arXiv:gr-qc/0207048v1 11 Jul 2002

SECOND ORDER PERTURBATIONS OF FLAT DUST FLRW UNIVERSES WITH A COSMOLOGICAL CONSTANT

FILIPE C. MENA*

Departamento de Matemática, Universidade do Minho,
Campus de Gualtar, 4710 Braga, Portugal

REZA TAVAKOL

Astronomy Unit, School of Mathematical Sciences, Queen Mary,
University of London, Mile End Road, London E1 4NS, U.K.

MARCO BRUNI

Institute of Cosmology and Gravitation, Mercantile House,
University of Portsmouth, Portsmouth PO1 2EG, U.K.

We summarize recent results concerning the evolution of second order perturbations in flat dust irrotational FLRW models with $\Lambda \neq 0$. We show that asymptotically these perturbations tend to constants in time, in agreement with the cosmic no-hair conjecture. We solve numerically the second order scalar perturbation equation, and very briefly discuss its all time behaviour and some possible implications for the structure formation.

Keywords: Mathematical cosmology; non-linear perturbations; cosmic no-hair conjecture; structure formation.

1. Introduction

First order perturbations of Friedmann-Lemaître-Robertson-Walker (FLRW) models fail to account for a number important relativistic properties that arise when non-linearities are taken into account. An important example of this is the so called mode coupling. Even if one considers purely scalar perturbations, a comoving scale k at second order is sourced by any other scale k'. In addition, the scalar, vector and tensor perturbations which are decoupled to the first order, become coupled once nonlinear perturbations are taken into account. Thus for example taking into account such couplings at second order results in the initial pure scalar metric perturbations to necessarily generate second order vector and tensor perturbation modes (see e.g.[1]). In view of this, it is important to develop higher order perturbation schemes which go beyond the first order and can thus account for the physical

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*email: fmena@math.uminho.pt

effects which are not taken into account up to the linear order. This would also improve the level of accuracy of results as well allowing the study of stability of the results obtained using first order perturbations.

Among concrete motivations for the use of a second order perturbative scheme is the need for more accurate results from the new generation of gravitational wave detectors as well as the increased computational precision required in order to analyse the cosmic microwave background anisotropies.[2,3] Both these problems are likely, in future, to require analyses that go beyond the linear order. In addition, the non-linear analysis in the context of cosmological modeling is also relevant on scales much smaller than the cosmological horizon where it might substantially modify the first order results[4] and therefore, as we shall also see below, become important for the study of the structure formation. Furthermore, as far as observations are concerned, second order effects can be important on small scales in order to account for non-linear effects such as the Rees-Sciama[5] and gravitational lensing effects (see e.g.[6]). For a list of references concerning these questions we refer the reader to[3].

Second order perturbations have so far not been widely studied in general relativity. In the cosmological context they seem to have been first employed by Tomita[7] to study the evolution of scalar perturbations in the Einstein–de-Sitter model using a synchronous gauge. This study was repeated by Mataresse et. al.[8], who obtained similar results using a comoving approach and Russ et al.[9] who have included second order terms resulting from a coupling between growing and decaying scalar perturbation modes. There have also been recent works[10] showing that the back reaction of the second order perturbations (to be precise, perturbations quadratic in first order terms) are important in early universe scenarios. Furthermore, it has been shown that in inflationary models the magnitude of the second order perturbations can be comparable to those in the first order and therefore a non-linear perturbative analysis may be crucial.[11]

The domain of applicability of the theory of second order perturbations that we shall describe is that of small perturbations about an homogeneous and isotropic background. We shall employ it as an approximate framework in order to study the effect of non-linearities in structure formation scenarios as well as the study of the stability of cosmological results obtained using first order perturbative schemes with respect to nonlinear perturbations.

In this paper we shall summarise recent results concerning the exact asymptotic solutions of the second order perturbation equations. An immediate consequence of these results was to prove the non-linear asymptotic stability of the de-Sitter attractor[12], and thus generalise the cosmic no-hair conjecture.[13] In general, however, the perturbation equations cannot be solved analytically. We shall show preliminary results concerning the numerical integration of the scalar perturbation equation for all times. Finally, we discuss possible implications on the structure formation. We use units in which $8\pi G = c = 1$. Greek indices take values $1, 2, 3$.

2. Perturbation Equations

We consider a homogeneous and isotropic background space-time given by a flat FLRW metric with an irrotational dust source field. The relevant equations for the expansion $\theta^\alpha{}_\beta$ in presence of the cosmological constant Λ are given by the evolution equation, the Raychaudhuri equation and the energy constraint:

$$\dot{\theta}_{\alpha\beta} + \theta\theta_{\alpha\beta} + R^*_{\alpha\beta} = \frac{1}{2}\rho\delta_{\alpha\beta} + \Lambda\delta_{\alpha\beta} \tag{1}$$

$$\dot{\theta} + \theta^{\alpha\beta}\theta_{\alpha\beta} + \frac{1}{2}\rho = \Lambda \tag{2}$$

$$\theta^2 - \theta^{\alpha\beta}\theta_{\alpha\beta} + R^* = 2(\rho + \Lambda) \tag{3}$$

where the dot denotes a derivative with respect to coordinate time t, ρ is the matter density and R^* is the 3-Ricci scalar. We use the notation of Matarrese, Mollerach and Bruni[1] and generalise their formalism to the case with nonzero Λ.

Using a conformal rescaling we can write the metric as

$$ds^2 = a^2(\tau)(-d\tau^2 + \gamma_{\alpha\beta}dx^\alpha dx^\beta),$$

where a is the scale factor. We work in the synchronous gauge (which for dust is also comoving) in which the first order metric perturbations can be written as

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta} + \gamma^{(1)}_{\alpha\beta} \qquad with \qquad \gamma^{(1)}_{\alpha\beta} = -2\phi\delta_{\alpha\beta} + D_{\alpha\beta}\chi + \pi_{\alpha\beta}, \tag{4}$$

where $D_{\alpha\beta} = \partial_\alpha\partial_\beta - \frac{1}{3}\delta_{\alpha\beta}\nabla^2$, the superscript $^{(1)}$ denotes first order quantities, ϕ and χ are the trace and tracefree parts of the scalar perturbation modes and the tensor modes are represented by $\pi_{\alpha\beta}$, which is transverse and divergence free. In the linear theory, the scalar, vector and tensor modes are independent and for an irrotational space-time one can set the vector modes equal to zero.

We recall that the evolution equation for the tensor modes can be obtained from the tracefree part of (1) in the form

$$\pi''_{\alpha\beta} + 2\frac{a'}{a}\pi'_{\alpha\beta} - \nabla^2\pi_{\alpha\beta} = 0, \tag{5}$$

where the prime denotes a derivative with respect to conformal time τ. Using tensor harmonics, which are eigenfunctions of the Laplace operator, one can set $\nabla^2\pi_{\alpha\beta} = -k^2\pi_{\alpha\beta}$, where k is the order of the harmonic.

The first order scalar perturbation equation can be obtained from the continuity equation (which in our case reads $\dot{\rho} + \theta\rho = 0$), together with (1) and (3), to give

$$\delta'' + \frac{a'}{a}\delta' - \frac{1}{2}a^2\rho_b\delta = 0, \tag{6}$$

where $\delta = (\rho - \rho_b)/\rho_b$ and ρ_b denotes the background matter density. One can further make use of the freedom of the synchronous coordinate system to specify $\delta = -\frac{1}{2}\nabla^2\chi$. This then completely fixes the gauge.

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Having recalled the first order perturbation equations we shall now consider the second order case. The conformal spatial metric tensor of the second order perturbed space-time can be written, as

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta} + \gamma_{\alpha\beta}^{(1)} + \frac{1}{2}\gamma_{\alpha\beta}^{(2)} \quad with \quad \gamma_{\alpha\beta}^{(2)} = -2\phi^{(2)}\delta_{\alpha\beta} + \chi_{\alpha\beta}^{(2)}, \tag{7}$$

where $\phi^{(2)}$ is the second order scalar perturbations and $\chi_{\alpha\beta}^{(2)}$ is trace-free. In order to obtain a simpler form for the second order perturbation equations we shall not, as in the first order case, separate the trace-free part of the $\chi_{\alpha\beta}^{(2)}$ perturbations. Substituting (7) in (2) we obtain[1,14] the following evolution equation for $\phi^{(2)}$

$$\phi^{(2)\prime\prime} + \frac{a'}{a}\phi^{(2)\prime} - \frac{\rho_b a^2}{2}\phi^{(2)} = \frac{1}{6}\gamma^{(1)\alpha\beta\prime}\left(2\frac{a'}{a}\gamma_{\alpha\beta}^{(1)} - \gamma_{\alpha\beta}^{(1)\prime}\right) +$$

$$+\frac{1}{6}\left[2\gamma^{(1)\alpha\beta}\left(2\gamma_{\alpha,\delta\beta}^{(1)\delta} - \nabla^2\gamma_{\alpha\beta}^{(1)} - \gamma_{\delta,\alpha\beta}^{(1)\delta}\right) - \gamma_\delta^{(1)\delta}\left(\gamma_{,\alpha\beta}^{(1)\alpha\beta} - \nabla^2\gamma_\alpha^{(1)\alpha}\right)\right] + \tag{8}$$

$$+\frac{\rho_b a^2}{6}\left[\frac{1}{4}\left(\gamma_\alpha^{(1)\alpha} - \gamma_{0\alpha}^{(1)\alpha}\right)^2 + \frac{1}{2}\left(\gamma^{(1)\alpha\beta}\gamma_{\alpha\beta}^{(1)} - \gamma_0^{(1)\alpha\beta}\gamma_{0\alpha\beta}^{(1)}\right) - \delta_0\left(\gamma_\alpha^{(1)\alpha} - \gamma_{0\alpha}^{(1)\alpha}\right)\right],$$

where the subscript "0" denotes the evaluation at an initial time $\tau = \tau_0$.

3. Exact Asymptotic Results

It is well known that the first order perturbations of flat dust FLRW models asymptotically tend to constants in time.[15] Now, in order to test the stability of this result with respect to second order perturbations we can solve the asymptotic form of (8). Keeping the lowest order terms on the right hand side of Eq. (8) we have

$$\phi^{(2)\prime\prime} - \frac{1}{\tau}\phi^{(2)\prime} + \frac{1}{2}\sqrt{\frac{\Lambda}{3}}\rho_0\tau\phi^{(2)} = F(\mathbf{x})\tau + O(\tau^2), \tag{9}$$

where F is a spatial function. The solutions to (9) have the form

$$\phi^{(2)}(\tau,\mathbf{x}) = \frac{F(\mathbf{x})}{A^2} + \tau C_3(\mathbf{x})J\left(\frac{2}{3},\frac{2}{3}A\tau^{3/2}\right) + \tau C_4(\mathbf{x})Y\left(\frac{2}{3},\frac{2}{3}A\tau^{3/2}\right), \tag{10}$$

where C_3 and C_4 are arbitrary functions and J and Y are Bessel functions. Therefore, asymptotically[a] $\phi^{(2)}$ approaches a constant value in time.

The evolution equation for $\chi_{\alpha\beta}^{(2)}$ can be obtained from (1) (see[14]) and its asymptotic form (for each α and β) is given by

$$\chi_{\alpha\beta}^{(2)\prime\prime} - \frac{2}{\tau}\chi_{\alpha\beta}^{(2)\prime} + k^2\chi_{\alpha\beta}^{(2)} = E_{\alpha\beta}(k) + O(\tau), \tag{11}$$

where $E_{\alpha\beta}$ depends on k. This equation can be solved to give

$$\chi_{\alpha\beta}^{(2)}(\tau,k) = \frac{E_{\alpha\beta}(k)}{k} + c_{\alpha\beta}(k\tau\cos(k\tau) - \sin(k\tau)) + d_{\alpha\beta}(k\tau\sin(k\tau) + \cos(k\tau)), \tag{12}$$

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[a]Note that $t \to \infty$ corresponds to $\tau \to 0$.

where $c_{\alpha\beta}$ and $d_{\alpha\beta}$ are arbitrary. As a result, as $\tau \to 0$, $\chi^{(2)}_{\alpha\beta}$ tends to a constant in time which depends on the asymptotic values of $\phi^{(2)}$ and $d_{\alpha\beta}$. On the other hand, these values are related to the first order initial free data so, at second order there are four independent functions.

To summarise, we showed that second order scalar and tensor perturbations asymptotically approach a constant value in time. This demonstrates that the asymptotic behaviour of the first order perturbations is stable to the presence of second order perturbations. In addition, using an invariant analysis (see[12]; cf. also[16]), we established the validity of the cosmic no-hair conjecture in nonlinear settings.

4. Numerical Results

Our analysis so far has concentrated on the asymptotic behaviour of the perturbations. The all time evolution of the second order perturbations is of potential importance for processes that evolve in time, such as the structure formation process. Here as a first step in this direction we numerically integrated the equation (8) for several initial conditions in order to investigate the evolution of $\phi^{(2)}$ in time. We found that even though asymptotically there is a single future attractor that determines the qualitative behaviour of $\phi^{(2)}$ at late times, the precise approach to this attractor can be different at intermediate times, as shown in Figure 1. The two curves represent the evolution of $\phi^{(2)}$ for different initial conditions. Both curves have initial values $\Lambda = 0.01$, $a(\tau_0) = 0.9$, $\phi^{(2)}(\tau_0) = 0.1$, $\phi^{(1)}(\tau_0) = 0.01$, $\phi^{(1)'}(\tau_0) = 0.01$ and $\rho_0 = 0.01$. The lower curve has $\phi^{(2)'}(\tau_0) = 0.01$, and the upper curve $\phi^{'(2)}(\tau_0) = 0.025$. Varying $\phi^{'(2)}(\tau_0)$ between 0.1 and 0.025 results in a family of curves lying between the two depicted curves. As can be seen, the evolution of $\phi^{(2)}$ can have distinct behaviours over intermediate times; having for example either one or three inflection points depending on the initial data.

This is an example of a dynamical system with the same attractor but different transient phases. This type of behaviour could result in different rates of structure formation over intermediate time scales, depending upon the choice of the initial data. In particular we could have a possible intermediate period where $\phi^{(2)}$ can be almost constant in time, as is the case with the lower curve in Figure 1.

5. Discussion and Conclusions

We have summarised our very recent results[12,14] on the evolution of second order perturbations in flat irrotational dust FLRW models. We have shown that second order perturbations tend asymptotically to constants in agreement with the cosmic censorship conjecture.

In order to obtain the behaviour of such perturbations at intermediate times, we have also integrated numerically the scalar perturbation equation and found different transient behaviours over intermediate time scales, depending upon the choice of the initial conditions. This can have interesting consequences for nonlinear phenomena that evolve in time, such as structure formation.
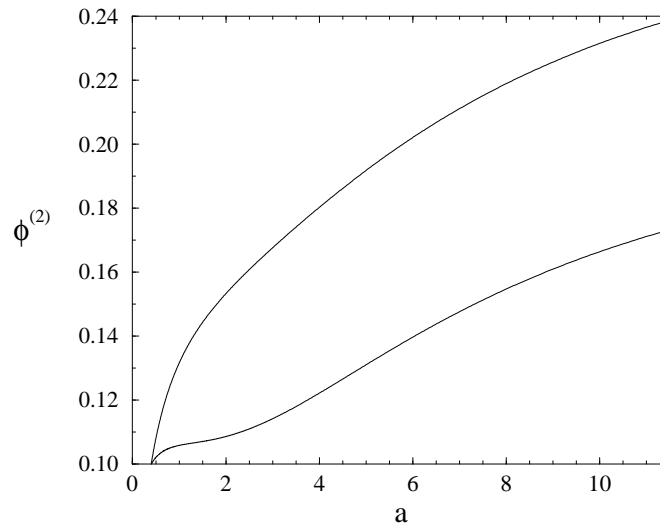
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Fig. 1.   Phase portrait representing the evolution of second order scalar perturbations $\phi^{(2)}$ in a comoving gauge for a flat FLRW dust model with $\Lambda = 0.01$.

Acknowledgements

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