Cosmology with positive and negative exponential potentials

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We present a phase-plane analysis of cosmologies containing a scalar field ϕ with an exponential potential $V \propto \exp(-\lambda\kappa\phi)$ where $\kappa^2 = 8\pi G$ and V may be positive or negative. We show that power-law kinetic-potential scaling solutions only exist for sufficiently flat ($\lambda^2 < 6$) positive potentials or steep ($\lambda^2 > 6$) negative potentials. The latter correspond to a class of ever-expanding cosmologies with negative potential. However we show that these expanding solutions with a negative potential are to unstable in the presence of ordinary matter, spatial curvature or anisotropic shear, and generic solutions always recollapse to a singularity. Power-law kinetic-potential scaling solutions are the late-time attractor in a collapsing universe for steep negative potentials (the ekpyrotic scenario) and stable against matter, curvature or shear perturbations. Otherwise kinetic-dominated solutions are the attractor during collapse (the pre big bang scenario) and are only marginally stable with respect to anisotropic shear.

I. INTRODUCTION

Scalar fields are ubiquitous in modern cosmology whether part of the matter sector, such as the Higgs field, or dilaton or moduli fields in the gravitational sector. They extend the range of qualitative behaviour found in homogeneous and isotropic Friedmann-Robertson-Walker (FRW) models compared with conventional perfect fluid models. Scalar fields with positive potential energy can drive an accelerated expansion and thus provide possible models for inflation in the early universe or dark energy at the present epoch.

In this paper we will consider models where the scalar field, ϕ, has a simple exponential potential

$$V = V_0 \exp(-\lambda\kappa\phi)\,, \qquad (1)$$

where $\kappa^2 \equiv 8\pi G_N$ is the gravitational coupling and λ is a dimensionless constant characterising the slope of the potential. Such potentials arise naturally in supergravity models [1]. The possible cosmological role of exponential potentials has been investigated before as a means of driving a period of early inflation [2–5], ekpyrotic or pre big bang collapse [6–9] or quintessence [10–13]. Although a period of accelerated expansion requires $V > 0$ we will also consider the case where $V < 0$. Negative potentials have recently been considered both in an expanding universe [14] and in a collapsing phase [6–8].

The scale-invariant form makes the exponential potential particularly simple to study analytically. There are well-known exact solutions corresponding to power-law solutions for the cosmological scale factor $a \propto t^p$ in a spatially flat FRW model [2], but more generally the coupled Einstein-Klein-Gordon equations for a single field can be reduced to a one-dimensional system which makes it particularly suited to a qualitative analysis [3,4,16–18,12,13]. Here we adopt a system of dimensionless dynamical variables [15] which have previously been used to study positive exponential potentials in isotropic [16,18] and anisotropic [17] cosmologies, and mostly recently the brane-world scenario [19].

In Section II we review the known results for the one-dimensional system with $V > 0$ and extend the analysis to include negative potentials. We then describe in section III the effect of introducing an additional term in the Friedmann equation to describe the effect of introducing a barotropic fluid, spatial curvature, or anisotropic shear in a Bianchi type I cosmology, thereby raising the system to two-dimensional phase-plane. We discuss the implications of our results for the stability of particular solutions in section IV, epsecially those solutions invoked in the epyrotic and other pre big bang scenarios.

II. ONE-DIMENSIONAL PHASE-SPACE

We consider first a scalar field with an exponential potential energy density evolving in a spatially-flat Friedmann–Robertson–Walker (FRW) universe. The Klein-Gordon equation for the scalar field in an FRW model is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,, \qquad (2)$$

where the Hubble parameter, H, is determined by the Friedmann constraint

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right)\,. \qquad (3)$$

A homogeneous scalar field has energy density $\rho_\phi = \dot{\phi}^2/2 + V$ and pressure $P_\phi = \dot{\phi}^2/2 - V$. We introduce dimensionless variables [16]

$$x \equiv \frac{\kappa \dot{\phi}}{\sqrt{6}\,H} \quad ; \quad y \equiv \frac{\kappa\sqrt{|V|}}{\sqrt{3}\,H} \ . \tag{4}$$

in which the Friedmann constraint (3) takes the simple form

$$x^2 \pm y^2 = 1 \ . \tag{5}$$

Throughout we will use upper/lower signs to denote the two distinct cases of $\pm V > 0$. x^2 measures the contribution to the expansion due to the field's kinetic energy density, while $\pm y^2$ represents the contribution of the potential energy. Their ratio determines the equation of state

$$\frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} = \frac{x^2 \mp y^2}{x^2 \pm y^2} \ . \tag{6}$$

For a positive potential this which ranges between a stiff fluid with $P_\phi = \rho_\phi$, when the kinetic energy dominates, and $P_\phi = -\rho_\phi$, when the potential dominates. For a negative potential we have an ultra-stiff fluid with $P_\phi > \rho_\phi$.

We will restrict our discussion of the existence and stability of critical points to the upper half plane $y \geq 0$, i.e., expanding cosmologies with $H > 0$, but the trajectories are symmetrical under time reversal, $H \to -H$, corresponding to reflection symmetry $x \to x$ and $y \to -y$. Thus early time solutions in an expanding universe are the same as to the late time limit in a collapsing cosmology. Without loss of generality we will consider only $\lambda \geq 0$ as the system is also symmetric under $\lambda \to -\lambda$ and $x \to -x$.

The evolution equation (2) can then be written as an autonomous system [15,16]:

$$x' = -3x(1-x^2) \pm \lambda\sqrt{\frac{3}{2}}y^2 \ , \tag{7}$$

$$y' = xy\left(3x - \lambda\sqrt{\frac{3}{2}}\right) \ , \tag{8}$$

where a prime denotes a derivative with respect to the logarithm of the scale factor, $N \equiv \ln(a)$. The constraint (5) then reduces the system to a one-dimensional phase-space corresponding to the unit circle for $V \geq 0$ or hyperbola for $V \leq 0$.

A. Critical points

Critical points correspond to fixed points (x_i, y_i) where both $x' = 0$ and $y' = 0$. These are self-similar solutions as dimensionless quantities are invariant under time-translation $N \to N + \Delta N$. For example we have $\dot{H}/H^2 =$ constant, where

$$\frac{\dot{H}}{H^2} = -3x^2 \ . \tag{9}$$

Integrating this equation shows that all critical points, where $x = x_i =$ non-zero constant, correspond to a power-law solution for the scale factor in terms of cosmic time:

$$a \propto |t|^p \ , \quad \text{where } p = \frac{1}{3x_i^2} \ . \tag{10}$$

The system (7) and (8) has at most three fixed points:

$A_\pm$ Two *kinetic-dominated* solutions exist for any form of the potential with $y_A = 0$ and $x_{A_+} = +1$ or $x_{A_-} = -1$. These are equivalent to stiff-fluid dominated evolution with $a \propto t^{1/3}$ irrespective of the nature of the potential. This is the form of cosmological evolution usually invoked approaching the singularity in the Einstein conformal frame in the pre big bang scenario [20].

B A *potential-kinetic-scaling* solution exists for $\pm(6 - \lambda^2) > 0$ with

$$x_B = \frac{\lambda}{\sqrt{6}} \ , \qquad y_B = \sqrt{1 - \frac{\lambda^2}{6}} \ . \tag{11}$$

This solution only exists for sufficiently flat potentials ($\lambda^2 < 6$) for positive potentials, or steep potentials ($\lambda^2 > 6$) for negative potentials.

The power-law exponent, $p = 2/\lambda^2$, depends on the slope of the potential. For positive potentials with $\lambda^2 < 2$ and $H > 0$ this solution corresponds to the well-known power-law inflation solutions with an accelerating scale factor $\ddot{a} > 0$ [2]. For negative exponential potential with $\lambda \gg 1$ and $H < 0$ this yields the accelerated collapse with $p \ll 1$ recently invoked in the ekpyrotic scenario [6,7].

B. Stability

$A_\pm$ Linear perturbations $x \to x + u$ about the points $x_{A_+} = +1$ and $x_{A_-} = -1$ have exponential behaviour $u \propto e^{mN}$ where $m_+ = \sqrt{6}(\sqrt{6} - \lambda)$ and $m_- = \sqrt{6}(\sqrt{6} + \lambda)$ respectively. Thus for positive λ, $x_{A_-} = -1$ is always unstable and $x_{A_+} = +1$ is stable for $\lambda^2 > 6$ but unstable for $\lambda^2 < 6$.

B Linear perturbations about the potential-kinetic scaling solution decay with eigenvalue $m = (\lambda^2 - 6)/2$ and this solution is thus always stable when this point exists for a positive potential, but unstable when this solution exists for a negative potential.

C. Behaviour at infinity

For positive potentials the one-dimensional system is compact, but for $V \leq 0$ the trajectories can go out to infinity with $x/y \to 1$ and the system reduces to
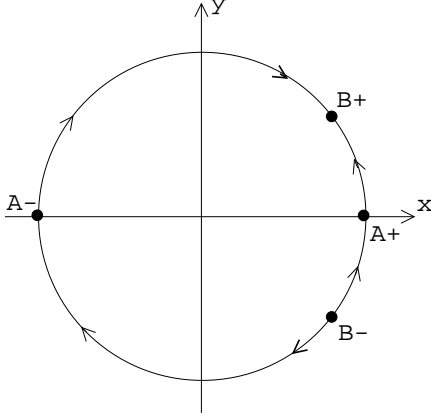
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FIG. 1. One-dimensional phase-space for flat positive potentials, $\lambda^2 < 6$. Arrows indicate evolution in cosmic time, t. Note that in the lower half-plane, $H < 0$, this has the opposite sense to $N \equiv \ln(a)$.

$$x' \to 3x^3 \qquad (12)$$

so trajectories always approach infinity with $x^2 = 1/6(N_* - N)$. Equation (9) can then be integrated to obtain a solution for H and hence the scale factor in terms of proper time

$$N_* - N \propto (t - t_*)^2 , \qquad (13)$$

showing that the behaviour at infinity always corresponds to recollapse, i.e., a maximum of the expansion at finite time t_*. Thus expanding solutions in the upper half-plane are linked to collapsing solutions in the lower half-plane at infinity. When H changes sign, both x and y also change sign, so solutions in Figures 3 and 4 exiting top-right are linked to solutions bottom-left and similarly top-left to bottom-right.

D. Qualitative evolution

a. *Flat positive potentials ($V > 0$, $\lambda^2 < 6$)* . All three critical points exist. Point B is the stable late-time attractor and points A_- and A_+ are unstable repellors. Hence generic solutions start in a kinetic-dominated regime and approach the kinetic-potential scaling solution at late times.

b. *Steep positive potentials ($V > 0$, $\lambda^2 > 6$)* . Only the two kinetic-dominated fixed points A_+ and A_- exist. For $\lambda > 0$, generic solutions start at $x_{A_-} = -1$ and approach $x_{A_+} = +1$ at late times.

c. *Flat negative potentials ($V < 0$, $\lambda^2 < 6$)* . Only the two kinetic-dominated fixed points A_+ and A_- exist and both are unstable repellors. Thus generic solutions begin in one of the kinetic dominated solutions and go out to infinity, i.e., recollapse. Thus these cosmological solutions have a finite lifetime, beginning in a kinetic-dominated big bang (point A_- or A_+), but recollapsing
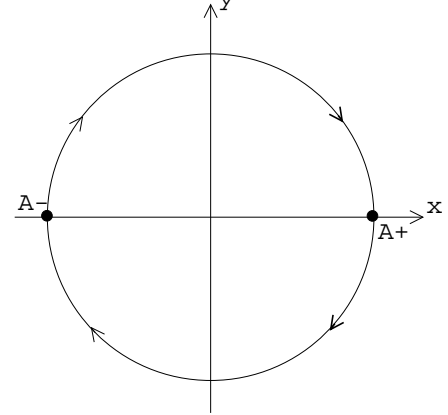

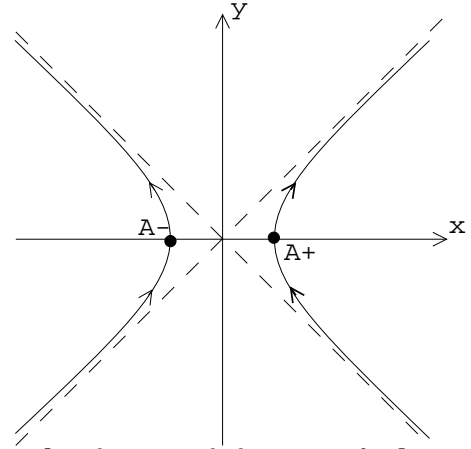FIG. 2. One-dimensional phase-space for steep positive potentials, $\lambda^2 > 6$.

FIG. 3. One-dimensional phase-space for flat negative potentials, $\lambda^2 < 6$.

after finite time and collapsing to a big crunch (at A_+ or A_-).

d. *Steep negative potentials ($V < 0$, $\lambda^2 > 6$)* . All three critical points exist. For $\lambda > 0$, point A_+ is the stable late-time attractor. Thus left hyberbola describes solutions that leave $x_{A_-} = -1$ and go to infinity. Generic solutions on the right hyperbola start in the kinetic-potential scaling solution at point B and then either become kinetic-dominated (approaching A_+) or go to infinity (recollapse).

III. TWO-DIMENSIONAL PHASE-SPACE

We now consider the effect of an additional component, not explicitly coupled to the scalar field, but gravitationally coupled via the Friedmann constraint for the Hubble expansion:

$$H^2 = \frac{\kappa^2}{3} \left(\frac{1}{2}\dot{\phi}^2 + V + \rho_\gamma \right) . \qquad (14)$$

Potential		Evolution		
Positive	flat	A_- big bang	$\rightarrow$ expansion	B_+ future infinity
		A_+ big bang	$\rightarrow$ expansion	B_+ future infinity
		B_- past infinity	$\rightarrow$ collapse	A_- big crunch
		B_- past infinity	$\rightarrow$ collapse	A_+ big crunch
	steep	A_- big bang	$\rightarrow$ expansion	A_+ future infinity
		A_+ past infinity	$\rightarrow$ collapse	A_- big crunch
Negative	flat	A_- big bang	$\rightarrow$ expansion and recollapse	A_+ big crunch
		A_+ big bang	$\rightarrow$ expansion and recollapse	A_- big crunch
	steep	B_+ big bang	$\rightarrow$ expansion	A_+ future infinity
		B_+ big bang	$\rightarrow$ expansion and recollapse	A_- big crunch
		A_- big bang	$\rightarrow$ expansion and recollapse	B_- big crunch
		A_+ past infinity	$\rightarrow$ collapse	B_- big crunch

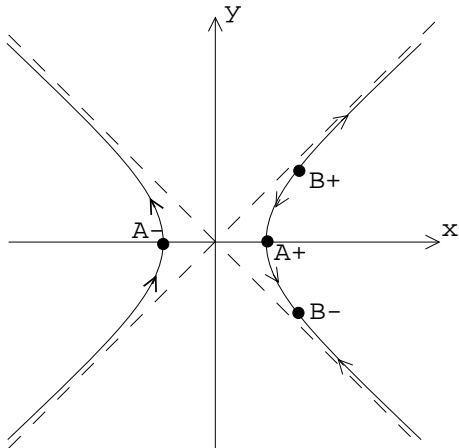TABLE I.   The evolution of one-dimensional systems.

FIG. 4.   One-dimensional phase-space for steep negative potentials, $\lambda^2 > 6$.

where ρ_γ obeys the continuity equation

$$\dot{\rho}_\gamma = -3H(\rho_\gamma + P_\gamma). \qquad (15)$$

and the scalar field obeys the same Klein-Gordon equation (2).

We will consider the simplest case of a barotropic fluid, with pressure $P_\gamma = (\gamma - 1)\rho_\gamma$ where γ =constant and $0 < \gamma < 2$ for conventional fluids (such as dust, $\gamma = 1$,

or radiation, $\gamma = 4/3$). In this case we again have an autonomous system [16] in terms of dimensionless variable x and y defined in Eq. (4) and a new dimensionless variable w

$$w \equiv \frac{\kappa\sqrt{\rho_\gamma}}{\sqrt{3}H} . \qquad (16)$$

The evolution equations (2) and (15) can then be written as an autonomous system [15,16]:

$$x' = -3x\left(1 - x^2 - \frac{\gamma}{2}w^2\right) \pm \lambda\sqrt{\frac{3}{2}}y^2 , \qquad (17)$$

$$y' = y\left(3x^2 + \frac{3\gamma}{2}w^2 - \lambda\sqrt{\frac{3}{2}}x\right) , \qquad (18)$$

$$w' = \frac{3}{2}w\left(-\gamma + 2x^2 + \gamma w^2\right) , \qquad (19)$$

where as before a prime denotes a derivative with respect to N.

The Friedmann constraint (14) takes the form

$$x^2 \pm y^2 + w^2 = 1 , \qquad (20)$$

and reduces the system to a two-dimensional phase-space corresponding to the unit sphere for $V \geq 0$ or hyperboloid for $V \leq 0$. Note that $w = 0$ remains an invariant

one-dimensional subspace. We will restrict our analysis to expanding cosmologies, $H > 0$, corresponding to the upper-quadrant $y \geq 0$ and $w \geq 0$, noting that the system is symmetric under $t \to -t$, $H \to -H$, $y \to -y$ and $w \to -w$.

The same system of equations can also be used to describe Bianchi type I cosmologies [17], where the shear appears in the constraint equation (14) like a fluid with $\gamma = 2$, or spatially curved FRW models [18], where the curvature appears in the Friedmann equation like a fluid density with $\gamma = 2/3$ (and $\rho_\gamma < 0$, for positive curved space, or $\rho_\gamma > 0$, for negative curvature). In particular the stability analysis for the barotropic fluids with $\gamma = 2$ or $\gamma = 2/3$ is also applicable to scalar fields with exponential potentials in Bianchi type I or curved FRW models, where the flat FRW models are an invariant one-dimensional subspace.

A. Critical points

Critical points correspond to fixed points (x_i, y_i, w_i) where $x' = 0$, $y' = 0$ and $w' = 0$, and these are self-similar solutions with, for example,

$$\frac{\dot{H}}{H^2} = -3x^2 - \frac{3\gamma}{2}w^2 \,, \tag{21}$$

This corresponds to a power-law solution for the scale factor

$$a \propto |t|^p \,, \quad \text{where } p = \frac{2}{6x_i^2 + 3\gamma w_i^2} \,. \tag{22}$$

The system (17), (18) and (8) has at most five fixed points given by the vacuum/scalar field solutions $A_\pm$, B, and two new points:

C A *fluid-dominated* solution

$$x_C = 0 \,, \quad y_C = 0 \,, \quad w_C = 1 \,, \tag{23}$$

always exists, for any form of potential, corresponding to a power-law solution given by Eq. (22) and $p = 2/3\gamma$.

D A *fluid-potential-kinetic-scaling* solution exists for steep positive potentials:

$$x_D = \sqrt{\frac{3}{2}}\frac{\gamma}{\lambda} \,, \qquad y_D = \sqrt{\pm\frac{3}{2}\frac{(2-\gamma)\gamma}{\lambda^2}} \,,$$

$$w_D = 1 - \frac{3\gamma}{\lambda^2} \,. \tag{24}$$

This solution only exists for potentials that are both positive ($V > 0$) and sufficiently steep ($\lambda^2 > 3\gamma^2$). The power-law exponent in Eq. (22), $p = 2/3\gamma$, is identical to that of the fluid-dominated solution and depends only on the barotropic index

and is independent of the slope of the potential. This is the classic example of 'tracking' behaviour which occurs for a wide variety of steep positive potentials [10,21].

B. Stability

The stability of $A_\pm$ and B with respect to perturbations in the $w = 0$ sub-space is unchanged, yielding one eigenmode and eigenvalue as discussed in sect. II B. Introducing fluid perturbations yields a second eigenmode $w' = 3(2x_i^2 - \gamma)w/2$:

$A_\pm$ Fluid perturbations evolve as $w \propto e^{mN}$ where the eigenvalue $m = 3(2 - \gamma)/2$. Thus the kinetic-dominated solutions are always unstable to fluid perturbations for $\gamma < 2$. For a stiff fluid with $\gamma = 2$ (or Bianchi type I shear) the kinetic-dominated solutions are marginally stable.

B Fluid perturbations evolve as $w \propto e^{mN}$ where the eigenvalue $m = (\lambda^2 - 3\gamma)/2$. Thus the kinetic-potential scaling solution is stable with respect to fluid perturbations for sufficiently flat potentials, $\lambda^2 < 3\gamma$, but unstable for $\lambda^2 > 3\gamma$. In particular, the kinetic-potential scaling solutions in spatially-flat FRW models are stable with respect to spatial curvature for $\lambda^2 < 2$, yielding with power-law inflation for positive potentials [2].

The two new points also have two eigenmodes:

C The fluid-dominated solution has two eigenmodes $x' = -3(2 - \gamma)x/2$ and $y' = 3\gamma y/2$. Thus the solution is stable to kinetic energy perturbations for $\gamma < 2$, but unstable to potential energy perturbations for $\gamma > 0$. Thus for $0 < \gamma < 2$ it is a saddle point and hence unstable to generic perturbations.

D Because point D can only exist for positive potential the phase-space reduces to that studied in Refs. [16] where it was shown that this scaling solution is always the late-time stable attractor when it exists.

C. Behaviour at infinity

For positive potentials the one-dimensional system is compact, but for $V \leq 0$ the trajectories can go out to infinity. We have

$$\left(\frac{x}{y}\right)' = -3\frac{x}{y} + \sqrt{\frac{3}{2}}\lambda y\left(\frac{x^2}{y^2} - 1\right) \,, \tag{25}$$

and hence as $y \to \infty$ we find that generic solutions approach $x/y \to -1$.
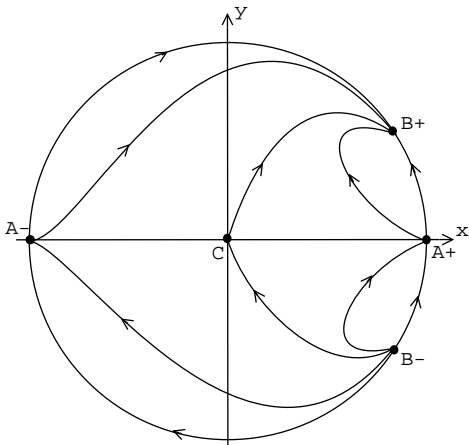
The remaining equations can then be written as

FIG. 5.   Two-dimensional phase-space for flat positive potentials, $\lambda^2 < 3\gamma$.

FIG. 6.   Two-dimensional phase-space for intermediate positive potentials, $3\gamma < \lambda^2 < 6$.

$$(x^2 + \frac{\gamma}{2}w^2)' \rightarrow 3(x^2 + \frac{\gamma}{2})^2 \qquad (26)$$

so trajectories approach infinity with $x^2 + (\gamma/2)w^2 = 1/6(N_* - N)$. Equation (21) can be integrated to obtain a solution for the scale factor in terms of proper time

$$N_* - N \propto (t - t_*)^2 \,, \qquad (27)$$

showing that the behaviour at infinity always corresponds to recollapse, i.e., a maximum of the expansion at finite time t_*. Thus expanding solutions in the upper quadrant $(w, y > 0)$ are linked to collapsing solutions in the lower quadrant $(w, y < 0)$ at infinity. When H changes sign, x, y and w all change sign.

FIG. 7.   Two-dimensional phase-space for steep positive potentials, $\lambda^2 > 6$.

D. Qualitative evolution

e. Flat positive potentials $(V > 0, \lambda^2 < 3\gamma)$. Four critical points exist in the upper quadrant $y \geq 0$. Points A_+ and A_- are unstable repellors, C is the unstable fluid dominated solution, while B is the stable late-time attractor. Generic solutions begin at one or other of the kinetic dominated solutions (A_+ or A_-) and approach the potential-kinetic solution (B) at late times. See Figure 5. They can approach arbitrarily close to the fluid dominated saddle point before reaching the potential-kinetic solution.

f. Intermediate positive potentials $(V > 0, 3\gamma < \lambda^2 < 6)$. All five critical points exist for $y \geq 0$. Generic solutions begin kinetic dominated (A_+ or A_-) and may approach the fluid dominated (C) or potential-kinetic (B) saddle points before going to the fluid-potential-kinetic scaling solution (D) at late times. See Figure 6.

g. Steep positive potentials $(V > 0, \lambda^2 > 6)$. Four critical points exist. Points A_+ and A_- are unstable repellors, C is the unstable fluid dominated solution, while D is the stable late-time attractor. Solutions can begin in the kinetic dominated solutions and may approach
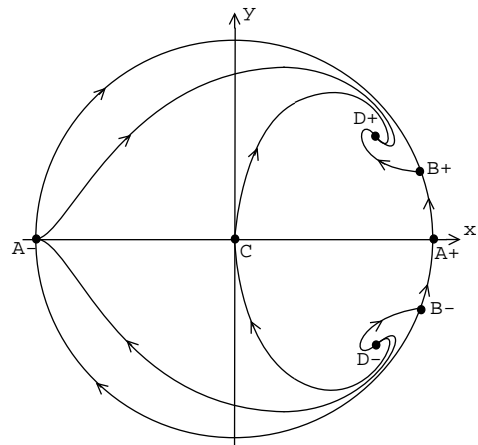
the fluid dominated saddle point (C) before going to the fluid-potential-kinetic scaling solution (D) at late times. See Figure 7. Those that start out kinetic dominated can approach the scaling solution via the fluid dominated solution.

h. Flat negative potentials $(V < 0, \lambda^2 < 6)$. Three critical points exist. Points A_+ and A_- are unstable repellors and C is the unstable fluid dominated solution. Generic solutions begin in one of the kinetic dominated solutions (A_+ or A_-) and go to infinity along the line $x = -y$, before recollapsing towards the kinetic dominated collapse solution A_+. See Figure 8. Expanding cosmologies that start out kinetic dominated can reach arbitrarily close to the fluid-dominated saddle point before recollapsing. Generic solutions have finite lifetime before recollapsing to a singularity.

i. Steep negative potentials $(V < 0, \lambda^2 > 6)$. Four critical points exist and all are unstable. Generic solutions begin either kinetic dominated (A_- for $\dot{\phi} < 0$) or kinetic-potential scaling (B for $\dot{\phi} > 0$) and go to infinity along the line $x = -y$, before recollapsing towards the
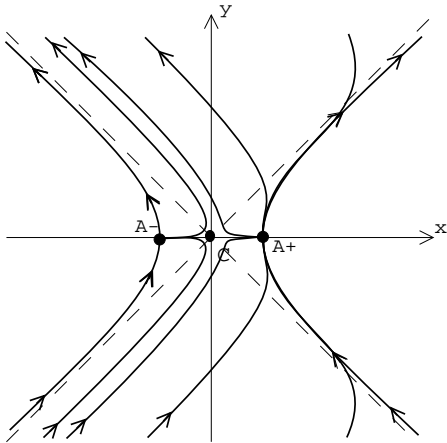
FIG. 8. Two-dimensional phase-space for flat negative potentials, $\lambda^2 < 6$.
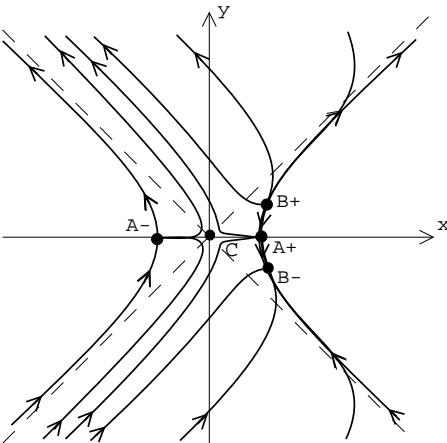

FIG. 9. Two-dimensional phase-space for steep negative potentials, $\lambda^2 > 6$.

kinetic-potential collapse solution (B). See Figure 9. If an expanding cosmology begins in the kinetic dominated solution (A_-) it may approach the fluid dominated saddle point (C) before going to infinity. If an expanding cosmology begins in the potential-kinetic solution (B) it may approach both the kinetic dominated saddle point (A_+) and the fluid dominated saddle point (C) before going to infinity. Generic solutions have finite lifetime before recollapsing to a singularity.

IV. DISCUSSION

We have presented a phase-space analysis of the qualitative evolution of cosmological models with a scalar field, ϕ, with positive or negative exponential potential, $V \propto \exp(-\lambda\kappa\phi)$. For positive potentials that are sufficiently flat ($\lambda^2 < 6$) there is a self-similar scaling solution where potential and kinetic energies remain proportional. This is the basis of the well-known power-law inflationary solutions [2] with $a \propto t^p$ and $p > 1/3$. There is no such scaling solution for flat negative potentials, but

sufficiently steep ($\lambda^2 > 6$) negative potentials do have a self-similar power-law solution with $p < 1/3$.

It has recently been found in the case of polynomial potentials [22] that expanding cosmologies with a negative potential always recollapse. By contrast, we find a expanding solutions with a sufficiently steep ($\lambda^2 > 6$) negative potential that start close to the scaling solution at early times but approach the kinetic-dominated solution at late times where the potential is negligible and expansion can continue indefinitely. Cosmologies with flat ($\lambda^2 < 6$) negative potentials do always recollapse.

The scaling solution is always the late-time attractor for flat positive potentials in an expanding universe, but the scaling solution with steep negative potentials is always unstable in an expanding universe. Conversely the kinetic-dominated solution with stiff equation of state ($P_\phi = \rho_\phi$) is always the attractor approaching a big crunch singularity in a collapsing universe with positive potential, but for steep negative potentials the attractor solution is the scaling solution with an ultra-stiff ($P_\phi > \rho_\phi$) equation of state. This is quite contrary to the usual expectation that any scalar field potential becomes negligible as the singularity is approached. For steep negative potentials the isotropic ultra-stiff scaling solution remains the attractor even in the presence of conventional matter perturbations, spatial curvature and anisotropic shear.

It is interesting to apply these results to recently proposed cosmological models based on exponential potentials in a 'pre big bang' phase. The ekpyrotic model [6,7] has a steep negative potential ($\lambda^2 \gg 6$). Thus the kinetic-potential scaling solution is the attractor solution approaching the big crunch singularity in a collapsing universe. It is stable to both conventional matter perturbations, curvature and shear perturbations. This echoes recent results found for anisotropic cosmologies in Randall-Sundrum-type brane-world scenarios [23], but is in contrast to the pre big bang scenario for a massless dilaton field [20] which is only marginally stable to anisotropic shear [24].

Another topical example is Finelli and Brandenberger's collapsing model [8] that gives a scale-invariant spectrum of comoving curvature perturbations during the collapse phase. This corresponds to a kinetic-potential scaling solution with $p = 2/3$, i.e., $\lambda^2 = 3$, so requires a flat, positive potential. Thus this solution is unstable in a collapsing universe where generic solutions approach the kinetic-dominated regime.

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