The Speciality Index as invariant indicator in the BKL Mixmaster Dynamics.

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Abstract. The speciality index, which has been mainly used in Numerical Relativity for studying gravitational waves phenomena as an indicator of the special or non-special Petrov type character of a spacetime, is applied here in the context of Mixmaster cosmology, using the Belinski-Khalatnikov-Lifshitz map. Possible applications for the associated chaotic dynamics are discussed.

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1. Introduction

The speciality index (SI) [1] is an invariant and dimensionless indicator of the special or non-special Petrov algebraic character of a given spacetime. This quantity has been mostly used in Numerical Relativity, for instance to study a black hole which is radiating gravitational waves, or black holes merging [1, 2]. In this Letter we use the SI in the context of Cosmology, to study the Belinski-Khalatnikov-Lifshitz (BKL) map [3, 4, 5], defining in this way an invariant indicator of the Petrov type "fluctuations" close to the singularity of the Mixmaster solution. Possible applications in the context of the associated chaotic dynamics are discussed.

2. Petrov classification: speciality index.

The algebraic properties of curvature are a very useful tool to obtain powerful insights into the character of a given spacetime metric. In particular those of the Weyl tensor, which is the trace free part of the Riemann tensor and in vacuum coincides with it, play a central role in Einstein's General Relativity theory. The Penrose-Debever equation $l_{[u}C_{p]qr[s}l_{t]}l^q l^r = 0$ states the existence of four distinct null eigenvectors for the most general spacetime: these are known as "*principal null directions*" (PND) [6].

If all the PND result distinct one has the algebraically general case (Type I). When some of them coincide, this gives rise to the algebraically special case summarized as follows: Type II (one pair of PND coincides), Type D (two pairs of PND coincide), Type III (three PND coincide), Type N(all four PND coincide) and Type O (no PND, because of conformal flatness).

Defining the complex tensor $\tilde{C}_{abcd} = C_{abcd} - i^*C_{abcd}$, one can introduce the two complex curvature invariants

$$I = \frac{1}{32}\tilde{C}_{abcd}\tilde{C}^{abcd}\,, \qquad J = \frac{1}{384}\tilde{C}_{abcd}\tilde{C}^{cd}{}_{mn}\tilde{C}^{mnab}\,. \tag{2.1}$$

These can be used to define the speciality index [1]

$$\mathcal{S} = 27J^2/I^3 \tag{2.2}$$

which marks, in an invariant way, the transition from certain algebraically special solutions ($\mathcal{S} = 1$) and the general Petrov type I ($\mathcal{S} \neq 1$) [6]. We point out that for some spacetimes this quantity might be not well defined [2] because of the possible vanishing of I and/or J, although for the vacuum Kasner spacetime [7], as shown in the following, this is not the case. This solution of Einstein equations is given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^{2p_1}\mathrm{d}x^2 - t^{2p_2}\mathrm{d}y^2 - t^{2p_3}\mathrm{d}z^2\,, \tag{2.3}$$

where

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \tag{2.4}$$

and the indices can take values in the interval $[-\frac{1}{3}, 1]$ only. The Kasner metric admits two special subcases when two of the p_i indices are equal: it then follows from (2.4) that either $p_1 = p_2 = 0$, $p_3 = 1$ (and permutations) and the spacetime is flat in this case, or $p_1 = -1/3$, $p_2 = p_3 = 2/3$ (and permutations) and one has the Kasner locally rotational symmetric type D solution, with a spindle-like singularity [6, 8]. For other choices of the parameter the Kasner spacetime is of Petrov type I. A simple calculation, using the constraints listed above, shows [9] that the SI for the vacuum Kasner spacetime is

$$\mathcal{S} = -\frac{27}{4}p_1p_2p_3 \equiv \frac{27}{4}p_3^2(1 - p_3)\,. \tag{2.5}$$

By direct inspection this quantity ranges from the value $\mathcal{S} = 1$ (the type D case) to $\mathcal{S} = 0$ (the flat case) and is well defined with continuity for any value of the parameter p_3.

3. Applications to BKL dynamics

The class of algebraically general Kasner spacetimes contains only algebraically special subcases of either type D and type O, and its SI has the simple time independent form (2.5). This allows us to use the latter to describe the Mixmaster dynamics as approximated by BKL Kasner epochs [3, 4, 5, 7]. By using the standard parametrization:

$$p_1 = -\frac{u}{u^2+u+1}\,, \quad p_2 = \frac{u+1}{u^2+u+1}\,, \quad p_3 = \frac{u(u+1)}{u^2+u+1}\,, \qquad u \in [1, +\infty) \tag{3.1}$$

satisfying the ordering

$$-\frac{1}{3} \le p_1 \le 0, \quad 0 \le p_2 \le \frac{2}{3}, \quad \frac{2}{3} \le p_3 \le 1\,, \tag{3.2}$$

with the sequence of Kasner epochs given by the rule (Gauss map):

$$u_{n+1} = u_n - 1 \quad \text{if} \quad 2 \leq u_n < \infty\,, \tag{3.3}$$

$$u_{n+1} = \frac{1}{u_n - 1} \quad \text{if} \quad 1 \leq u_n \leq 2\,, \tag{3.4}$$

we obtain from (2.5) an "n"-dependent speciality index labelled by each epoch

$$\mathcal{S}_n = \frac{27}{4}\,\frac{u_n^2(u_n+1)^2}{(u_n^2+u_n+1)^3}\,. \tag{3.5}$$

Although irrational initial values u_1 only should be considered [7], in numerical simulations this requirement results clearly an abstraction, because truncated rational numbers only can be handled. Using as an example the sequence given by Berger [10] which starts "close" to the flat spacetime configuration with $u_1 = 7.2328$, we get the speciality index evolution shown in Figure (1).

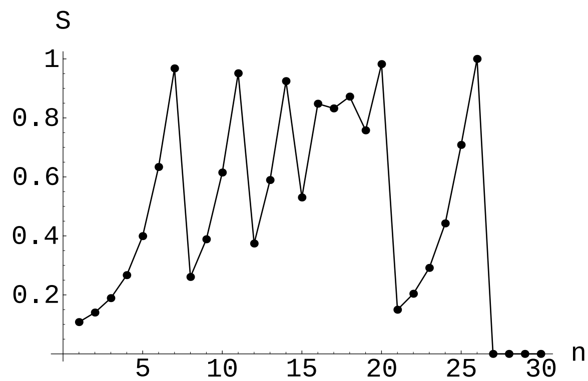

Figure 1. Speciality Index for the vacuum BKL map with $u_1 = 7.2328$ in terms of epochs n. Interpolation for visual clarification only. The flattening on the right is due to the map sensibility.

We point out that as soon as the system gets close to the type D configuration, it moves rapidly towards to the type O region, and then gently evolves until it gets close to the type D case again, and so on. The flattening of the trend in Figure (1) is due to the well known sensibility of the Gauss map which can generate very long periods of monotonic behavior until oscillations start again.

Pictorially, using the representation of the Mixmaster as the motion on a contracting triangular potential well in the time direction moving away from the initial singularity, the motion close to the type D case corresponds to an almost perpendicular bounce on the middle of the side of the triangle, with the incoming free-motion phase before the bounce corresponding to the flat space Kasner indices and the outgoing free-motion phase after the bounce corresponding to the Kasner indices.

The bounce is equivalent to a transition from the one set of Kasner indices to the other, for the asymptotic behavior away from the straight wall but still far from its time spent at the opposite end in the "channel" corner of the potential where the exact Taub solution originates and then returns, and space curvature effects remain important since the system point is always close to the potential walls of the channel.

We point out that the BKL parametrization has cancelled any information concerning the specific direction in which the motion is happening, leaving in our case an invariant dynamics in an "abstract Petrov space".

4. Conclusions

We have introduced the speciality index in Mixmaster BKL dynamics, in analogy with its use in the numerical treatment of gravitational wave sources. Because of its gauge invariant nature, time independence and adimensionality, the Kasner SI and its derived BKL version can be used in the sophisticated numerical simulations of Mixmaster, to get useful invariant information concerning chaos [11, 12]. In particular it would be useful to study the probability associated on the various regions of the segment $[0, 1]$ in which the SI ranges during its evolution. As pointed out in a recent review on the subject [13] (see also references therein), "a remaining open question is how closely an actual Mixmaster evolution is approximated by a single BKL sequence." A direct comparison of the BKL SI (which approximates the Mixmaster dynamics) with the corresponding exact Bianchi IX one will be the appropriate way for approaching the problem numerically, to give an answer to this question.

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