The consistency of codimension-2 braneworlds and their cosmology

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We study axially symmetric codimension-2 cosmology for a distributional braneworld fueled by a localised four-dimensional perfect fluid, in a six-dimensional Lovelock theory. We argue that only the matching conditions (dubbed topological) where the extrinsic curvature on the brane has no jump describe a pure codimension-2 brane. If there is discontinuity in the extrinsic curvature on the brane, this induces inevitably codimension-1 distributional terms. We study these topological matching conditions, together with constraints from the bulk equations evaluated at the brane position, for two cases of regularisation of the codimension-2 defect. First, for an arbitrary smooth regularisation of the defect and second for a ring regularisation which has a cusp in the angular part of the metric. For a cosmological ansatz, we see that in the first case the coupled system is not closed and requires input from the bulk equations away from the brane. The relevant bulk function, which is a time-dependent angular deficit, describes the energy exchange between the brane and the six-dimensional bulk spacetime. On the other hand, for the ring regularisation case, the system is closed and there is no leakage of energy in the bulk. We demonstrate that the full set of matching conditions and field equations evaluated at the brane position are consistent, correcting some previous claim in the literature which used rather restrictive assumptions for the form of geometrical quantities close to the codimension-2 brane. We analyse the modified Friedmann equation and we see that there are certain corrections coming from the non-zero extrinsic curvature on the brane. We establish the presence of geometric self-acceleration and a possible curvature domination wedged in between the period of matter and self-acceleration eras as signatures of codimension-2 cosmology.

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I. INTRODUCTION

Codimension-2 distributional defects[1] are infinitesimally thin matter filaments which are particularly interesting, subtle and elusive objects in gravitational theories. In four-dimensional general relativity, codimension-2 defects of infinitesimal size describe cosmic strings which are the singular relativistic versions of finite size vortices in condensed matter systems. Strings were thought to play a dominant role in the structure formation history of our Universe [1]. They were widely studied in differing contexts, but mostly at the zero thickness limit and in the test approximation; in other words, neglecting their self-gravity, *i.e.* the effect of their energy-momentum tensor on their proper evolution. When one takes into account their self-gravity, at least for a straight distributional string, one finds that they are described by a codimension-2 conical singularity [2] sourcing the constant tension of the defect (see also [3] for constant curvature bulk). Their effect is a global, topological one, since locally the induced gravitational field is flat. Indeed, each point of the string worldsheet generates a two dimensional cone with a non-trivial deficit angle.

When considering however, non-trivial geometries, one was confronted by a paradox [4] which can be summed up in the following way: test cosmic strings are described using the Nambu effective action which leads to a local minimal area requirement, or equivalently a minimal surface spanned by the string worldsheet. This is geometrically described by the trace of the second fundamental form being equal to zero (for examples see [5]) and can be visualised as the surface (with boundary) of a soap film obtained upon retrieving slowly a circular wire from liquid soap. To the Nambu action one can find specific extrinsic and intrinsic curvature corrections [6] in order to take into account the finite width of the defect. It is natural therefore to expect, that upon considering the *self*-gravitating field of the cosmic string, a small correction should result on top of the minimal surface motion. Surprisingly however, self-gravity of the distributional string leads to trivial motion or non-bending of the worldsheet with the *full* second fundamental form

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[1] Codimension denotes the number of independent normal vectors to the worldsheet swept by the defect. We concentrate on the codimension since it is this, rather than the number of tangential dimensions of the defect, that essentially quantifies its dynamical properties.

being identically zero. Geometrically the string worldsheet is a totally geodesic surface. In other words self-gravity makes the defect completely rigid not allowing it to bend in the ambient spacetime [7] giving back again essentially the straight string metric (see also [8]).

This puzzling property of gravitational rigidity reemerged in the context of codimension-2 braneworlds embedded in a six-dimensional spacetime. Unlike four-dimensional General Relativity (GR), where one could convincingly argue that self-gravity of cosmic strings is negligible, this was far more subtle for a braneworld given that its motion is by construction fueled by its proper matter density. Indeed, not surprisingly, early work on codimension-2 distributional braneworlds showed that they were pure tension objects [9]. In other words, one could not find a distributional solution (see [10] for a mathematical discussion on distributional sources in GR) for a non constant energy-momentum tensor such as that, say, of a perfect fluid necessary to describe braneworld cosmology for example. One had to introduce finite thickness effects [11], which made the problem considerably more difficult [12], and more importantly, plagued the generality of the result and hence its physical relevance. Indeed, the key point when studying distributional sources is that they describe the important main features of brane dynamics for an arbitrary family of finite width regularisations at the limit of infinitesimal thickness. At the absence of a distributional description the dynamics are reguralisation dependent. The distributional description for example, leads to considerable simplification when one looks at codimension-one braneworlds where one can solve the full system of brane-bulk equations at least for cases with enhanced symmetry such as homogeneous and isotropic cosmology [13].

Activity on this subject concentrated mainly on the interesting topological properties of codimension-2 defects in relation to the self-tuning paradigm [14]. The key point to resolve the gravitational rigidity puzzle was to understand that it was not the defect construction which was problematic, rather the gravity theory itself did not have the relevant differential complexity in order to describe complicated distributional solutions[2]. In other words, it is not strings that cannot bend, it is just Einstein equations that cannot describe the curving of strings infinitesimally. Even ordinary codimension-1 junction conditions fail in cases where the spacetime is of lesser symmetry, such as when pasting together the Kerr metric with flat spacetime, where the spacetime metric fails to be continuous.

In fact, given that the defect carries a distributional energy-momentum tensor, the gravitational field equations have to be second order in such a way as to have piecewise continuous first metric derivatives. Although four-dimensional GR is the unique second order tensor theory with this property, in five or six dimensions one has to add an extra term, the Gauss-Bonnet term, to the action in order to have the most general second derivative (and not higher than second derivative) field equations. The generic second order derivative tensor gravity theory is in fact well-known to be Lovelock's theory [15] (for a review see [16], [17]). In this gravity theory each $2n+1$ dimensional manifold picks up in the gravitational action a novel Langrangian density, which is a specific combination of the nth power of spacetime curvature. This gravitational term stems from the Euler-Poincaré characteristic of the manifold in $2n$ dimensions and hence is purely topological at this dimension. As such, the Einstein-Hilbert and the Gauss-Bonnet terms are the Euler-Poincaré characteristics for two-dimensional and four-dimensional manifolds without boundary.

The clear-cut hint that the completion of GR in higher dimensions may be relevant to codimension-2 braneworlds came with the work of Bostock et al. [18], where it was noticed that upon considering the general second derivative gravity theory, one could have, at least in principle, a non trivial energy momentum tensor fueling geometric junction conditions for a codimension-2 conical defect. Furthermore, it was shown that the higher order gravity term in question, the Gauss-Bonnet term, was generating on the brane an induced Einstein-Hilbert term plus extrinsic curvature corrections, whereas the Einstein-Hilbert term in the bulk was giving a pure induced cosmological constant. This observation was urged further in [19] (see also [20]) where it was shown that Lovelock theory can source defects with codimension even higher than two, unlike ordinary GR. A simple geometric explanation was given relating by a simple sum rule each nth bulk Lovelock density with the distributional brane's even codimension $2m$ and the induced brane Lovelock density, $n-m$. Therefore, the six-dimensional bulk Gauss-Bonnet term ($n=2$) is "reduced" to a codimension-2 Einstein-Hilbert term on the brane $n-m=1$ and so forth. The same $n=2$ Gauss-Bonnet term in eight dimensions would "reduce" to a pure tension term on a codimension-4 brane $n-m=0$ and so on.

This activity came to a halt when under some quite mild symmetry hypothesis (axial symmetry) it was claimed that the full set of junction plus bulk field equations at the location of the brane led to an inconsistent system of differential equations for a non-trivial distributional codimension-2 defect [21]. Furthermore, the junction conditions were shown not to be unique [19], without any obvious particular physical difference but with only differing mathematical regularity. Lastly, certain restrictive initial conditions on the braneworld surface [18] introduced important constraints [22] on the admissible matter on the brane.

The aim of this paper is to falsify or explain all points in the previous paragraph. We will show that the system of equations is not only consistent, but has in general a free degree of freedom in the face of a varying deficit

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[2] We thank Brandon Carter for early, enlightening discussions on the subject of gravitating distributional defects.

angle function. This will result from carefully considering the correct and all relevant geometric expansions for the braneworld geometry. We will show that differing mathematical regularity leading to differing junction conditions inevitably leads to *distinct* physical setups (following up work by [20]). We shall show that the unique junction conditions tailored for a pure codimension-2 defect have the *same* mathematical regularity as those of GR and are the topological matching conditions of [19]. The matching conditions introduced in [18] inevitably lead to extra codimension-1 distributional matter. Having established these basic facts, we will go on to show that one can obtain the modified Lemaître-Friedmann-Robertson-Walker (LFRW) brane equations by solving the full system of the field equations, as well as the junction conditions in the brane neighborhood. We will find that the cosmological equations, apart from the ordinary LFRW term, have several interesting features including a geometric self-acceleration (in agreement with the maximally symmetric solutions of [23]), brane bending effects depending on the equation of state of matter, and energy exchange between the bulk and the brane triggered by a dynamical conical deficit angle.

The organisation of the paper is as follows: we will give the general set-up and a careful treatment of the differing matching conditions (topological, topological with ring regularisation and general) in the next section. We will then demonstrate consistency of the whole set-up and go on to investigate the cosmological equations. We will conclude in the last section.

II. GENERAL SETUP AND MATCHING CONDITIONS

Let us consider the general system of six-dimensional Lovelock gravity coupled to localised brane sources. If we describe the regular bulk matter by $S_{bulk}[\mathsf{g}]$ and the distributional brane matter by $S_{brane}[g]$, the total action of the system is,

$$S = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-|\mathsf{g}|}\,\left(\mathcal{R} + \frac{\alpha}{6}\,\mathcal{L}_{GB}\right) + S_{bulk}[\mathsf{g}] + S_{brane}[g]\ , \tag{2.1}$$

with the Gauss-Bonnet Lagrangian density

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{MN}\,\mathcal{R}^{MN} + \mathcal{R}_{MNK\Lambda}\,\mathcal{R}^{MNK\Lambda}\ . \tag{2.2}$$

In the above, the calligraphic quantities refer to the bulk metric tensor g, while the regular ones to the brane metric tensor g. The field equations arising from the action (2.1) are

$$\mathcal{G}_M^N - \frac{\alpha}{6}\mathcal{H}_M^N = \kappa_6^2\mathcal{T}_M^N + \kappa_6^2\,T_M^N\ , \tag{2.3}$$

where $\mathcal{T}_{MN}$ is a regular bulk energy-momentum tensor and T_{MN} is the distributional brane energy-momentum tensor. The most plausible assumption is that the bulk contains only cosmological constant $\mathcal{T}_{MN} = -(\Lambda_6/\kappa_6^2)\ \mathsf{g}_{MN}$, however, for the sake of generality we leave for the moment $\mathcal{T}_{AB}$ arbitrary and non-vanishing (we will however assume later that it is regular at the position of the brane). The Gauss-Bonnet contribution to the equations of motion is explicitly

$$\mathcal{H}_{MN} \equiv \frac{1}{2}\mathcal{L}_{GB}\ \mathsf{g}_{MN} - 2\mathcal{R}\mathcal{R}_{MN} + 4\mathcal{R}_{MK}\mathcal{R}_N{}^K + 4\mathcal{R}_{MKN\Lambda}\mathcal{R}^{K\Lambda} - 2\mathcal{R}_{MK\Lambda\Xi}\mathcal{R}_N{}^{K\Lambda\Xi}\ . \tag{2.4}$$

In the following, we will fix the notation to $3\kappa_6^2 = 4\pi$ to simplify the equations.

Let us now consider that there is axial symmetry in the bulk, so that the bulk metric ansatz can be written in the brane Gaussian-Normal coordinates as

$$ds_6^2 = dr^2 + L^2(x,r)d\theta^2 + g_{\mu\nu}(x,r)dx^\mu dx^\nu = dr^2 + g_{ab}(x,r)dx^a dx^b\ , \tag{2.5}$$

where θ has the standard periodicity 2π, and the braneworld metric $g_{\mu\nu}(x,0)$ is assumed to be regular everywhere, with the possible exception of isolated singular points on the brane. We denote by y collectively the transverse coordinates r,θ.

To study in detail the matching conditions we must first differentiate between two distinct cases:

1. The first case is *topological matching conditions*, discussed in [19], which have a geometric origin based on the distributional version of the Chern-Gauss-Bonnet theorem [24]. They assume everywhere smooth intrinsic and extrinsic tangential sections.

2. The second case, which admits the lesser mathematical regularity, was introduced in [18] and assumes, not only a conical deficit based on the normal geometry, but also a distributional jump in the extrinsic tangential sector,

namely a specific combination of extrinsic curvature quantities[3]. We will call this possibility the *general case* since for regular extrinsic parts it reduces to the topological case outlined above.

As we shall see later on, the second type of matching conditions inevitably lead to additional codimension-1 distributional singularities [20], and hence additional matter sectors *have* to be introduced in order to close the system mathematically. This is not too surprising, since mathematical discontinuities in the extrinsic sector are expected to produce codimension-1 defects [4], [25]. On the other hand, the topological type of matching conditions lead to regular induced metrics unlike the former case.

A. Topological matching conditions: Smooth regularisation

Let us start with discussing the case of the *topological matching conditions*. In this case, there is no jump of the extrinsic curvature inside and outside the thin defect. As usually done when dealing with distributional sources, the matching conditions are derived by integrating around the singular space. Here, we seek to integrate over the conical space, say $\mathcal{C}$, with internal metric

$$ds_2^2 = dr^2 + L^2 d\theta^2 \ , \tag{2.6}$$

$$\text{with} \quad L(x,r) = \beta(x)r + \frac{1}{2}\beta_2(x)r^2 + \frac{1}{6}\beta_3(x)r^3 + \mathcal{O}(r^4) \ , \tag{2.7}$$

where $\delta = 2\pi(1-\beta(x))$ is the conical deficit. Notice, the coefficients are allowed to depend on the "brane" coordinates x^μ. The angle θ varies in the interval $[0, 2\pi)$.

In the present paper, we are interested in the case that the distributional brane energy-momentum tensor supports a codimension-2 Dirac singularity and therefore can be written as

$$T_{MN} \equiv T_{\mu\nu}\frac{\delta(r)}{2\pi L}\delta^\mu_M \delta^\nu_N \ . \tag{2.8}$$

In general, there can be also codimension-1 $\delta(r)$ parts in the energy momentum tensor, which, as we will see later, is unavoidable if the extrinsic curvature has a jump across the interface layer of the defect.

Then, by integrating the equations of motion, the only non-zero contribution yielding the codimension-2 matching conditions reads [19],

$$\alpha\,\frac{1}{4\pi}\int d^2y\sqrt{g_2}\,R_2\,\left[G_{\mu\nu} + W_{\mu\nu} - \frac{3}{2\alpha}g_{\mu\nu}\right] = T_{\mu\nu} \ , \tag{2.9}$$

where $R_2 = -2L''/L$ is the curvature of the two-dimensional internal space, and a prime denotes ∂_r . The Einstein equation is corrected with the following contribution from the extrinsic curvature $K_{\mu\nu} \equiv \frac{1}{2}\partial_r g_{\mu\nu}$ on the brane

$$W_{\mu\nu} = K^\lambda_\mu K_{\nu\lambda} - KK_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(K^2 - K^2_{\kappa\lambda}) \ , \tag{2.10}$$

which is identical to the one in [18].

In order to evaluate the integral of (2.9) which is singular at the tip, we assume a family of *smooth* regularisations C_λ of the cone, parametrised by λ, with smooth C^2 everywhere caps (for details see [8]), pasted smoothly to the rest of the cone at a boundary ∂C, which we assume does not depend on the parameter λ (see left part of Fig.1). The distributional limit is obtained at the $\lambda \to \lambda_c$ limit, where the cap tends to the singular conical space. The width of the regularised defect is supposed to be small enough so that the r-dependent brackets can be approximated with the leading term in their r-expansion, which is just the value of the brackets at $r = 0$. Then the integral has to be performed over only the internal curvature. The Chern-Gauss-Bonnet theorem relates the geometric curvature for arbitrary λ, to the Euler characteristic χ of the regularised cap and the integral of the geodesic curvature k_g of the boundary ∂C. Then, for every cap labeled by λ, the following holds

$$\frac{1}{4\pi}\int_{C_\lambda} d^2y\sqrt{g_\lambda}\,R_\lambda = \chi - \frac{1}{2\pi}\int_{\partial C_\lambda} k_g\,d\theta \ . \tag{2.11}$$

[3] In [18] a particular jump of the extrinsic curvature was assumed, namely that the extrinsic curvature vanishes at the core of the defect.

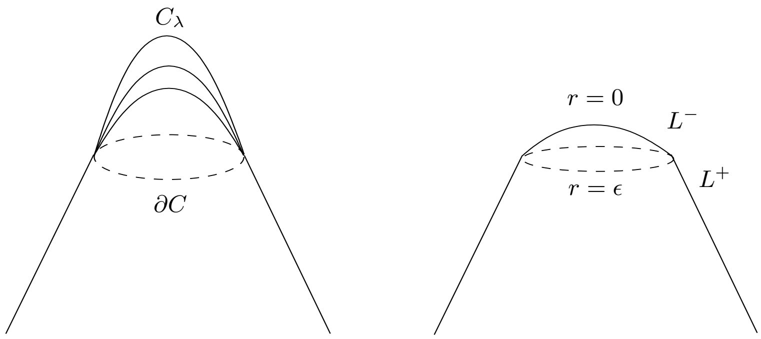

FIG. 1: On the left, the conical singularity is regularised by a family of regular interiors C_λ smoothly connected to the exterior solution of the cone at the boundary ∂C. On the right, a ring regularisation is chosen where the internal space function L is not smooth, but has a cusp across the ring at $r = \epsilon$.

Since the λ-labeled caps are topologically equivalent to a disk which has $\chi = 1$ and since $k_g = L'$, the theorem gives that for every kind of smooth regularisation of the conical singularity, one obtains

$$\frac{1}{4\pi}\int_{C_\lambda} d^2y\sqrt{g_\lambda}\,R_\lambda = 1-\beta \ . \tag{2.12}$$

The above result only used the fact that the expansion of L close to the conical defect has the expansion (2.7) and is independent of the smooth regularisation of the interior of the defect. Finally, with the use of the above theorem, the codimension-2 matching conditions for the topological case is

$$G_{\mu\nu} + W_{\mu\nu} - \frac{3}{2\alpha}g_{\mu\nu} = \frac{1}{\alpha(1-\beta)}T_{\mu\nu} \ . \tag{2.13}$$

Let us now look at the leading $\mathcal{O}(1/r)$ terms of the equations of motion. These come from the contributions of terms of the equations of motion which are multiplied by L'/L, or $\nabla_\mu L'/L$. With ∇_μ we denote covariant differentiation with respect to the metric $g_{\mu\nu}$. Then, according to the expansion (2.7), these terms contribute as

$$\left.\frac{L'}{L}\right|_{r\to 0} = \frac{1}{r} + \mathcal{O}(1) \quad , \quad \left.\frac{\nabla_\mu L'}{L}\right|_{r\to 0} = \frac{\nabla_\mu \beta}{\beta}\cdot\frac{1}{r} + \mathcal{O}(1) \ . \tag{2.14}$$

With the natural assumption that the energy momentum tensor $\mathcal{T}_{MN}$ does not blow up close to the distributional singularity (otherwise the singularity would not be distributional), we obtain that the $\mathcal{O}(1/r)$ terms of the (rr) and the $(r\mu)$ equations yield respectively

$$K^{\mu\nu}\left(G_{\mu\nu} + W_{\mu\nu} - \frac{3}{2\alpha}g_{\mu\nu}\right) = 0 \ , \tag{2.15}$$

$$\frac{\nabla^\nu\beta}{\beta}\left(G_{\mu\nu} + W_{\mu\nu} - \frac{3}{2\alpha}g_{\mu\nu}\right) + \nabla^\nu W_{\mu\nu} = 0 \ . \tag{2.16}$$

These two equations act as constraint equations for the two quantities $K_{\mu\nu}$, β, which have to be determined and substituted back to the brane Einstein equation (2.13). In the general case, there are 11 independent unknown functions, but there are 5 independent constraint equations in (2.15), (2.16). Therefore, there will in general be functions which cannot be determined by the local equations around the brane, but need to be determined by the bulk solution. This, in fact, is clearly visible in the case of isotropic cosmology on the brane, as we will see in the next section.

Finally, if we differentiate the brane Einstein equation (2.13) and use the constraint (2.16), we can derive the energy conservation equation

$$\nabla^\nu T_{\mu\nu} = -\frac{\nabla^\nu\beta}{\beta(1-\beta)}T_{\mu\nu} \ . \tag{2.17}$$

From the above, we see that the energy is not strictly conserved on the brane, but can radiate in the bulk if the deficit angle changes. We can therefore anticipate that cosmological evolutions, close to the standard four dimensional one, will have β almost constant. We will come to this point in a later section.

B. Topological matching conditions: Ring regularisation

In the previous analysis, we chose an arbitrary smooth regularisation of the distributional part of the geometry and the source. In this section, we will analyse a more specific case, where the codimension-2 source is taken by the limit of a sharp codimension-1 source with infinitesimal radius. Therefore, we replace the tip of the cone with a cap glued to the rest of the compactification with a ring interface (see right part of Fig.1). The extrinsic curvature is still continuous across the ring, but the angular part of the internal space metric L has a cusp. Inside the cap, since we will finally take the limit of its thickness to zero, the normal derivatives in the center of it tend to the values of the corresponding quantities just inside the ring.

Outside the ring, we assume that the geometry has an expansion as

$$L^{+}(x,r)= \beta(x)r + \sum_{n=2}^{\infty}\frac{1}{n!}\beta_n(x)r^n \ , \tag{2.18}$$

which is the standard conical singularity structure in the limit of zero ring thickness. On the other hand, inside the ring, the geometry should tend to two-dimensional flat space

$$L^{-}(x,r)=\mathcal{C}(x,\epsilon) + r + \sum_{n=2}^{\infty}\frac{1}{n!}\bar{\beta}_n(x)r^n \ . \tag{2.19}$$

with the constant $\mathcal{C}(x,\epsilon)=\epsilon[\beta(x)-1]+\sum_{n=2}^{\infty}\frac{1}{n!}(\beta_n(x)-\bar{\beta}_n(x))\epsilon^n$ to ensure continuity of L across the ring. In the above, ϵ is the width of the cap. As we will see later on, it is important for the consistency of the model that the higher order coefficients in the expansion of L (at least β_2) are different inside and outside the ring. The extrinsic curvature across the ring is continuous as in the previous subsection. We will assume that the derivative of the extrinsic curvature is also continuous, although this will turn up to be a fact, rather than an assumption, when we study the consistency of the setup. The crucial difference with the previous subsection is that the metric expansion inside the ring (2.19) is specified and L' has a definite jump.

In general, there are two types of distributional singularities that one can obtain in the $\epsilon\to 0$ thin limit. One is that of codimension-2 which has the structure $\delta^{(2)}(r)=\frac{\delta(r)}{2\pi r}$ and originates from the terms $\frac{\delta(r-\epsilon)}{2\pi\epsilon}$. There can exist, however, in the same limit a codimension-1 defect with singularity structure $\delta(r)$, which comes simply from $\delta(r-\epsilon)$. To calculate the distributional pieces of the equations of motion, we will make use of the following identities where we denote by "distr" the distributional parts of the corresponding quantities

$$\begin{aligned}
\mathrm{distr}(L''g_{\mu\nu}) &= (\beta-1)[g_{\mu\nu}+g'_{\mu\nu}\epsilon]\delta(r-\epsilon) \ , &(2.20)\\
\mathrm{distr}(L''G_{\mu\nu}) &= (\beta-1)[G_{\mu\nu}+G'_{\mu\nu}\epsilon]\delta(r-\epsilon) \ , &(2.21)\\
\mathrm{distr}(L''W_{\mu\nu}) &= (\beta-1)[W_{\mu\nu}+W'_{\mu\nu}\epsilon]\delta(r-\epsilon) \ . &(2.22)
\end{aligned}$$

Here, we also note the next to leading order terms in the ϵ-expansion, since these distributional parts are divided by $L\sim\epsilon$ at the ring position. Because of this well defined ring structure in the internal space, there could be codimension-1 contributions of the equations of motion surviving in the $\epsilon\to 0$ limit. It is clear to see that the only such possible contributions come from the $(\mu\nu)$ equations. To find the codimension-1 contributions, we should be careful with the distributional definition of T_{MN} in (2.8). This is because, inside the cap, the energy momentum tensor which becomes distributional in the $\epsilon\to 0$ limit, may have an expansion of the type

$$T_{\mu\nu}=T^{(0)}_{\mu\nu}+T^{(1)}_{\mu\nu}\ r+\mathcal{O}(r^2) \ . \tag{2.23}$$

Expanding (2.8) at $r=\epsilon$, we obtain

$$T_{MN}=\left[T^{(0)}_{\mu\nu}\frac{\delta(r-\epsilon)}{\epsilon}+\left(T^{(1)}_{\mu\nu}-\frac{1}{2}\frac{\beta_2}{\beta}T^{(0)}_{\mu\nu}\right)\delta(r-\epsilon)\right]\frac{1}{2\pi\beta}\delta^{\mu}_{M}\delta^{\nu}_{N} \ . \tag{2.24}$$

In the following we will use $T_{\mu\nu}$ to denote $T^{(0)}_{\mu\nu}$.

From the $\delta(r-\epsilon)/\epsilon$ singular pieces of equations (2.3), we obtain the matching conditions for the codimension-2 singularity. Due to the topological origin of the dimensionally extended Lovelock densities, the Gauss-Bonnet combination in the bulk theory will give an Einstein brane term plus extrinsic curvature corrections, at the level of the junction conditions. The latter then read as before

$$G_{\mu\nu}+W_{\mu\nu}-\frac{3}{2\alpha}g_{\mu\nu}=\frac{1}{\alpha(1-\beta)}T_{\mu\nu} \ . \tag{2.25}$$

From the $\delta(r-\epsilon)$ part of the $(\mu\nu)$ equations, we obtain the codimension-1 matching conditions at $r=\epsilon$

$$\left[G_{\mu\nu} + W_{\mu\nu} - \frac{3}{2\alpha} g_{\mu\nu}\right]' = \frac{1}{\alpha(1-\beta)} T^{(1)}_{\mu\nu} \ . \tag{2.26}$$

As we will comment at the end, we need always have non-zero $T^{(1)}_{\mu\nu}$ in order not to overconstrain the system. Let us note that (2.26) is a consequence of the ring regularisation. Such kind of codimension-1 equation did not arise in the the smooth regularisation in the topological case, since then, only two-dimensional distributions are well defined.

Let us now look at the $\mathcal{O}(1/\epsilon)$ terms of the equations of motion. We will make again the natural assumption that the non-distributional energy-momentum tensor $\mathcal{T}_{AB}$ inside and outside the rings is regular and has no $\mathcal{O}(1/\epsilon)$ singularity. In evaluating the $\mathcal{O}(1/\epsilon)$ terms, we should be careful to do it both infinitesimally outside of the ring and also infinitesimally inside of it. This is because the function L' has a jump. This function enters in the equations of motion in two forms, as L'/L and as $\nabla_\mu L'/L$. Then, according to the expansions (2.18) and (2.19), these terms contribute as

$$\frac{L'}{L}\bigg|_+ = \frac{1}{\epsilon} + \mathcal{O}(1) \ , \quad \frac{\nabla_\mu L'}{L}\bigg|_+ = \frac{\nabla_\mu \beta}{\beta}\cdot\frac{1}{\epsilon} + \mathcal{O}(1) \ , \tag{2.27}$$

$$\frac{L'}{L}\bigg|_- = \frac{1}{\beta}\cdot\frac{1}{\epsilon} + \mathcal{O}(1) \ , \quad \frac{\nabla_\mu L'}{L}\bigg|_- = \frac{\nabla_\mu \bar{\beta}_2}{\beta} + \mathcal{O}(\epsilon) \ . \tag{2.28}$$

The (rr) equation has only L'/L pieces, therefore the $\mathcal{O}(1/\epsilon)$ terms give us the equation

$$K^{\mu\nu}\left(G_{\mu\nu} + W_{\mu\nu} - \frac{3}{2\alpha} g_{\mu\nu}\right) = 0 \ . \tag{2.29}$$

On the other hand, the $(r\mu)$ equation has both L'/L and $\nabla_\mu L'/L$ pieces. The difference of the contributions of these two pieces is a $\nabla^\nu W_{\mu\nu}$ term, which therefore has to vanish

$$\nabla^\nu W_{\mu\nu} = 0 \ . \tag{2.30}$$

The $\mathcal{O}(1/\epsilon)$ part of the the $(r\mu)$ equation just inside the ring gives by itself

$$\frac{\nabla^\nu \beta}{\beta}\left(G_{\mu\nu} + W_{\mu\nu} - \frac{3}{2\alpha} g_{\mu\nu}\right) = 0 \ . \tag{2.31}$$

Counting again the constraints of (2.29), (2.30), (2.31), we find that there are 9 independent equations to determine the 11 unknown quantities, $K_{\mu\nu}$ and β. Therefore, there will again be in general functions which cannot be determined by the local equations around the brane, but need to be determined by the bulk solution. However, in the case of isotropic cosmology on the brane, as we will see in the next section, all the unknown functions entering in the brane Einstein equation (2.25) are completely determined (modulo integration constants) and the system therefore closes.

Had we taken $T^{(1)}_{\mu\nu} = 0$ in the codimension-1 matching conditions (2.26), which have 10 independent components, these matching conditions would act as constraints and therefore in principle overconstrain the system. This is indeed the case for the cosmological ansatz case that we will discuss later.

Finally, let us again differentiate the brane Einstein equation (2.25). Using the constraint (2.30), we see that the brane energy momentum tensor is always conserved

$$\nabla^\nu T_{\mu\nu} = 0 \ . \tag{2.32}$$

Therefore, the ring regularisation, *i.e.* the fact that we consider not a C^2 but a C^1 cap C_λ at $r=\epsilon$, freezes brane radiation.

C. General matching conditions

To study the general matching conditions, we will use a similar regularisation as in the previous subsection, where in addition, the tangential sector will have in principle distributional jumps. The regularisation of the codimension-2 singularity is then depicted in Fig.2. In more details, we assume that both the angular metric function L and the extrinsic curvature $K_{\mu\nu}$ have jumps when going from the exterior to the interior of the ring. Inside the ring, since we

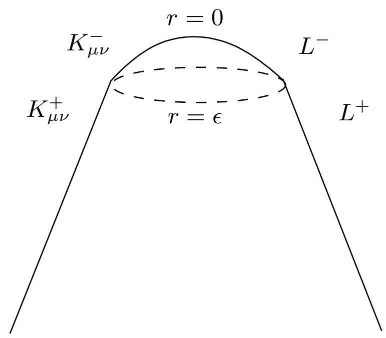

FIG. 2: The conical singularity regularised by a cap glued to the bulk at a ring intersurface at $r=\epsilon$. Both $K_{\mu\nu}$ and L have jumps across at the ring.

will finally take the limit of its thickness to zero, the normal derivatives in the center of the cap tend to the values of the corresponding quantities just inside the ring.

In this general setup, we would like to investigate the possibility of having as before a codimension-2 distributional energy-momentum tensor as in (2.8). We consider the case of having a codimension-1 distributional source as out of the scope of this paper. The answer to the former question can easily be seen in the following to be negative.

The extrinsic curvature inside and outside the ring brane have in general a non-trivial relation between them, depending on the details of the internal cap structure. Without loss of generality of our following analysis, we can make the simplified approximation that the two extrinsic curvatures are proportional as

$$K^-_{\mu\nu} = \eta K^+_{\mu\nu} \ , \tag{2.33}$$

with $\eta=$ const. In the following, we will simplify the notation denoting with $K_{\mu\nu}$ the exterior extrinsic curvature $K^+_{\mu\nu}$ (and similarly for other quantities constructed by it). Furthermore, we assume that higher orders in the r-expansion of the metric are continuous across the ring brane. This means in particular that the non-distributional part of $K'_{\mu\nu}$ is continuous, in correspondence with β_2 in the expansion of L.

When looking at the distributional part of the equations of motion, since also $K_{\mu\nu}$ can now have a jump, we should modify the distributional parts in (2.20)-(2.22) as following

$$\begin{aligned}
\mathrm{distr}(L'g_{\mu\nu})' &= [(\beta-1)g_{\mu\nu} + (\beta-\eta)g'_{\mu\nu}\epsilon]\delta(r-\epsilon) \ , && (2.34)\\
\mathrm{distr}(L'G_{\mu\nu})' &= [(\beta-1)G_{\mu\nu} + (\beta-\eta)G'_{\mu\nu}\epsilon]\delta(r-\epsilon) \ , && (2.35)\\
\mathrm{distr}(L'W_{\mu\nu})' &= [(\beta-\eta^2)W_{\mu\nu} + (\beta-1)W'_{\mu\nu}\epsilon]\delta(r-\epsilon) \ , && (2.36)\\
\mathrm{distr}K'_{\mu\nu} &= (1-\eta)K_{\mu\nu}\delta(r-\epsilon) \ . && (2.37)
\end{aligned}$$

The difference in the ϵ-terms in the brackets has to do with the assumption that $K'_{\mu\nu}$ is continuous across the ring. All the derivatives in the above expressions are computed at the exterior of the ring. From the above we can first write down the codimension-2 matching conditions from the $\delta(r-\epsilon)/\epsilon$ terms of the $(\mu\nu)$ equations of motion

$$G_{\mu\nu} + \frac{\eta^2-\beta}{1-\beta}W_{\mu\nu} - \frac{3}{2\alpha}g_{\mu\nu} = \frac{1}{\alpha(1-\beta)}T_{\mu\nu} \ . \tag{2.38}$$

For $\eta=0$ we recover the matching conditions of [18]. The codimension-1 $\delta(r-\epsilon)$ part of the $(\mu\nu)$ equations of motion provide us with the codimension-1 matching conditions which, in agreement with [20], give after some simplifications

$$\begin{aligned}
&\left[G_{\mu\nu} + \frac{1-\beta}{\eta-\beta}W_{\mu\nu} - \frac{3}{2\alpha}g_{\mu\nu}\right]' + \frac{3\beta}{2\alpha}\,\frac{\eta-1}{\eta-\beta}(K_{\mu\nu}-Kg_{\mu\nu}) - \beta\,\frac{\eta^3-1}{\eta-\beta}\left(K^\kappa_\mu W_{\kappa\nu} - \frac{1}{3}K^{\kappa\lambda}W_{\kappa\lambda}g_{\mu\nu}\right)\\
&+\beta\,\frac{\eta-1}{\eta-\beta}\left[\frac{\nabla_\kappa\nabla_\nu L}{L}K^\kappa_\mu + \frac{\nabla_\mu\nabla_\kappa L}{L}K^\kappa_\nu - \frac{\Box L}{L}K_{\mu\nu} - \frac{\nabla_\mu\nabla_\nu L}{L}K + \left(\frac{\Box L}{L}K - \frac{\nabla_\kappa\nabla_\lambda L}{L}K^{\kappa\lambda}\right)g_{\mu\nu}\right]\\
&+\beta\,\frac{\eta-1}{\eta-\beta}\left(R_{\mu\kappa\lambda\nu}K^{\kappa\lambda} + G_{\kappa\lambda}K^{\kappa\lambda}g_{\mu\nu} + R_{\mu\nu} + \frac{1}{2}RK_{\mu\nu} - R_{\kappa\nu}K^\kappa_\mu - R_{\mu\kappa}K^\kappa_\nu\right) = \frac{1}{\alpha(\eta-\beta)}T^{(1)}_{\mu\nu} \ . \qquad (2.39)
\end{aligned}$$

Note that the above equation for $\eta=1$ yields the codimension-1 matching condition (2.26) of the previous section. The codimension-1 $\delta(r-\epsilon)$ part of the $(\theta\theta)$ equation, on the other hand, will not have an energy-momentum contribution because we assume that the defect is of pure codimension-2 nature. Therefore, in agreement with [20], we obtain

$$(1-\eta)K - \frac{2\alpha}{9}(1-\eta^3)K^{\mu\nu}W_{\mu\nu} - \frac{2\alpha}{3}(1-\eta)K^{\mu\nu}G_{\mu\nu} = 0 \ . \tag{2.40}$$

From the above constraint equation there are two possibilities. The obvious solution is $\eta = 1$, and which gives us the topological matching conditions. There could be of course some solution for $K_{\mu\nu}$ satisfying the above equation for $\eta \neq 0$. However, as we will see in the following section for the cosmological ansatz, it turns out that these codimension-1 parts overconstrain the system.

Regarding the $\mathcal{O}(1/\epsilon)$ terms of the equations of motion in this general conditions case, we obtain the constraint (2.29) from the (rr) equation outside the ring brane

$$K^{\mu\nu}\left(G_{\mu\nu} + W_{\mu\nu} - \frac{3}{2\alpha}g_{\mu\nu}\right) = 0 \ , \tag{2.41}$$

however, inside the ring, because the extrinsic curvature has a jump, we obtain by substraction of (2.41)

$$\eta(\eta^2-1)K^{\mu\nu}W_{\mu\nu} = 0 \ . \tag{2.42}$$

Moreover, from the $\mathcal{O}(1/\epsilon)$ terms of the $(r\mu)$ equation, we obtain inside the ring

$$\frac{\nabla^\nu\beta}{\beta}\left(G_{\mu\nu} + \eta^2 W_{\mu\nu} - \frac{3}{2\alpha}g_{\mu\nu}\right) = 0 \ , \tag{2.43}$$

and for the difference of the corresponing equation outside the brane from the one inside it

$$\nabla^\nu W_{\mu\nu} = (\eta^2-1)\frac{\nabla^\nu\beta}{\beta}W_{\mu\nu} \ . \tag{2.44}$$

For general η, we see that, apart from the constraints (2.41), (2.43) and (2.44) that we also had in the topological ring regularised case, there are 2 more constraints from (2.40) and (2.42). Thus, since the total number of constrains is 11, we can in principle determine the 11 unknowns, $K_{\mu\nu}$ and β. However, it turns out that in cases of symmetry as the cosmological ansatz that we will study in the following, the system for general η becomes overconstrained.

III. COSMOLOGICAL EQUATIONS AND CONSISTENCY

In this section, we will study the cosmological equations for a codimension-2 brane. We will first study in which cases the system of the brane equations studied in the previous section closes. We will then move on by checking the consistency of the next order in the ϵ-expansion of the equations of motion. For this purpose we adopt the following metric ansatz for the four-dimensional metric $g_{\mu\nu}$ describing LFRW cosmology

$$ds^2 = -n^2(t,r)dt^2 + a^2(t,r)ds_\kappa^2 + dr^2 + L^2(t,r)d\theta^2 \ , \tag{3.1}$$

with the spatial metric for $\kappa = \pm 1, 0$ curvatures

$$ds_\kappa^2 = \frac{d\rho^2}{1-\kappa\rho^2} + \rho^2(d\phi^2 + \sin^2\phi \ d\omega^2) \ . \tag{3.2}$$

The expansion of n and a is

$$n(t,r) \ = \ 1 + Nr + \frac{1}{2}N_2 r^2 + \frac{1}{6}N_3(x)r^3 + \mathcal{O}(r^4) \ , \tag{3.3}$$

$$a(t,r) \ = \ a\left[1 + Ar + \frac{1}{2}A_2 r^2 + \frac{1}{6}A_3(x)r^3 + \mathcal{O}(r^4)\right] \ , \tag{3.4}$$

where with $a = a(t)$ we denote the scale factor on the brane. We will assume that the energy-momentum tensor on the brane is the one of perfect fluid $T_\mu^\nu = (-\rho, P, P, P)$. The energy density and pressure may contain contributions from the vacuum energy of the brane. We will not factor out these contributions until the next section.

A. The brane equations of motion

Let us now write the system of brane equations for the different cases discussed in the previous section. By the term brane equations, we mean the matching conditions supplemented with the constraints from the $\mathcal{O}(1/\epsilon)$ parts of the (rr) and $(r\mu)$ equations, and the codimension-1 constraints when appropriate.

For the case of topological boundary conditions with a smooth regularisation, we obtain that the equations (2.13), (2.15), (2.16) give the following system

$$\frac{\rho}{3\alpha(1-\beta)} = H^2 + \frac{\kappa}{a^2} + \frac{1}{2\alpha} - A^2 \ , \tag{3.5}$$

$$-\frac{\rho+3P}{6\alpha(1-\beta)} = \frac{\ddot{a}}{a} + \frac{1}{2\alpha} - fA^2 \ , \tag{3.6}$$

$$f \equiv \frac{N}{A} = \frac{3P}{\rho} \ , \tag{3.7}$$

$$2A\dot{A} + 2H(1-f)A^2 = \frac{\dot{\beta}}{\beta}\frac{\rho}{3\alpha(1-\beta)} \ . \tag{3.8}$$

In order to derive the equations (3.7) and (3.8), we have used the first one (3.5). The above system of equations has evidently one free function, which we can take to be $\beta(t)$. This function is expected to be fixed by the asymptotic dynamics of the bulk and cannot be determined by the local equations of motion.

For the case of topological boundary conditions with ring-type regularisation, we find that (3.8) is split in two parts. The equations (2.30) and (2.31) now give

$$2A\dot{A} + 2H(1-f)A^2 = 0 \ , \tag{3.9}$$

$$\dot{\beta}\,\rho = 0 \ , \tag{3.10}$$

From the last one, we will in the following choose the solution $\dot{\beta} = 0$, i.e. that the deficit angle is constant, since the other possible solution demands that the brane energy density vanishes ($\rho = 0$). In this case, the system of (3.5), (3.6), (3.7), (3.9) and (3.10) is closed, the cosmology on the brane is uniquely determined, and is not depending on the bulk dynamics but only on the local structure of the equations of motion.

The general boundary conditions on the other hand, have corresponding equations with the ones of the previous case, plus two more which have to do with the constraint (2.42) and the codimension-1 constraint (2.40). These read

$$\frac{\rho}{3\alpha(\eta^2-\beta)} = \frac{1-\beta}{\eta^2-\beta}(H^2 + \frac{\kappa}{a^2} + \frac{1}{2\alpha}) - A^2 \tag{3.11}$$

$$-\frac{\rho+3P}{6\alpha(\eta^2-\beta)} = \frac{1-\beta}{\eta^2-\beta}(\frac{\ddot{a}}{a} + \frac{1}{2\alpha}) - fA^2 \tag{3.12}$$

$$f \equiv \frac{N}{A} = \frac{3P - 3\alpha A^2(\eta^2-1)}{\rho + 6\alpha A^2(\eta^2-1)} \tag{3.13}$$

$$2A\dot{A} + 2H(1-f)A^2 = \frac{\dot{\beta}}{\beta}(\eta^2-1)A^2 \tag{3.14}$$

$$\dot{\beta}\left(\rho + 3\alpha\beta(1-\beta)(\eta^2-1)A^2\right) = 0 \tag{3.15}$$

$$\eta(\eta^2-1)(1+2f) = 0 \tag{3.16}$$

$$(1-\eta)\left[f + 3 - \frac{2\alpha}{3}(1+\eta+\eta^2)A^2(1+2f) + 2\alpha H^2(1+f) - 4\alpha\frac{\ddot{a}}{a}\right] = 0. \tag{3.17}$$

Obviously, for the topological case we arrive at the generic conclusions we explained in the previous section. Let us concentrate on the general case. From (3.16) we see that can either have $\eta = 0$ or $f = -1/2$ with general η. The choice $\eta = 0$ is inconsistent for general matter on the brane because of the codimension-1 constraint (3.17). On the other hand the choice $f = -1/2$ fixes the equation of state to $w = -1/6$ (for tensionless branes - for branes with tension the equation of state will be fixed to another constant value), therefore it is again not so interesting.

From the above we deduce that the only consistent boundary conditions with a purely codimension 2 defect are the topological ones. This theorem lies on two assumptions: first that the cosmological ansatz (3.1) does not contain any g_{tr} or $\mathrm{g}_{r\theta}$ components (see [26] for criticism of the (3.1) ansatz) and second that the metric is analytic at $r = 0$ so that it can be expanded in powers or r as in (2.18), (2.19), (3.3), (3.4). Let us also stress that, if we had had a true codimension-1 defect with non-zero $T_{\theta\theta}$, there would be no consistency issue *for any* η, since then the constraint (3.17) would have a matter part in the right hand side.

	JC	$\mathcal{O}(1/\epsilon)$	$\mathcal{O}(1)$
(rt)		★	■
(rr)		★	■
($\theta\theta$)			■
(tt)	★	■	▲
(ij)	★	■	▲

	Functions
★	a, N, A, β
■	N_2, A_2, β_2, $\bar{\beta}_2$ + the above
▲	N_3, A_3, β_3, $\bar{\beta}_3$ + the above

FIG. 3: The grouping of equations according to the various functions that are to be determined. On the left table, the first column has the codimension-2 junction conditions, while the other columns have the various orders in r of the Einstein equations. The $\mathcal{O}(1/\epsilon)$ component of the $(\theta\theta)$ Einstein equation vanishes identically.

B. Consistency of the next order equations

Up to this point, we have looked at the matching conditions supplemented by some equations of order $1/\epsilon$. These included apart from the brane metric, the deficit angle β and the extrinsic curvature $K_{\mu\nu}$. In the cosmological case the corresponding functions were the following: a, β, A and N. It is easy to see that in general the various terms in the ϵ-expansion of the equations of motion break into groups according to the unknown functions that they involve. The resulting grouping can be easily seen from Fig.3.

The next group of equations (which we call second order) is the one with the $\mathcal{O}(1/\epsilon)$ terms of the $(\mu\nu)$ equations and the $\mathcal{O}(1)$ equations of the $(\theta\theta)$, (rr) and (rt). These form a set of five equations which involve the next order variables A_2, N_2 and β_2. We will study them in the cosmological case to see if they are consistent. The strategy is to solve for them from three of these equations and then check that the two extra equations do not overconstrain the system. This check is done in view of [21], where it was claimed that the system is inconsistent (overconstrained). Here, we find that a careful inclusion of the $(L'/L)K_{\mu\nu}$ terms omitted in [21] gives consistency. Without loss of generality, we will make the simplifying assumption for this proof that the bulk is empty, *i.e.* $\mathcal{T}_{MN} = 0$, although some matter could be necessary in the interior of the defect for the ring regularised case. We will comment at the end of the section about what happens if we drop this assumption.

Let us look first for the topological case without a specified regularisation. We can first solve for A_2, N_2 and β_2 from the $\mathcal{O}(1/\epsilon)$ parts of (tt) and (ij) equations and from the $\mathcal{O}(1)$ part of $(\theta\theta)$ equation. As it can be seen in Appendix A, these equations of motion form a 3×3 system of linear algebraic in the second order variables equations. Using the brane equations (3.5)-(3.8), we can easily solve for A_2, N_2 and β_2 as

$$A_2 = \frac{15(1-\beta)^2 + 2(1-\beta)(\rho+9P) + 4\rho P - \frac{2\rho^2}{A^2}\frac{\dot{\beta}^2}{\beta^2}}{4\alpha(1-\beta)(\rho+9P)} \tag{3.18}$$

$$N_2 = \frac{-\rho(\rho+3P)^2 - \frac{45}{2}(1-\beta)^2(\rho+6P) + 3(1-\beta)\rho(\rho+9P) - 6\rho^2 P + \frac{\rho^3}{A^2}\frac{\dot{\beta}^2}{\beta^2}}{6\alpha(1-\beta)\rho(\rho+9P)} \tag{3.19}$$

$$\beta_2 = \beta A \frac{45(1-\beta)^2 - 2\rho(\rho+3P) - \frac{6\rho^2}{A^2}\frac{\dot{\beta}^2}{\beta^2}}{2\alpha\rho(\rho+9P)}. \tag{3.20}$$

The next step is to make sure that the $\mathcal{O}(1)$ parts of (rr) and (rt) equations do not overconstrain the system. These equations are written down in Appendix A. It is a tedious, but straightforward exercise to see that in fact they are automatically satisfied and therefore the system is consistent.

The next check we have to do concerns the ring regularisation in the topological matching conditions. Here we need to compute the equations also in the interior in order to see if the system is consistent. In this case, a simplification which will help us is that the equations for the first order variables imply that $\dot{\beta} = 0$. This simplifies considerably the equations. In the interior, having assumed that A_2 and N_2 are the same (continuity of $K'_{\mu\nu}$), we have the extra variable $\bar{\beta}_2$. The algebraic system in the interior of the $\mathcal{O}(1/\epsilon)$ parts of (tt) and (ij) equations and from the $\mathcal{O}(1)$ part of $(\theta\theta)$ equation, is the same as for the exterior with the only substitution

$$\bar{\beta}_2 = \frac{\beta_2}{\beta} . \tag{3.21}$$

The next step for the consistency of the system in the interior is to see the $\mathcal{O}(1)$ parts of (rr) and (rt) equations. The $\mathcal{O}(1)$ part of the (rt) equation is indeed then identically satisfied. However, the $\mathcal{O}(1)$ part of (rr) equation is not satisfied without the inclusion of a $\mathcal{T}_{rr}^{(in)}$ matter in the interior. In more detail, the matter we have to include in the interior of the ring in order not to overconstrain the system is given by the latter equation to be

$$\mathcal{T}_{rr}^{(in)} = \frac{-45(1-\beta)^2 + 6(1-\beta)(\rho - 3P) - 2\rho(\rho + 3P)}{24\pi\alpha\beta(1-\beta)} \equiv X \,. \tag{3.22}$$

Had we not taken a $\mathcal{T}_{rr}^{(in)}$, then the equations would only be satisfied for a specific relation of ρ and P of the brane matter. This would lead to overconstraining the system, and in the spirit of [21] to inconsistency. However, it is natural to assume that the thickening out of the thin codimension-2 defect gives rise to a non-zero energy momentum in the interior of it.

As far as the third order equations of the table in Fig.3 are concerned, there is no question about consistency. This is because there are three new variables A_3, N_3 and β_3 with two equations (in the smooth regularisation case), the $\mathcal{O}(1)$ parts of the (tt) and the (ij) equations. In the ring regularisation case, there is one more variable $\bar{\beta}_3$, but two more equations, as the number of equations is doubled (evaluated inside and outside the ring), so again every function can be determined. Therefore, we will not study them in more details.

Finally, let us comment about the differences in the above approach if we had allowed for general matter $\mathcal{T}_{MN}$ in the bulk. It is obvious that in this case the above solutions for the second order variables are altered. Then, for the smooth regularisation, we would obtain from the $\mathcal{O}(1)$ parts of the (rr) and (rt) equations that

$$\mathcal{T}_{r\theta} = 0 \ \ , \ \ \mathcal{T}_r^r = \mathcal{T}_\theta^\theta \ . \tag{3.23}$$

The same would hold for the exterior to the ring matter in the ring regularisation case. For the interior, the equations of motion are satisfied if

$$\mathcal{T}_{r\theta}^{(in)} = 0 \ \ , \ \ \mathcal{T}_\theta^{(in)\ \theta} = \mathcal{T}_\theta^\theta \ \ , \ \ \mathcal{T}_r^{(in)\ r} = X + \frac{1}{\beta}\mathcal{T}_\theta^\theta \ , \tag{3.24}$$

where X is the interior rr component of the energy-momentum tensor in the absence of other matter given in (3.22).

C. Vanishing extrinsic curvature limit

In the previous section, one could wonder about what happens in the limit of vanishing extrinsic curvature ($K_{\mu\nu}$), since several expressions are divided by A or N. In this subsection, we will look back to the original equations without dividing them by components of the extrinsic curvature. First the brane equations read for both types of regularisation that we considered

$$\frac{\rho}{3\alpha(1-\beta)} \ = \ H^2 + \frac{\kappa}{a^2} + \frac{1}{2\alpha} \ , \tag{3.25}$$

$$-\frac{\rho + 3P}{6\alpha(1-\beta)} \ = \ \frac{\ddot{a}}{a} + \frac{1}{2\alpha} \ , \tag{3.26}$$

$$\dot{\beta}\ \rho \ = \ 0 \ . \tag{3.27}$$

From the above we have two distinct choices. First that $\rho = 0$. Differentiating the Friedmann equation and using the acceleration equation, we find then that also $P = 0$. The solution then for the scale factor is de-Sitter space in all possible foliations ($\kappa = \pm 1, 0$). The second order constraints for the $\mathcal{O}(1)$ parts of the (00), (ij) equations and the $\mathcal{O}(1)$ part of the (rt) equation are trivially satisfied. On the other hand, the $\mathcal{O}(1)$ parts of the $(\theta\theta)$ and the (rr) equations are only satisfied with the inclusion of bulk matter. In both types of regularisations (smooth and ring) and for both inside and outside the ring in the ring regularisation case, the later equations give

$$\mathcal{T}_r^r = \mathcal{T}_r^r = \frac{15}{8\pi\alpha} \ , \tag{3.28}$$

which corresponds (when also looking at the $\mathcal{O}(1)$ parts of the $(\mu\nu)$ equations) to cosmological constant $\Lambda_6 = -5/2\alpha$. This value of the bulk cosmological constant in fact realises the Born-Infeld limit [27] as noted in [23].

The second distinct case is when $\dot{\beta} = 0$. Notice that then, the equations (3.25), (3.26) are the 4-dimensional standard LFRW equations[4]. Then, the second order constraints for the $\mathcal{O}(1)$ parts of the (00), (ij) equations and the

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[4] Parts of these observations can be found in the PhD thesis of Paul Bostock [28].

$\mathcal{O}(1)$ part of the (rt) equation are trivially satisfied for $\beta_2 = \bar{\beta}_2 = 0$. On the other hand, the $\mathcal{O}(1)$ parts of the $(\theta\theta)$ and the (rr) equations give constraints. In the smooth regularisation, they coincide and give a relation between A_2 and N_2. In the ring regularisation, the same holds for the outside problem. Inside the ring, the $\mathcal{O}(1)$ part of the $(\theta\theta)$ equation gives the same relation between A_2 and N_2. On the other hand, the $\mathcal{O}(1)$ part of the (rr) equation needs some extra $\mathcal{T}_{rr}$.

It is instructive to see the limit of this special case for Minkowski brane vacua, where $H = 0$ and $\kappa = 0$, with cosmological constant Λ_6 in the bulk. Then, the brane has tension $T_4 = \rho = -P = 3(1-\beta)/2$. In the smooth regularisation case, the relation between A_2 and N_2 is rather simple

$$N_2 = -3A_2 - \Lambda_6 \; . \tag{3.29}$$

In the ring regularisation, the $\mathcal{O}(1)$ part of the (rr) equation dictates that

$$\mathcal{T}_{rr}^{(in)} = -\frac{3}{4\pi}\left[\Lambda_6 + \Lambda_6 \frac{1-\beta}{\beta}\right] \; , \tag{3.30}$$

where the first addendum is the bulk cosmological constant contribution and the second is an extra contribution necessary for the regularisation to work. We can compare then, this special case with the known exact solution of [23] of the double Wick rotated black hole. As we see in Appendix B, the above reproduce the correct relation between the bulk cosmological constant and the coefficients of $K'_{\mu\nu}$. This example is important, because it violates the assumption of the form of the expansion of $K_{\mu\nu}$ that was considered in [18] (namely $K'_{\mu\nu} = 0$). The same assumption was subsequently used in [22] and led to erroneous constraints for the brane matter.

IV. CODIMENSION-2 COSMOLOGICAL EVOLUTION

Let us now revisit the brane equations and try to understand what kind of cosmological evolution can be obtained on the codimension-2 brane. Let us stress here that, although we proved that the equations of motion locally around the purely codimension-2 defect are consistent for the topological boundary conditions, one has to make sure that there are no singularities introduced when integrating the equations of motion away from the defect. Here, we will not carry out this formidable task of the bulk integration, but we will content ourselves with deriving the possible four dimensional cosmologies from just the local equations on the brane. As discussed in the previous sections, the brane equations of motion for the smoothly regularised topological case are

$$\frac{\rho}{3\alpha(1-\beta)} = H^2 + \frac{\kappa}{a^2} + \frac{1}{2\alpha} - A^2 \; , \tag{4.1}$$

$$-\frac{\rho+3P}{6\alpha(1-\beta)} = \frac{\ddot{a}}{a} + \frac{1}{2\alpha} - fA^2 \; , \tag{4.2}$$

$$f \equiv \frac{N}{A} = \frac{3P}{\rho} \; , \tag{4.3}$$

$$2A\dot{A} + 2H(1-f)A^2 = \frac{\dot{\beta}}{\beta}\frac{\rho}{3\alpha(1-\beta)} \; . \tag{4.4}$$

Here, we recognize the first two equations as a modified version of the four-dimensional Friedmann and acceleration equations. Note that β is generically a function of time which can make the effective four dimensional Planck scale time varying. A second important point is the appearance of an effective cosmological constant in the face of α, which is of geometric origin. A last difference from the standard four dimensional equations is the presence of the extrinsic curvature correction parametrised by A.

On the other hand, the continuity equation (2.17), which is not independent from the above, gives

$$\dot{\rho} + 3H(\rho + P) + \frac{\rho\,\dot{\beta}}{\beta(1-\beta)} = 0. \tag{4.5}$$

Note that, in general, there is not energy conservation for the brane matter since a varying β means inevitable energy exchange between the bulk and brane. This is in contrast with the $\eta = 0$ case where there is energy conservation [29]. The energy conservation equation (4.5) can be rewritten as

$$\frac{d}{dt}\left(\frac{\rho\beta}{1-\beta}\right) + 3H\frac{(\rho+P)\beta}{1-\beta} = 0, \tag{4.6}$$

which looks like the standard energy conservation equation for a redefined energy density and pressure as $(\rho, P) \to (\rho, P)\beta/(1-\beta)$. In fact, the five (four independent) coupled equations of motion, (4.1), (4.2), (4.3), (4.4), (4.5), which are the full information available to us at $r = 0$, are consistent with each other and constitute a non-closed system. In other words, one needs extra information coming from the bulk geometry in order to fix one of the functions, e.g. β, and then to solve fully the system. The solution in the bulk is no longer unique as in the case of codimension-1 brane cosmology and one has a family of bulk solutions parametrised by the angular deficit function β. Not all these bulk solutions will be acceptable. Certain of them will inevitably carry singularities away from the brane for example, and not only in the position of the brane at $r = 0$.

From the matching conditions (2.13), we can read the effective four-dimensional gravitational constant as

$$\kappa_4^2 = \left(\frac{3\kappa_6^2}{4\pi}\right)\frac{1}{\alpha(1-\beta)} \ , \tag{4.7}$$

where we have momentarily reintroduced the six-dimensional gravitational constant according to (2.1). As noted before, an energy exchange between bulk and brane means a time-varying gravitational constant through the variation of β. This variation is constrained during the early cosmology by the primordial abundances at the nucleosynthesis epoch, with a limit of this variation approximately $\frac{|\dot{G}|}{GH} \lesssim 0.2$ for $G = 8\pi\kappa_4^2$ (see [30] for details of this limit). This is constraining the variation of β as

$$\left|\frac{\dot{\beta}}{(1-\beta)H}\right| \lesssim 0.2 \ , \tag{4.8}$$

which is not a rather strong constraint. Further constraints come from the fact that the theory with varying β is similar to a scalar-tensor theory and therefore there will be strong constraints from solar system observations. Not knowing the full family of solutions in the bulk, we choose to consider the case where β is *approximately constant*. This, in fact, will be exactly the case if we restrict our analysis to the ring topological regularisation, where there is no energy exchange between the defect and the bulk. This case is bound to give us, at least seemingly, acceptable four-dimensional cosmology, and has the merit that the system of equations is closed and does not depend on undetermined (by the matching conditions) functions.

When β is constant, we can solve for the extrinsic curvature A for a given, constant equation of state for the brane matter. For that purpose, we split the energy density and pressure to a part which has the form of brane vacuum energy λ and part which describes brane matter

$$\rho = \lambda + \rho_m \quad , \quad P = -\lambda + P_m \ , \tag{4.9}$$

with the brane matter equation of state defined as $w = P_m/\rho_m$. Then, for $w \neq -1$, we can integrate (4.4) as

$$A^2 = \frac{C^2}{(\lambda+\rho_m)^2}\ \rho_m^{\frac{8}{3(1+w)}} \ , \tag{4.10}$$

with C^2 a positive integration constant. With this expression, the Friedmann and acceleration equations (4.1), (4.2) become

$$H^2 + \frac{\kappa}{a^2} = \frac{\kappa_4^2}{3}\rho_m + \left(\frac{\kappa_4^2}{3}\lambda - \frac{1}{2\alpha}\right) + \frac{C^2}{(\lambda+\rho_m)^2}\ \rho_m^{\frac{8}{3(1+w)}} \ , \tag{4.11}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa_4^2}{6}(1+3w)\rho_m + \left(\frac{\kappa_4^2}{3}\lambda - \frac{1}{2\alpha}\right) + 3\frac{C^2}{(\lambda+\rho_m)^3}(w\rho_m - \lambda)\rho_m^{\frac{8}{3(1+w)}} \ , \tag{4.12}$$

which show a non-trivial correction to the cosmological equations as a function of w. The magnitude of this correction depends on the integration constant C^2. In the following, we will try to see whether it is possible for large enough C^2, consistent with observations, to have some interesting modification to cosmology. From the integration of the continuity equation (4.6) for β =const. we obtain

$$\rho_m = \frac{\rho_0(1-\beta)}{\beta a^{3(1+w)}} \ , \tag{4.13}$$

which holds also for the special $w = -1$ case.

For $C^2 \neq 0$, our six-dimensional bulk geometry has a genuine curvature singularity at $r = 0$ (apart from the distributional one). In fact, this is to be expected from purely geometrical considerations (see [31]). Higher codimension defects, when considered in their zero width limit, will develop curvature singularities. These are expected

w	ρ_m	A^2
-1	const.	a^{-8}
$-2/3$	a^{-1}	a^{-6}
$-1/3$	a^{-2}	a^{-4}
0	a^{-3}	a^{-2}
$1/3$	a^{-4}	const.
1	a^{-6}	a^{4}

FIG. 4: The functional dependence on the scale factor of the energy density and the extrinsic curvature correction to the Friedmann equation for different equations of state of the brane matter in the self-accelerating case. From top to bottom we have the equations of state for cosmological constant, domain walls, cosmic strings, dust, radiation and stiff matter.

to be smoothable once we take finite width corrections into account. Therefore, we expect finite width effects to be important at the UV sector and our distributional approximation to break down even though this will not show up necessarily in the field equations themselves!

Note that equations (4.11), (4.12) are valid not only for β exactly constant, but also for a $\beta(t)$ which is slightly oscillating around a constant value. To see this, we note from (4.6) that (as far as $\rho_m + P_m > 0$) the quantity $u = \rho\beta/(1-\beta)$ has opposite monotonicity than a, and therefore, u can be used as a good time parameter. Note that for this case the solution (4.13) is not valid in general. From equations (4.1)-(4.4), we can obtain a differential equation for $d(H^2)/du$ containing also the function β. Using the small oscillatory behaviour of β, we get an autonomous equation for $d(H^2)/du$, whose integration gives equations (4.11), (4.12).

Let us now analyse some particular cases of interest.

A. Self-accelerating branes

The first interesting case is the one in which the tension of the brane is vanishing $\lambda = 0$. In this case, we see from (4.11) that there is an *asymptotic accelerating phase* for $\alpha < 0$. This acceleration is purely due to the geometric Gauss-Bonnet term and therefore is a case of self-acceleration. This corresponds to generalisation of the self-accelerating solutions of [23] with the addition of matter. Before proceeding, let us note that in order that these self-accelerating solutions are viable, one has to check that they are free from ghosts that are typical in five dimensional self-accelerating braneworlds [32]. The correction to the Friedmann equation is given by

$$A^2 = C^2\ \rho_m^{\frac{2(1-3w)}{3(1+w)}} = \frac{\tilde{C}^2}{a^{2(1-3w)}}\ . \tag{4.14}$$

Note that the second equality holds also for $w = -1$. In the table in Fig.4 we list the form of corrections that we obtain for different constant equations of state. It is worth noting four interesting limits of the cosmological evolution in this case.

First is the one where the brane is dominated by cosmological constant. This may go against the initial assumption that the tension of the brane vanishes, because of the ambiguity to separate the vacuum part of the energy density from the matter energy density. If a matter component behaving as inflaton or dark energy dominates at some period of the history of the Universe, it will have the behaviour that we note here. So, for this case the extrinsic curvature plays the role of a dark radiation squared term a^{-8} that will dominate the cosmology at early times.

A second interesting limit is when the extrinsic curvature on the brane vanishes $A = 0$. This limit is of particular interest since we have a regular six-dimensional geometry at the location of the brane at $r = 0$. Then, the codimension-2 cosmology is exactly the four-dimensional LFRW plus an effective geometric cosmological constant $\frac{1}{2\alpha}$ which is positive for negative α. This is the generalised version of the solutions of [23] which were obtained in the case of pure tension. We see clearly that even in the absence of matter $\rho = 0$ and $\kappa = 0$ we have a non-zero H^2, hence these solutions are self-accelerating.

A third interesting limit is when the matter equation of state is that of radiation $w = 1/3$. Then, the extrinsic curvature correction is of the form of constant vacuum energy. One would be tempted to use this vacuum energy to drive early Universe inflation. This seems to be possible, but around matter-radiation equality the Universe will be dominated by this new vacuum energy, making the phenomenology of the model problematic.

A fourth interesting case is when the matter equation of state is that of dust $w = 0$. Then the extrinsic curvature correction is of the form of curvature (a^{-2}). In this case, we can see a possibility of having an observable signature

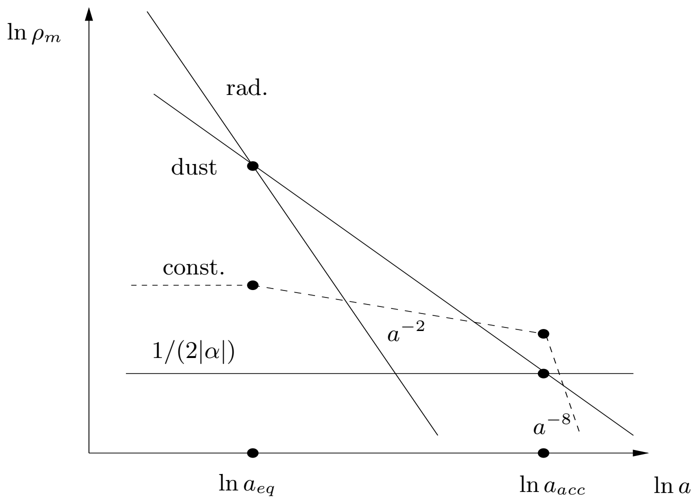

FIG. 5: The log-log evolution of the energy densities as a function of the scale factor. With solid lines, there are the contributions of the standard matter and the geometrically induced $1/(2|\alpha|)$ cosmological constant. With dashed lines are the evolutions of the extra component ($\propto A^2$) in the Friedmann equation due to the extrinsic curvature of the brane. The dependence of this extra component on the scale factor is noted. This dependence changes whenever a new matter component becomes dominant, *i.e.* at matter domination at a_{eq} and at the cosmological constant domination at a_{acc}.

of the codimension-2 cosmology, if the constant C^2 is chosen so that there is a brief period of curvature domination around the vacuum energy domination period. This scenario is depicted in Fig. 5. In order to obtain the standard epochs, *i.e.* matter domination and then cosmological constant domination, we need to have that $\rho_m(a_{eq}) > 1/(2|\alpha|)$ and $(\kappa_4^2/3)\rho_m(a_{eq}) > A^2(a_{eq})$, where a_{eq} is the scale factor at radiation-matter equality and a_{acc} the one at the moment of cosmological constant domination. In order that the brief period of curvature domination is observable, we need $A^2(a_{acc}) > 1/(2|\alpha|)$. Therefore combining these inequalities we see that we need

$$\frac{1}{2|\alpha|} < A^2(a_{acc}) < \frac{\kappa_4^2}{3}\rho_m(a_{eq}) \ . \tag{4.15}$$

Then, a straightforward manipulation with the help of (4.13), (4.14) provides us with the following constraint on C^2

$$\left(\frac{\rho_0(1-\beta)}{\beta}\right)^{-2/3} \frac{a_{acc}^2}{2|\alpha|} < C^2 < \frac{\kappa_4^2}{3}\left(\frac{\rho_0(1-\beta)}{\beta}\right)^{1/3}\left(\frac{a_{acc}^2}{a_{eq}^3}\right) \ . \tag{4.16}$$

It is not yet clear if such a brief period of curvature domination is phenomenologically viable and it requires certainly further study to see the potential observational signatures of such a case.

B. Self-tuning branes

A second interesting case is when self-tuning can be realised. For that possibility, the topological quantity β is used to accurately cancel the vacuum energy contribution of the brane to the effective cosmological constant. In more details, as discussed in [23], by tuning β one can make the two vacuum energy contributions, the geometrical one $1/2|\alpha|$ and the brane tension one λ, cancel

$$\frac{\kappa_4^2}{3}\lambda - \frac{1}{2\alpha} = 0 \ . \tag{4.17}$$

This limit is orthogonal to the self-accelerating case that we studied before. Such kind of self-tuning solutions, found in [23], have of course to be checked for their stability. The effective Friedmann equation for the present case is given by the expression

$$H^2 + \frac{\kappa}{a^2} = \frac{\kappa_4^2}{3}\rho_m + \frac{C^2}{(\frac{3}{2\alpha\kappa_4^2}+\rho_m)^2}\ \rho_m^{\frac{8}{3(1+w)}} \ , \tag{4.18}$$

which includes a non-trivial correction beyond the standard linear to energy density term. Let us note here that the actual self-tuning is not visible from this Friedmann equation. This is because the present is valid for $\beta = \text{const.}$ Instead, one should consider the time-varying deficit angle case in order to see how (4.17) can be dynamically achieved at the early Universe evolution. Since before nucleosynthesis there are no constraints on the variation of Newton's constant, one could have an acceptable cosmology with varying β. However, then, the brane equations do not close and the mechanism of self-tuning should be dictated by bulk boundary conditions. The study of such a case is beyond the scope of the present paper.

V. CONCLUSIONS

In this paper we have demonstrated that a distributional treatment of codimension-2 branes is consistent and possible if one considers the full Lovelock gravity in six dimensions. Higher dimensional GR cannot describe infinitesimally thin codimension-2 defects[5] for it does not present the differential complexity to allow such distributional terms [19]. Furthermore, we have shown that it is the higher order gravity terms in the bulk action that guarantee the presence of ordinary GR gravity on the brane. This, up to now, was hinted by the general form of the junction conditions [18], [19], but never demonstrated as a consistent solution of the full bulk and brane field equations. A nice example where the importance of the full bulk field equations becomes essential is the "apparent" choice of junction conditions [19]. In principle, at the level of distributions, differing mathematical regularity can give differing matching conditions, both a priori completely consistent. It is only upon considering the bulk field equations (in the spirit of [20]) that we see that the general junction conditions introduced in [18] inevitably need codimension-1 matter sources. We have therefore established a genuine physical difference between the topological and general junction conditions. The former are pure codimension-2 junction conditions, whereas the latter are mixed codimension-1 and codimension-2 junction conditions. It would be interesting to see the relation of the general matching conditions with work on intersecting branes and cascading cosmologies [33].

We have studied the cosmology of conical branes and have found several interesting features. Firstly, quite generically the higher order terms give ordinary LFRW equations, as though one included an induced gravity term on the brane [34]. This is quite different from codimension-1 cosmology [13], where the ordinary behaviour is recovered only at late times. Corrections to this standard evolution are threefold: there is a geometric acceleration scale related to $1/\alpha$, which restates the problem of a minute cosmological constant as a gravitational hierarchy between the bulk Gauss-Bonnet and Einstein-Hilbert term. Secondly, extrinsic curvature corrections are apparent and they are dependent of the equation of state of the perfect-fluid matter. In other words, as the brane evolves in the bulk, a dust equation of state can lead to an extra component behaving as curvature, or a radiation equation of state to a cosmological constant term, and so on. To put it in a nutshell, each matter fluid introduces two differing fluid components in the modified LFRW equations. Last, but not least, as the brane evolves it can generically radiate in the bulk. Again, this is unlike the vacuum bulk codimension-1 braneworlds.

The above characteristics can be used to constrain the model in question with respect to cosmological observations. Furthermore, on the theoretical side it would be interesting to have particular bulk solutions manifesting the cosmology evolution we have found, and in particular setting the boundary conditions in order to fix β. The most important information we have given here in this direction is that it is pure six-dimensional Gauss-Bonnet gravity that gives the dynamical LFRW-like evolution on the brane. The six-dimensional Einstein term gives the possibility for a phase of geometric acceleration. Furthermore, a varying β could have interesting consequences for a cosmological self-tuning scenario following equation (4.17). However, note that the evolution in β would then leave an imprint on the cosmological evolution equations. These are amongst the open, interesting questions that this work puts forward and which we hope will be answered in the near future.

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[5] There is a slight caveat in this argument as it is unknown mathematically if breaking of axial symmetry may remedy this problem.

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APPENDIX A: THE SECOND ORDER EQUATIONS OF MOTION

In this Appendix we will list the equations of motion which involve the second order variables A_2, N_2 and β_2. As we saw in the main text, they depend on the matching conditions and also the regulatisation that we have taken. For this Appendix, we assume that the bulk is empty, *i.e.* $\mathcal{T}_{MN}=0$, and furthermore that the three dimensional space is flat, *i.e.* $\kappa=0$.

Let us first see for the topological matching conditions without an explicit regularisation. The $\mathcal{O}(1/\epsilon)$ parts of (tt) and (ij) equations and the $\mathcal{O}(1)$ part of $(\theta\theta)$ equation form an algebraic set of equations for the three second order variables. We can write the system in the suggestive form

$$D\begin{pmatrix} A_2 \\ N_2 \\ \beta_2 \end{pmatrix} = C \ , \tag{A1}$$

with the following coefficient matrix

$$D=\begin{pmatrix} 2 & 0 & \frac{1}{A\beta}\left(H^2+\frac{1}{2\alpha}-A^2\right) \\ 2\left(1+\frac{A}{N}\right) & 2\frac{A}{N} & -\frac{1}{N\beta}\left(H^2+\frac{3}{2\alpha}-A^2-2AN+2\frac{\ddot a}{a}\right) \\ \frac{2\alpha}{3}\left(H^2+\frac{3}{2\alpha}-A^2-2AN+2\frac{\ddot a}{a}\right) & \frac{2\alpha}{3}\left(H^2+\frac{1}{2\alpha}-A^2\right) & 0 \end{pmatrix}\ , \tag{A2}$$

and the following vector of the right hand side

$$C=\begin{pmatrix} H^2+\frac{3}{2\alpha}-A^2 \\ H^2+\frac{3}{2\alpha}-3A^2+2\frac{A}{N}\left(\frac{\ddot a}{a}+\frac{3}{2\alpha}\right) \\ H^2-A^2+\frac{\ddot a}{a}-AN+\frac{2\alpha}{3}\left[A^3N+H^2(2A^2+2N^2-5AN)+4(A+N)\dot A H+2\dot A^2+\frac{\ddot a}{a}(A^2+H^2)\right] \end{pmatrix}\ . \tag{A3}$$

The solution of this system is given in Sec.III B. There are two more equations that have to be verified from the grouping of Fig.3. First, the $\mathcal{O}(1)$ part of the (rr) equation. This is again an algebraic equation in the set of (β_2,N_2,A_2) variables

$$\begin{aligned}
&\left[1-2\alpha A(A+2\alpha N)+\frac{2\alpha}{3}\left(H^2+2\frac{\ddot a}{a}\right)\right]A_2+\left[1-6\alpha A^2+2\alpha H^2\right]N_2 \\
&\qquad +\frac{A}{3\beta}\left[\frac{3}{2}-\alpha A^2-3\alpha AN+\frac{1}{2}\frac{N}{A}+\alpha H^2+\alpha\frac{N}{A}H^2+2\alpha\frac{\ddot a}{a}\right]\beta_2= \\
&=H^2-AN+\frac{1}{3}N^2+\frac{\ddot a}{a}+\frac{1}{3}\frac{\ddot\beta}{\beta}-2\alpha A^2\left(A^2+\frac{5}{3}AN+N^2\right)+2\alpha H^2\left(\frac{1}{3}A^2+\frac{1}{3}AN+N^2\right) \\
&\qquad -4\alpha H\left(A\dot A+\frac{1}{3}N\dot A-\frac{1}{3}A\dot N\right)+\frac{2\alpha}{3}\frac{\ddot a}{a}\left(A^2+4AN+H^2\right)-\frac{4\alpha}{3}A\ddot A+\frac{2\alpha}{3}\frac{\ddot\beta}{\beta}(H^2-A^2)\ .
\end{aligned}\tag{A4}$$

On the other hand, the $\mathcal{O}(1)$ part of the (rt) equations is a differential (Bianchi) equation with respect to the second order variables

$$\begin{aligned}
&4\alpha\dot A_2-\frac{2\alpha}{A\beta}\left(H^2+\frac{1}{2\alpha}-A^2\right)\dot\beta_2+4\alpha\left(\frac{\dot A}{A}+\left(2-\frac{N}{A}\right)H\right)A_2-4\alpha HN_2+\frac{2\alpha A}{\beta}\left(H\left(1+\frac{N}{A}\right)+\frac{\dot A}{A}\right)\beta_2= \\
&\qquad =3H\left(1-\frac{N}{A}\right)+3\frac{\dot A}{A}+2\alpha H\left(H^2\left(1-\frac{N}{A}\right)+3A^2-2N^2-AN\right)+2\alpha\frac{\dot A}{A}(H^2+3A^2+2AN)\ .
\end{aligned}\tag{A5}$$

For the case of ring regularisation of the topological case, we have as second order variables the set (A_2, N_2, β_2, $\bar{\beta}_2$). In the exterior of the ring, the system of equations in the same as above, with the simplification that $\dot{\beta}=0$. In the interior of the ring, by taking the simple relation (3.21), we can trade $\bar{\beta}_2$ for β_2. Then, we can easily see that we have again the same system (A1), with (A2), (A3). On the other hand, the $\mathcal{O}(1)$ part of the (rr) equation changes to

$$\left[1-2\alpha A(A+2\alpha N)+\frac{2\alpha}{3}\left(H^2+2\frac{\ddot{a}}{a}\right)\right]A_2+\left[1-6\alpha A^2+2\alpha H^2\right]N_2$$

$$+\frac{A}{3\beta}\left[\frac{3}{2}-\alpha A^2-3\alpha AN+\frac{1}{2}\frac{N}{A}+\alpha H^2+\alpha\frac{N}{A}H^2+2\alpha\frac{\ddot{a}}{a}\right]\beta_2=$$

$$=\beta H^2-\beta AN+\frac{1}{3}N^2+(1-\beta)A^2+\beta\frac{\ddot{a}}{a}-2\alpha A^2\left(A^2+\frac{(6-\beta)}{3}AN+N^2\right)+2\alpha H^2\left(\frac{1}{3}A^2+\frac{(2-\beta)}{3}AN+N^2\right)$$

$$-4\alpha H\left(A\dot{A}+\frac{1}{3}N\dot{A}-\frac{1}{3}A\dot{N}\right)+\frac{2\alpha}{3}\frac{\ddot{a}}{a}\left((2-\beta)A^2+4AN+\beta H^2\right)-\frac{4\alpha}{3}A\ddot{A}\ . \quad \text{(A6)}$$

Finally, the $\mathcal{O}(1)$ part of the (rt) equation also changes to

$$\frac{4\alpha}{\beta}\dot{A}_2-\frac{2\alpha}{A\beta^2}\left(H^2+\frac{1}{2\alpha}-A^2\right)\dot{\beta}_2+\frac{4\alpha}{\beta}\left(\frac{\dot{A}}{A}+\left(2-\frac{N}{A}\right)H\right)A_2-\frac{4\alpha}{\beta}HN_2+\frac{2\alpha A}{\beta^2}\left(H\left(1+\frac{N}{A}\right)+\frac{\dot{A}}{A}\right)\beta_2=$$

$$=3H\left(1-\frac{N}{A}\right)+3\frac{\dot{A}}{A}+2\alpha H\left(H^2\left(1-\frac{N}{A}\right)+\left(\frac{4}{\beta}-1\right)A^2-\frac{2}{\beta}N^2-\left(\frac{2}{\beta}-1\right)AN\right)$$

$$+2\alpha\frac{\dot{A}}{A}(H^2+\left(\frac{4}{\beta}-1\right)A^2+\frac{2}{\beta}AN)\ . \quad \text{(A7)}$$

APPENDIX B: COMPARISON WITH THE DOUBLE WICK ROTATED BLACK HOLE SOLUTION

In this Appendix we will compare the explicit simple solution of a system where the bulk has a Gauss-Bonnet term and cosmological constant, with the constraint that we noted in sec. III C. We will show that firstly, in this solution it is $K'_{\mu\nu}\neq 0$, violating the assumptions of [18] and furthermore that the leading terms in the r expansion of $K_{\mu\nu}$ is the one found in the main text from the local consistency conditions.

The solution of a Gauss-Bonnet AdS black hole with horizon of toroidal topology is

$$ds^2 = -V(R)dt^2+\frac{dR^2}{V(R)}+R^2\delta_{mn}dx^m dx^n\ , \quad \text{(B1)}$$

$$V(R) = \frac{R^2}{2\alpha}\left[1-\sqrt{1-4\alpha\left(k^2-\frac{\mu}{R^5}\right)}\right]\ , \quad \text{(B2)}$$

with $k^2=-\Lambda_6/10$. Let us assume that $\alpha>0$, $\mu>0$, $4\alpha k^2<1$. Then, there is only one singularity at $r=0$, shielded by an horizon at $R_H=(\mu/k^2)^{1/5}$.

Let us now make a double Wick rotation $t\to i\theta$ and $x^0\to it$. Then, we obtain the metric

$$ds^2=R^2\eta_{\mu\nu}dx^\mu dx^\nu+\frac{dR^2}{V(R)}+V(R)d\theta^2\ . \quad \text{(B3)}$$

We can go to a Gaussian-Normal system by means of the transformation $R'=\sqrt{V}$ and obtain the metric

$$ds^2=R(r)^2\eta_{\mu\nu}dx^\mu dx^\nu+dr^2+V(R(r))d\theta^2\ . \quad \text{(B4)}$$

Then, the comparison with the previous presentation is straightforward

$$L=\sqrt{V}\ , \quad \text{(B5)}$$

$$K_{\mu\nu}=\frac{1}{2}(R^2)'\eta_{\mu\nu}=R\sqrt{V}\eta_{\mu\nu}\ , \quad \text{(B6)}$$

with the requirement that $R_H=1$ (thus $\mu=k^2$) in order to have the induced metric $g_{\mu\nu}|_{R_H}=\eta_{\mu\nu}$. The coordinate transformation can be written locally as

$$R=1+\frac{5}{4}k^2r^2+\mathcal{O}(r^4)\ . \quad \text{(B7)}$$

Therefore, the above quantities are

$$L = \frac{5}{2}k^2 r + \mathcal{O}(r^3) \ , \tag{B8}$$

$$K_{\mu\nu} = \frac{5}{2}k^2 r \eta_{\mu\nu} + \mathcal{O}(r^2) \ , \tag{B9}$$

from which we confirm that $\beta_2 = 0$ and verify also the correctness of (3.29). This is because $K'_{\mu\nu}|_{r=0} = (-N_2, A_2, A_2, A_2)$, with $N_2 = A_2$ because of the Minkowski symmetry. Then, the relation (3.29) gives

$$A_2 = N_2 = -\frac{\Lambda_6}{4} \ . \tag{B10}$$

Moreover, from (B8) we obtain β, in the case that θ is normalized so that $\theta \in [0, 2\pi)$.

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