

Are braneworlds born isotropic?

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It has recently been suggested that an isotropic singularity may be a generic feature of brane cosmologies, even in the inhomogeneous case. Using the covariant and gauge-invariant approach we present a detailed analysis of linear perturbations of the isotropic model \mathcal{F}_b which is a past attractor in the phase space of homogeneous Bianchi models on the brane. We find that for matter with an equation of state parameter $\gamma > 1$, the dimensionless variables representing generic anisotropic and inhomogeneous perturbations decay as $t \rightarrow 0$, showing that the model \mathcal{F}_b is asymptotically stable in the past. We conclude that brane universes are born with isotropy naturally built-in, contrary to standard cosmology. The observed large-scale homogeneity and isotropy of the universe can therefore be explained as a consequence of the initial conditions if the brane-world paradigm represents a description of the very early universe.

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INTRODUCTION

A classic problem of cosmology is finding ways to explain the very high degree of isotropy observed in the Cosmic Microwave Background (CMB). In general relativity, where isotropy is a special rather than generic feature of cosmological models, we need a dynamical mechanism which is able to produce isotropy. One of the most efficient mechanisms for isotropising the universe is Inflation [30] but it requires sufficiently homogeneous initial data in order to start at all [2]. Although one can adopt the view that it is sufficient to have one such homogeneous enough patch in an otherwise inhomogeneous initial universe to explain what we observe, this seems somehow unsatisfactory. In our view the isotropy problem remains open to debate in standard cosmology.

Over the past few years the brane-world paradigm has received considerable attention as a possible candidate for string inspired cosmology (see [3] for a recent review). In this scenario the observable universe is a 4-dimensional slice, *the brane*, embedded in a higher dimensional spacetime called *the bulk*. Here we consider the formulation developed in [4] in order to generalise a previous model by Randall and Sundrum [5], where the bulk is 5-dimensional and contains only a cosmological constant, assumed to be negative.

In a series of recent papers a number of authors [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] have presented a detailed description of the dynamics of homogeneous and anisotropic brane-worlds, finding a remarkable result: unlike standard general relativity, where the generic cosmological singularity is anisotropic, the past attractor for homogeneous anisotropic models in the brane is a simple Robertson-Walker model \mathcal{F}_b . More significantly, in [9, 10] it was shown that this result holds true in

Bianchi IX models, as well as for some simple inhomogeneous models. Since the Belinski-Lifshitz-Khalatnikov (BLK) conjecture [16] suggests that the Bianchi IX behaviour in the vicinity of the singularity is general, i.e. that the approach to the cosmological singularity in a generic inhomogeneous universe model should locally be the same as in Bianchi IX, it has been suggested that the isotropic singularity could be a generic feature of brane cosmological models.

If this conjecture [9, 10] could be proved correct, brane cosmology would have the very attractive feature of having isotropy built in. Inflation in this context would still be the most efficient way of producing the fluctuations seen in the CMB, but there would be no need of special initial conditions for it to start [17]. Also the Penrose conjecture [18] on gravitational entropy and an initially vanishing Weyl tensor would be satisfied in these models (c.f. [19, 20]).

In this paper we prove that this conjecture is true, within a perturbative approach and in the large-scale and high energy regime, as justified below. We arrive at this result through a detailed analysis of generic linear inhomogeneous and anisotropic perturbations [21, 22, 23, 24] of this past attractor \mathcal{F}_b . This is done by using the full set of linear 1+3 covariant *propagation* and *constraint* equations for this background, which describe the kinematics of the fluid flow and the dynamics of the gravitational field (see eqns. 87-100 in [25]). These equations are then split into their *scalar*, *vector* and *tensor* contributions giving three sets of linear propagation and constraint equations which govern the complete perturbation dynamics.

From a dynamical systems point of view the past attractor \mathcal{F}_b for brane homogeneous cosmological models found in [9, 10] is a fixed point in the phase space of these

models. This phase space may be thought of as an invariant submanifold within an higher dimensional phase space for more general inhomogeneous models. Our analysis can be thought of as an exploration of the neighbourhood of \mathcal{F}_b out of the invariant submanifold explored in [9, 10].

We restrict our analysis to large-scales, at a time when physical scales of perturbations are much larger than the Hubble radius, $\lambda \gg H^{-1}$ (equivalent to neglecting Laplacian terms in the evolution equations). This may at first glance seem restrictive, but it is not the case for the non-inflationary perfect fluid models that are relevant to our discussion. Indeed, it is well known that any wavelength $\lambda < H^{-1}$ at a given time becomes much larger than H^{-1} at early enough times. Because of this crucial property of perturbations for non-inflationary models our analysis is completely general, i.e. valid for any λ as $t \rightarrow 0$.

In what follows we restrict our analysis to the case of vanishing background dark radiation ($\mathcal{U} = 0$).

DIMENSIONLESS VARIABLES AND HARMONICS

We define dimensionless expansion normalised quantities for the shear σ_{ab} , the vorticity ω_a , the electric E_{ab} and magnetic H_{ab} parts of the Weyl tensor [23]:

$$\Sigma_{ab} = \frac{\sigma_{ab}}{H}, \quad W_a = \frac{\omega_a}{H}, \quad \mathcal{E}_{ab} = \frac{E_{ab}}{H^2}, \quad \mathcal{H}_{ab} = \frac{H_{ab}}{H^2}, \quad (1)$$

where H is the Hubble parameter $H = \dot{a}/a$ and a is the scale factor.

It turns out to be convenient to use the dimensionless variable

$$W_a^* = a\omega_a \quad (2)$$

to characterise the vorticity of the fluid flow. Here $\omega_a = \eta_{abc}\omega^{bc}$ is the usual vorticity vector.

The appropriate dimensionless density and expansion gradients which describe the *scalar* and *vector* parts of density perturbations are given by (see [25] for details of definitions)

$$\Delta_a = \frac{a}{\rho} D_a \rho, \quad Z_a^* = \frac{3a}{H} D_a H, \quad (3)$$

and for the brane-world contributions we define the following dimensionless gradients describing inhomogeneity in the non-local quantities

$$U_a = \frac{\kappa^2 \rho}{H^2} U_a, \quad Q_a^* = \frac{\kappa^2 a \rho}{H} Q_a, \quad (4)$$

where U_a and Q_a are defined in equation (27) of [26].

When undertaking a complete analysis of the perturbation dynamics it turns out useful to define a set of auxiliary variables corresponding to the curls of the standard

quantities defined above:

$$\bar{W}_a^* = \frac{1}{H} \text{curl} W_a^*, \quad \bar{\Sigma}_{ab} = \frac{1}{H} \text{curl} \Sigma_{ab}, \quad (5)$$

$$\bar{\mathcal{E}}_{ab} = \frac{1}{H} \text{curl} \mathcal{E}_{ab}, \quad \bar{\mathcal{H}}_{ab} = \frac{1}{H} \mathcal{H}_{ab}, \quad (6)$$

$$\bar{Q}_a^* = \frac{1}{H} \text{curl} Q_a^*. \quad (7)$$

Finally, it is useful to use the dimensionless time derivative $d/d\tau = d/d(\ln a)$ (denoted by a prime) to analyse the past evolution of the perturbation dynamics.

We use the harmonics defined in [22] to expand the above tensors X_a, X_{ab} in terms of scalar (S), vector (V) and tensor (T) harmonics Q . This yields a covariant and gauge invariant splitting of the evolution and constraint equations for the above quantities into three sets of evolution and constraint equations for scalar, vector and tensor modes respectively:

$$X = X^S Q^S \quad (8)$$

$$X_a = k^{-1} X^S Q_a^S + X^V Q_a^V \quad (9)$$

$$X_{ab} = k^{-2} X^S Q_{ab}^S + k^{-1} X^V Q_{ab}^V + X^T Q_{ab}^T. \quad (10)$$

PERTURBATION EQUATIONS AND THEIR SOLUTIONS

We begin by giving the *evolution* equations for *scalar perturbations* (suppressing the label S):

$$\Sigma' = (3\gamma - 2)\Sigma - \mathcal{E}, \quad (11)$$

$$\mathcal{E}' = (6\gamma - 3)\mathcal{E} - 3\gamma\Sigma, \quad (12)$$

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z^*, \quad (13)$$

$$Z^{*'} = (3\gamma - 2)Z^* - 3(3\gamma + 1)\Delta - U^*, \quad (14)$$

$$Q^{*'} = (3\gamma - 3)Q^* - \frac{1}{3}U^* - 6\gamma\Delta, \quad (15)$$

$$U^{*'} = (6\gamma - 4)U^*. \quad (16)$$

The *scalar* constraint equations are:

$$2\Sigma = 2Z^* - 3Q^*, \quad (17)$$

$$2\mathcal{E} = 6\Delta - 3Q^* + U^*. \quad (18)$$

Vector perturbations on large-scale are described by the following *evolution* equations for the basic variables

$$\Sigma' = (3\gamma - 2)\Sigma - \mathcal{E} - 2(\gamma - 1)\bar{W}^*, \quad (19)$$

$$\mathcal{E}' = (6\gamma - 3)\mathcal{E} - 3\gamma\Sigma + 6\gamma\bar{W}^*, \quad (20)$$

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z^*, \quad (21)$$

$$Z^{*'} = (3\gamma - 2)Z^* - (9\gamma + 3)\Delta - U^* - (6\gamma - 6)\bar{W}^* \quad (22)$$

$$Q^{*'} = (3\gamma - 3)Q^* - \frac{1}{3}U^* - 6\gamma\Delta, \quad (23)$$

$$U^{*'} = (6\gamma - 4)U^*, \quad (24)$$

$$W^{*'} = (3\gamma - 4)W^*, \quad (25)$$

$$\mathcal{H}' = (6\gamma - 3)\mathcal{H} - 12\gamma W^* + \bar{Q}^*, \quad (26)$$

TABLE I: Large-scale behaviour or geometric and kinematic quantities in the high energy limit. Here $p = 3\gamma - 3$, $q = 6\gamma - 2$, $r = -3$, $s = 6\gamma - 4$, $u = 6\gamma - 5$, $v = 6\gamma - 3$

Quantity	Scalar contribution	Vector Contribution	Tensor Contribution
Σ	$-\frac{3}{2}Q_0^*a^p, -\frac{6\gamma+1}{\gamma(3\gamma+1)}\Delta_1a^q$	$-2Q_0^*a^p, -\frac{4}{3\gamma}\frac{1+6\gamma}{1+3\gamma}\Delta_1a^q$	Σ_0a^p, Σ_1a^q
\mathcal{E}	$-\frac{3}{2}Q_0^*a^p, \frac{3(6\gamma+1)}{3\gamma+1}\Delta_1a^q$	$-2Q_0^*a^p, \frac{4(1+6\gamma)}{1+3\gamma}a^q$	$\Sigma_0a^p, -3\gamma\Sigma_1a^q$
Δ	$\Delta_1a^q, \frac{1}{2}Q_1^*a^r, -\frac{\gamma}{2(6\gamma-1)}U_0^*a^s$	$\Delta_1a^q, \frac{1}{2}Q_1^*a^r, -\frac{\gamma}{2}U_0^*a^s$	0
Z^*	$-\frac{3\gamma+1}{\gamma}\Delta_1a^q, \frac{3}{2}Q_1^*a^r, \frac{3\gamma-1}{2(6\gamma-1)}U_0^*a^s$	$-\frac{3\gamma+1}{\gamma}\Delta_1a^q, \frac{3}{2}Q_1^*a^r, \frac{3\gamma-1}{2(6\gamma-1)}U_0^*a^s$	0
Q^*	$Q_0^*a^p, -\frac{6\gamma}{3\gamma+1}\Delta_1a^q, Q_1^*a^r, \frac{3\gamma-1}{3(6\gamma+1)}U_0^*a^s$	$Q_0^*a^p, -\frac{6\gamma}{3\gamma+1}\Delta_1a^q, Q_1^*a^r, \frac{1}{3}\frac{3\gamma-1}{6\gamma-1}U_0^*a^s$	0
U^*	$U_0^*a^s$	$U_0^*a^s$	0
W	0	W_0a^u	0
\mathcal{H}	0	\mathcal{H}_0a^s	$\mathcal{H}_0a^s, \mathcal{H}_1a^{\frac{3}{2}q}$
\bar{W}^*	0	0	0
\bar{Q}^*	0	$-\mathcal{H}_0a^s, 12\gamma W_0^*a^u$	0
$\bar{\Sigma}$	0	\mathcal{H}_0a^s	$\mathcal{H}_0a^s, \mathcal{H}_1a^{\frac{3}{2}q}$
$\bar{\mathcal{E}}$	0	\mathcal{H}_0a^s	$\mathcal{H}_0a^s, -3\gamma\mathcal{H}_1a^{\frac{3}{2}q}$
$\bar{\mathcal{H}}$	0	0	0

and their curls

$$\bar{W}^{\prime} = (6\gamma - 5)\bar{W}^*, \quad (27)$$

$$\bar{Q}^{\prime} = (6\gamma - 4)\bar{Q}^* - 36\gamma^2 W^*, \quad (28)$$

$$\bar{\mathcal{H}}^{\prime} = (9\gamma - 4)\bar{\mathcal{H}} - 6\gamma\bar{W}^* \quad (29)$$

$$\bar{\mathcal{E}}^{\prime} = (9\gamma - 4)\bar{\mathcal{E}} - 3\gamma\bar{\Sigma}, \quad (30)$$

$$\bar{\Sigma}^{\prime} = (6\gamma - 3)\bar{\Sigma} - \bar{\mathcal{E}}. \quad (31)$$

The *constraints* for the basic variables are

$$3\Sigma = 4Z^* - 6Q^* + 6\bar{W}^*, \quad (32)$$

$$3\mathcal{E} = 12\Delta - 6Q^* + 2U^*, \quad (33)$$

$$\mathcal{H} = 12\gamma W^* - \bar{Q}^*, \quad (34)$$

$$\bar{\Sigma} = \mathcal{H}, \quad (35)$$

and the curl constraints are

$$\bar{\mathcal{H}} = 6\gamma\bar{W}^*, \quad (36)$$

$$\bar{\Sigma} = 12\gamma W^* - \bar{Q}^*, \quad (37)$$

$$\bar{\mathcal{E}} = 12\gamma W^* - \bar{Q}^*. \quad (38)$$

Finally, the large-scale evolution of *tensor perturbations* are governed by propagation equations for the tensor contributions of the shear Σ_{ab} , the electric \mathcal{E}_{ab} and magnetic \mathcal{H}_{ab} parts of the Weyl tensor:

$$\Sigma^{\prime} = (3\gamma - 2)\Sigma - \mathcal{E}, \quad (39)$$

$$\mathcal{E}^{\prime} = (6\gamma - 3)\mathcal{E} - 3\gamma\Sigma + \bar{\mathcal{H}}, \quad (40)$$

$$\mathcal{H}^{\prime} = (6\gamma - 3)\mathcal{H} - \bar{\mathcal{E}}, \quad (41)$$

together with propagation equations for their curls

$$\bar{\Sigma}^{\prime} = (6\gamma - 3)\bar{\Sigma} - \bar{\mathcal{E}}, \quad (42)$$

$$\bar{\mathcal{E}}^{\prime} = (9\gamma - 4)\bar{\mathcal{E}} - 3\gamma\bar{\Sigma}, \quad (43)$$

$$\bar{\mathcal{H}}^{\prime} = (9\gamma - 4)\bar{\mathcal{H}}. \quad (44)$$

The only tensorial contribution to the constraints is

$$\bar{\Sigma} = \mathcal{H}. \quad (45)$$

Solutions to these three sets of equations are presented in Table I. On close inspection of the exponents we conclude that \mathcal{F}_b is stable in the past (as $t \rightarrow 0$) to generic inhomogeneous and anisotropic perturbations provided the matter is described by a non-inflationary perfect fluid with γ -law equation of state parameter satisfying $\gamma > 1$ and we use the large-scale approximation ($\lambda \gg H^{-1}$). In particular it can be easily seen that the expansion normalised shear vanishes as $a \rightarrow 0$, signalling isotropization.

DISCUSSION OF RESULTS

We have considered here only the case of vanishing background Weyl energy density, $\mathcal{U} = 0$. This assumption considerably simplifies the analysis, but it is expected that our results will remain true for $\mathcal{U} \neq 0$. Indeed, when $\mathcal{U} \neq 0$ \mathcal{F}_b still remains a past attractor of the isotropic models. In other words, our analysis is restricted to the invariant submanifold $\mathcal{U} = 0$ of the larger phase space with $\mathcal{U} \neq 0$, but this submanifold is asymptotically stable against $\mathcal{U} \neq 0$ perturbations. A more complete analysis including this case will be the subject of a future investigation.

In a related paper the dynamics of a class of *spatially inhomogeneous* G_2 cosmological models in the brane-world scenario has been studied [27]. Area expansion normalized scale-invariant dependent variables, the timelike area gauge and an effective logarithmic time t were employed, and the initial singularity occurs for $t \rightarrow -\infty$. The resulting governing system of evolution equations of the spatially inhomogeneous G_2 brane cosmological

models can then be written as a constrained system of autonomous first-order partial differential equations in two independent variables. The local dynamical behaviour of this class of spatially inhomogeneous models close to the singularity was then studied *numerically*. It was found that the area expansion rate increases without bound (and hence the Hubble rate $H \rightarrow \infty$ as $t \rightarrow -\infty$, so that there always exists an initial singularity). For $\gamma > 4/3$, the numerics indicate isotropization towards \mathcal{F}_b as $t \rightarrow -\infty$ for *all* initial conditions. In the case of radiation ($\gamma = 4/3$), the models were still found to isotropize as $t \rightarrow -\infty$, albeit slowly. From the numerical analysis we find that there is an initial isotropic singularity in all of these G_2 spatially inhomogeneous brane cosmologies for a range of parameter values which include the physically important cases of radiation and a scalar field source. The numerical results are supported by a qualitative dynamical analysis and a detailed calculation of the past asymptotic decay rates [27].

Finally, we note that this result corrects an earlier paper [28] in which two of the authors claimed the contrary to be true (i.e., brane-worlds do not isotropize). The claim in [28] was based on the existence of the decaying mode $\propto a^{-3}$ (growing in the past) in the solutions for Δ and Q^* which when substituted into the shear (17) and $divE$ constraints (18) gave the same mode in the expansion normalised shear Σ and electric part of the Weyl tensor \mathcal{E} . This resulted from using an incorrect propagation equation for the non-local energy flux in [26] (eqn. 32 in this paper) which gave different solutions for Q^* [31]. The error in [26] results in a wrong decaying mode that is totally irrelevant for the future evolution of perturbations studied in that paper, but changes dramatically their behaviour as $t \rightarrow 0$. It is easy to see that when the corrected solutions (in Table 1) are substituted into (17) and (18) the a^{-3} mode actually cancels out in Σ and \mathcal{E} . Note also that this mode does not appear in the solution for the magnetic part of the Weyl tensor \mathcal{H} .

Since Σ , \mathcal{E} and \mathcal{H} give a better description of the geometry, their past evolution represents the true behaviour of anisotropies close to the initial singularity, and not Δ . In fact one can argue that the existence of the a^{-3} mode in Δ (defined in equation (3)) results from it not being the most appropriate measure of inhomogeneity at high energies since the dominant background energy density is not ρ but $\rho^{tot} \propto \rho^2$ (see equation (11) in [29]). More precisely, if we define Δ^{HE} (High Energy) normalising with ρ^{tot} instead of ρ then $\Delta^{HE} \propto \Delta/\rho$ and the decaying mode in Δ becomes a mode $\propto a^{-3+3\gamma}$. Since $-3+3\gamma > 0$ for $\gamma > 1$, we remove the decaying mode in the density inhomogeneity.

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jecture, i.e. classical perturbations in inflationary models with a scalar field or cosmological constant are swept out, as well as (local) proofs for homogeneous and inhomogeneous models (see e.g. [1] and references therein).

[31] The authors of [26] agree with our findings; R. Maartens, private communication.