frac{\kappa_4^2}{6}\delta\rho_E = -\ddot{\mathcal{R}} - 3H\dot{\mathcal{R}} + H\dot{A} + 3H^2A - \frac{1}{3}\frac{k^2}{a_o^2}(\mathcal{R}+A), \tag{112}$$

$$\kappa_4^2\delta\pi_E = -\frac{1}{a_o^2}(\mathcal{R}+A). \tag{113}$$

In order to evaluate the effect of metric perturbations, it is useful to use the Mukhanov-Sasaki variable,

$$\mathcal{Q} = \delta\phi - \frac{\dot{\phi}}{H}\mathcal{R}. \tag{114}$$

In terms of our slow-roll expansion, Eq. (109), we have $\mathcal{Q}_0 = \delta\phi_0$ and $\mathcal{Q}_1 = \delta\phi_1 - (\dot{\phi}/H)\mathcal{R}$. Then following [41], we use the effective Einstein equations to derive the equation for $\mathcal{Q}$,

$$\ddot{\mathcal{Q}}_1 + 3H\dot{\mathcal{Q}}_1 + \frac{k^2}{a_o^2}\mathcal{Q}_1 = -V''\mathcal{Q}_0 - 6\dot{H}\mathcal{Q}_0 + \mathcal{J}, \tag{115}$$

where

$$\mathcal{J} = -\frac{\kappa_4^2\dot{\phi}}{3H}\left(k^2\delta\pi_E + \delta\rho_E\right) \tag{116}$$

$$= -\frac{\dot{\phi}}{H}\frac{k^2}{6a_o^3}\left(\ddot{\Omega} - H\dot{\Omega} + \frac{k^2}{a_o^2}\Omega\right). \tag{117}$$

Equation (115) is the same as the standard 4D equation except for the term $\mathcal{J}$, which describes the corrections from the 5D bulk perturbations. We can evaluate the right-hand side using Eq. (103) for Ω on the brane.

In order to evaluate $\mathcal{J}$, we need to handle the infinite sum of modes. However, on large scales, it is possible to use the $m^2 = 2H^2$ mode, Eq. (105), to rewrite $\mathcal{J}$ in terms of $\mathcal{Q}_0$ as

$$\mathcal{J} = \frac{\kappa_5^2\mu\dot{\phi}^2}{9H^2}(\mu y_h)\frac{k^4}{a_o^4\mu^2}\mathcal{Q}_0. \tag{118}$$

Here we should note that the leading-order time behaviour of the $m^2 = 2H^2$ mode on large scales, Eq. (105), disappears in $\mathcal{J}$, so we need to take into account the next order solution. Then the $m^2 = -4H^2$ mode gives a comparable contribution, but it gives qualitatively the same contribution as Eq. (118), so we neglect it. To compare this correction with the standard correction term $-6\dot{H}\mathcal{Q}_0$, we use the background equation

$$\dot{H} = -\frac{1}{2}\kappa_{4,\mathrm{eff}}^2\dot{\phi}^2, \tag{119}$$

and $\mu y_h = \sinh^{-1}(\mu/H)$, to evaluate the ratio of these two corrections:

$$\frac{\mathcal{J}}{\dot{H}\mathcal{Q}_0} \sim \frac{k^4}{a_o^4\mu^2 H^2}\sinh^{-1}\frac{\mu}{H}\left[1+\left(\frac{H}{\mu}\right)^2\right]^{-1/2}. \tag{120}$$

At low energies, $H/\mu \ll 1$, this ratio is very small on super-Hubble scales. Even at high energies, $H/\mu \gg 1$, the ratio is suppressed on super-Hubble scales,

$$\frac{\mathcal{J}}{\dot{H}\mathcal{Q}_0} \sim \frac{k^4}{a_o^4 H^4}. \tag{121}$$

Thus we conclude that the corrections that come from bulk metric perturbations are always small compared with the corrections to the de Sitter geometry, i.e., $\dot{H}/H^2$ corrections, on super-Hubble scales.

But on sub-Hubble scales, the situation changes significantly. The correction $\mathcal{J}$ becomes significant, and an infinite number of modes in Ω should be taken into account, because all modes become comparable. This indicates that the quantum theory of the Mukhanov-Sasaki variable on small scales is quite different from the standard 4D results that take into account slow-roll corrections. This is in line with the expectation that high-energy particles on the brane can couple to massive bulk gravitons and will be sensitive to the higher-dimensional geometry.

VI. CONCLUSIONS

In this paper, we investigated bulk scalar metric perturbations about a de Sitter brane. In the absence of matter perturbations, we have confirmed that there are no normalizable light modes (with $m^2 < \frac{9}{4}H^2$) for a single brane, but in the presence of a second brane there is a normalizable "radion" mode.

In the 5D longitudinal gauge the coordinate positions of vacuum branes are unperturbed and the radion mode appears as a discrete bulk mode with $m^2 = 2H^2$. In a Gaussian normal gauge, with transverse-tracefree condition, the radion appears as a relative perturbation of the coordinate position of the two branes. This "brane-bending" mode obeys a canonical 4D wave equation with a tachyonic effective mass, $m^2 = -4H^2$, as reported in previous analyses [19, 21, 23]. The brane-bending supports a discrete bulk mode in the GN gauge [39], which again has $m^2 = 2H^2$.

We have shown the equivalence of the descriptions in the two different gauges by evaluating the projected Weyl tensor on the brane. The radion appears as an "instability" for two de Sitter branes [19, 21, 23], but the effect of the radion "instability" on the brane measured by the projected Weyl tensor becomes small on large scales [24].

We then considered the bulk metric perturbations excited by scalar field perturbations on a *single* de Sitter brane. The $m^2 = 2H^2$ mode together with the zero-mode and an infinite ladder of discrete tachyonic modes in the 5D longitudinal gauge, become normalizable. Nonetheless the boundary condition requires the self-consistent 4-dimensional evolution of scalar field perturbations on the brane, so that the dangerous growing modes are not allowed. These normalizable discrete modes introduce corrections to the scalar perturbations computed in an effectively 4-dimensional approach. On super-Hubble scales, the $m^2 = 2H^2$ mode is dominant and the correction is smaller than the slow-roll corrections to the de Sitter background. Thus we have verified that there exists a scalar curvature perturbation ζ defined by

$$\zeta = \frac{H}{\dot{\phi}}\mathcal{Q}\,, \tag{122}$$

that is constant on large scales [12, 13], even including lowest-order metric backreaction at high energies. However, on short scales, all the infinite ladder of discrete tachyonic modes become comparable. Thus the effect of backreaction could be large. This is consistent with the expectation that high-energy particles on the brane can probe the higher-dimensional gravity.

In 4-dimensional gravity we can incorporate first-order metric perturbations along with the field fluctuations in a gauge-invariant combination, the Sasaki-Mukhanov variable, Eq. (114), which on small scales then obeys the wave equation for a single free field in flat spacetime. In 5-dimensional gravity the short-wavelength field fluctuations on the brane are coupled to an infinite ladder of bulk metric perturbations when we try to reduce it to an effective 4-dimensional theory.

A possible consequence could be the damping of the amplitude of quantum field fluctuations on small scales during inflation on the brane, due to the excitation of the infinite ladder of discrete modes. To quantify this effect, we need to be able to handle the infinite summation of modes. This requires further investigation and we hope to report results in a separate publication.

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