Non-integrability of geodesic flow on certain algebraic surfaces

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Abstract. This paper addresses an open problem recently posed by V. Kovloz in this journal: a rigorous proof of the non-integrability of the geodesic flow on the cubic surface $xyz = 1$. We prove this result using the Morales-Ramis theorem and Kovacic algorithm. We also consider some consequences and extensions of this result.

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1. Introduction

In two recent papers [6] and [7] Kozlov posed the following open problem: to rigorously prove the non-integrability (in the sense of Louiville) of the geodesic flow on the surface $xyz = 1$. In what follows we will exploit the Hamiltonian nature of the geodesic equations by examining the variational equations about a planar geodesic. The crucial theorem we shall make use of is due to Morales-Ruiz and Ramis, which we quote from [9]:

Theorem (Morales-Ramis). *For a $2n$ dimensional Hamiltonian system assume there are n first integrals which are meromorphic, in involution and independent in the neighborhood of some non-constant solution. Then the identity component of the differential Galois group of the normal variational equation (NVE) is an abelian subgroup of the symplectic group.*

On two-dimensional manifolds the normal variational equation (which we shall derive in the next section) is a second order linear ordinary differential equation with meromorphic coefficients. In particular, we will see that the NVE's of interest are Fuchsian. To show that the geodesic flow is not meromorphic-integrable it suffices to show that the NVE is not solvable in the sense of differential Galois theory: the identity component of the differential Galois group of the NVE is not abelian. This means we cannot "build" the solutions from the field of meromorphic functions by adjoining integrals, exponentiation of integrals, or algebraic functions of elements of the field of meromorphic functions. To test this we make use of the Kovacic algorithm.

Before we state the Kovacic algorithm we note that this algorithm is very robust and can treat any second order linear ODE with rational coefficients, however in the present work we will only need a very limited portion of the algorithm. Thus to save space we will present a much abbreviated version and refer the reader to the original article of Kovacic [5] and the reduced version appropriate for Fuchsian ODE's in Churchill and Rod [4], whose notation we will follow most closely below.

Consider a linear ode of the following form

$$\frac{d^2\xi}{dy^2} = \xi'' = r(y)\xi, \quad r(y) \in \mathbb{C}(y). \tag{1}$$

If the equation is Fuchsian, that is it admits only regular singular points, then we can decompose $r(y)$ as

$$r(y) = \sum_{j=1}^{k} \frac{\beta_j}{(y-a_j)^2} + \sum_{j=1}^{k} \frac{\delta_j}{y-a_j},$$

where k is the number of finite regular singular points at locations $y = a_j$. When $\sum \delta_j = 0$ then $y = \infty$ is also a regular singular point, with $\beta_\infty = \sum(\beta_j + \delta_j a_j)$. The indicial exponents are

$$\tau_j^{\pm} = \tfrac{1}{2}\left(1 \pm \sqrt{1+4\beta_j}\right), \quad \tau_\infty^{\pm} = \tfrac{1}{2}\left(1 \pm \sqrt{1+4\beta_\infty}\right)$$

at $y = a_j$ and $y = \infty$ respectively.

Kovacic proved in [5] that there are only 4 possible cases for the differential Galois group of (1). We will see in Section 3 that we can rule out two of these cases immediately (cases I and III), and as such we will present only necessary conditions for these cases.

Theorem (Kovacic). *Let $\mathcal{G}$ be the differential Galois group associated with* (1)*, and note $\mathcal{G} \subset SL(2,\mathbb{C})$. Then only one of four cases can hold:*

(I) $\mathcal{G}$ is triangulisable (or reducible). A necessary condition for this case to hold is that, defining the 'modified exponents' $\alpha^{\pm}$ as

$$\alpha_j^{\pm} = \tau_j^{\pm} \text{ if } \beta_j \neq 0; \quad \alpha_j^{\pm} = 1 \text{ if } \beta_j = 0 \text{ and } \delta_j \neq 0; \quad \alpha_j^{\pm} = 0 \text{ if } \beta_j = \delta_j = 0,$$
$$\alpha_\infty^{\pm} = \tau_\infty^{\pm} \text{ if } \beta_\infty \neq 0; \quad \alpha_\infty^{+} = 1, \alpha_\infty^{-} = 0 \text{ if } \beta_\infty = 0,$$

there is some combination

$$d = \alpha_\infty^{\pm} - \sum_{j=1}^{k} \alpha_j^{\pm} \in \mathbb{N}_0 = 0, 1, 2, 3, \ldots$$

(II) $\mathcal{G}$ is conjugate to a subgroup of the 'DP' group, in the terminology of Churchill and Rod. A necessary and sufficient condition for this case to hold is that, defining the following sets

$$\begin{aligned} &E_j = \{2 + e\sqrt{1+4\beta_j}, e = 0, \pm 2\} \cap \mathbb{Z} \text{ if } \beta_j \neq 0; \\ &E_j = \{4\} \text{ if } \beta_j = 0, \delta_j \neq 0; \quad E_j = \{0\} \text{ if } \beta_j = \delta_j = 0; \\ &E_\infty = \{2 + e\sqrt{1+4\beta_\infty}, e = 0, \pm 2\} \cap \mathbb{Z} \text{ if } \beta_\infty \neq 0; \\ &E_\infty = \{0, 2, 4\} \text{ if } \beta_\infty = 0, \end{aligned} \tag{2}$$

there is some combination of $e_j \in E_j$ *and* $e_\infty \in E_\infty$*, not all even integers, so that*

$$d = \frac{1}{2}\Big(e_\infty - \sum_{j=1}^{k} e_j\Big) \in \mathbb{N}_0$$

and there exists a monic polynomial $P(y)$ *of degree* d *which solves the following ODE:*

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0, \quad \theta = \frac{1}{2}\sum_{j=1}^{k}\frac{e_j}{y - a_j}. \tag{3}$$

(III) $\mathcal{G}$ *is finite. A necessary condition for this case to hold is that all indicial exponents* $\tau_j^{\pm}$ *and* $\tau_\infty^{\pm}$ *are rational.*

(IV) $\mathcal{G} = SL(2, \mathbb{C})$*, whose identity component is not abelian and therefore* (1) *is not solvable.*

In the next section we will derive the NVE about a planar geodesic on the Monge patch with a plane of symmetry; in Section 3 we will prove, using the theorems presented in this Section, that the NVE is not solvable and therefore the geodesic flow is not Louiville integrable on the surface $xyz = 1$. In Section 4 we will consider some extensions and consequences of this result, and in Section 5 we finish with some conclusions.

We note that the approach followed in this paper has been used to prove the non-integrability of a number of problems in mechanics and celestial mechanics (see, for example, [10], [11], [3], [14], [1], [12], [8], [2], to name but a few), but with the exception of another work by the author [15] this approach is novel in examining geodesic flow.

2. Derivation of the NVE

The key feature of the surface $xyz = c$, where w.l.o.g. we can set $c = 1$, which facilitates this analysis is that by a simple rearrangement and rotation of $\pi/4$ about the z-axis we can write $z = 1/(x^2 - y^2)$, or more generally

$$z = f(x, y), \quad f_{,x}(0, y) = 0. \tag{4}$$

This means the surface is a Monge patch (or graph) with a plane of symmetry (or invariant plane), the y-z plane. The surface is actually made up of 4 identical components, and to demonstrate the non-integrability of the geodesics of the surface we need only demonstrate this property on one component. We will restrict our attention to the quadrant

$$\{x, y, z \in \mathbb{R}^3 : |x| > |y|, x > 0\}$$

which immediately rules out any possible divergences in f.

To keep the approach of this section general and to facilitate the analysis of Section 4 we will derive the NVE on the Monge patch with a plane of symmetry as in (4).

Lemma 1. *The normal variational equation about the planar geodesic on the Monge patch* (4) *is*

$$\ddot{\xi} + \mathcal{K}|_0 \xi = 0,$$

where $\mathcal{K}|_0$ *is the Gauss curvature evaluated along the planar geodesic.*

Proof. Using the standard parameterisation $(x, y, f(x, y))$ and resulting line element we may calculate the Christoffel symbols and geodesic equations:

$$\ddot{x}^a + \sum_{b,c} \Gamma^a_{bc}\, \dot{x}^b \dot{x}^c = 0, \quad x^a = (x, y)$$

where a dot denotes differentiation w.r.t. arc-length s. The $x = 0$ plane is invariant since $\Gamma^x_{yy} = 0$ when $f_{,x}(0, y) = 0$, and thus there is a planar geodesic $(0, \tilde{y}(s), f(0, \tilde{y}(s)))$ where $\tilde{y}$ solves $(1 + f_{,y}(0, \tilde{y})^2)\dot{\tilde{y}}^2 = 1$. Linearizing the geodesic equations about this planar geodesic the normal variational equation will simply be the variation in the x-direction, namely

$$\ddot{\xi} + \left(\Gamma^x_{yy,x}\dot{\tilde{y}}^2\right)\Big|_0 \xi = \ddot{\xi} + \left(\frac{f_{,xx}f_{,yy}}{(1 + f_{,y}^2)^2}\right)\Big|_0 \xi = \ddot{\xi} + \mathcal{K}|_0 \xi = 0.$$

□

The problem with this equation is that the coefficient is a function of $\tilde{y}(s)$, which is defined implicitly as a solution of $(1 + f_{,y}{}^2)\dot{\tilde{y}}^2 = 1$. Clearly $\tilde{y}$ will also parameterise the planar geodesic and thus we make the change of independent variable to $\tilde{y}$, calculating derivatives such as (dropping the tildes)

$$\frac{d}{ds} = \dot{y}\frac{d}{dy} = \frac{1}{\sqrt{1 + f_{,y}{}^2}}\frac{d}{dy}$$

and so on, to arrive at the NVE (dropping the 0 subscript)

$$\xi'' - \left(\frac{f_{,y}f_{,yy}}{1 + f_{,y}{}^2}\right)\xi' + \left(\frac{f_{,yy}f_{,xx}}{1 + f_{,y}{}^2}\right)\xi = 0. \tag{5}$$

If, for a given surface $z = f(x, y)$ with $f_{,x}(0, y) = 0$, this NVE is not solvable in the sense of differential Galois theory as described in Section 1, then the geodesic flow on that surface is not integrable. This is precisely what we will show in the next Section for the surface $xyz = 1$. Before we do however, we can make a comment about (5):

Notice that the equation is of the form $\xi'' - f_{,y}Q(y)\xi' + f_{,xx}Q(y)\xi = 0$. If $f_{,xx}/f_{,y} = 1/y$, then (5) would have the simple solution $\xi_1 = c_1 y$ from which we could construct a second solution via integrals and exponentiation of integrals of meromorphic functions. This leads us to consider solutions of the PDE

$$y f_{,xx} - f_{,y} = 0 \tag{6}$$

(where we evaluate the derivatives of f at $x = 0$) as candidates for surfaces with integrable geodesic flow. Examples include well-known integrable surfaces such as $f(x, y) = f(x^2 + y^2)$ and more interesting surfaces such as $f = \cos(\omega x)e^{-\omega^2 y^2/2}$. But we should not divert too much attention to (6): a surface which solves this equation need not have integrable geodesic flow, it merely passes this integrability test.

3. Non-integrability of the surface $xyz = 1$

In the case of $z = f(x, y) = (x^2 - y^2)^{-1}$, the NVE takes the form

$$\xi'' - \frac{18(2 + 3y^6)}{y^2(y^6 + 4)^2}\xi = 0, \tag{7}$$

where we have removed the ξ' term from (5) via the standard transformation [5], and we extend the independent variable to the complex domain. There are 8 regular singular points, $a_j = \{0, \rho_1, \ldots, \rho_6,\}$ and ∞ where ρ_i denotes the 6 roots of $y^6 + 4 = 0$ symmetrically distributed about the circle of radius $\sqrt[6]{4}$ centred on the origin. We find the β coefficients are

$$\beta_j = \left\{ -\frac{9}{4}, \frac{5}{16}, \ldots, \frac{5}{16} \right\}, \quad \beta_\infty = 0,$$

and only $\delta_\infty = 0$. We can see immediately that $\tau_0^\pm = \frac{1}{2}(1 \pm i\sqrt{8})$, and therefore case III of the Kovacic algorithm can be ruled out (the finite case). What's more, none of the other $\tau_j^\pm$ are complex so case I of the algorithm can also be ruled out (the triangulisable case).

Case II is more problematic. The sets described in (2) are

$$E_0 = \{2\}, \quad E_{1\ldots6} = \{2, 5, -1\}, \quad E_\infty = \{0, 2, 4\}.$$

There are 21 combination of the elements of these sets leading to each of $d = 0$ and $d = 1$, and 1 leading to each of $d = 2, 3, 4$ (for example, $d = \frac{1}{2}\big(4-(2-1-1-1-1-1-1)\big) = 4$). For each of these combinations we must attempt to construct a monic polynomial of order d that satisfies (3). This can be done using a computer algebra system such as Mathematica; the calculations are tedious rather than difficult. By checking each combination we can see that there is no polynomial P satisfying (3). We can now state the main result of this paper:

Theorem 1. *The geodesic flow on the surface $xyz = 1$ is not integrable in the sense of Louiville with meromorphic first integrals.*

Proof. The differential Galois group of the normal variational equation (7) does not fall into case I, II or III of Kovacic's algorithm, as we have shown above. Therefore we must have $\mathcal{G} = SL(2, \mathbb{C})$, the identity component of which (also $SL(2, \mathbb{C})$) is not abelian. By the Morales-Ramis theorem of Section 1 this means the geodesic equations are not Louiville integrable with meromorphic first integrals. $\square$

4. Extensions and limitations

It seems natural to ask can we use the same techniques to examine other surfaces similarly defined. We will consider two generalisations, $x^n y^n z^n = 1$ and $x^n y^n z = 1$.

4.1. Surfaces of the form $x^n y^n z^n = 1$

While it might seem "obvious" that $xyz = 1$ and $(xyz)^n = 1$ are "the same", care needs to be taken. If n is an even integer then the surface will have twice as many components as when n is odd; for example the point $(1, 1, -1)$ is on $x^2y^2z^2 = 1$ but not on $xyz = 1$. To show they are isometric would require the calculation of the first fundamental form, which is not well-defined for algebraic surfaces, i.e. surfaces defined implicitly by $F(x, y, z) = c$. We can calculate the Gauss curvature using the following expression [13] (here ∇F and $H(F)$ are the gradient and Hessian of F respectively, and the norms are w.r.t. the ambient Euclidean space)

$$K = -\frac{\begin{vmatrix} H(F) & \nabla F \\ \nabla F^T & 0 \end{vmatrix}}{|\nabla F|^4}$$

which we find to be independent of the value of n, but having the same Gauss curvature at identified points is a necessary but not sufficient condition for isometry. Instead, we can generate the geodesic equations themselves on the algebraic surfaces in question, and prove the following theorem.

Theorem 2. *The geodesic flow on the algebraic surface $x^n y^n z^n = 1$ with $n \in \mathbb{R}$ is not Louiville integrable with meromorphic first integrals.*

Proof. The geodesic equations on the algebraic surface $F(\boldsymbol{r}) = c$ where $\boldsymbol{r} = (x, y, z)$ are given by [6]

$$\ddot{\boldsymbol{r}} = \lambda \nabla F, \quad \lambda = -\frac{\big(H(F)\dot{\boldsymbol{r}}\big).\dot{\boldsymbol{r}}}{|\nabla F|^2}.$$

Taking $F = x^n y^n z^n$ we find the geodesic equations are independent of n, i.e. the geodesic equations are the same for all values of n (we need to make use of the fact that $\dot{F} = \nabla F.\dot{\boldsymbol{r}} = 0$). We have shown the geodesic equations are not Louiville integrable when $n = 1$ in the previous Section, and therefore they are not integrable for $n \in \mathbb{R}$. □

4.2. Surfaces of the form $x^n y^n z = 1$

It might be hoped that we could generalize the surface considered in Section 3 to Monge patches of the form

$$z = \frac{1}{(x^2 - y^2)^n}, \quad n \in \mathbb{N}. \tag{8}$$

Unfortunately the techniques described in this paper do not allow for a uniform treatment, for the following reason.

Using the methods of Section 2 the NVE of the planar geodesic of (8) is

$$\xi'' + \frac{2n^2(2n+1)(4n^3 - 10n^2 - y^{4n+2}(4n+5)}{y^2(4n^2 + y^{4n+2})^2}\xi = 0.$$

As before, there are regular singular points at $0, \infty$ and the $4n+2$ roots of $4n^2+y^{4n+2} = 0$ which are distinct and symmetrically distributed along a circle centred on the origin of the complex plane. The β coefficients are

$$\beta_j = \left\{ \frac{(1+2n)(2n-5)}{4}, \frac{5}{16}, \ldots, \frac{5}{16} \right\}, \quad \beta_\infty = 0,$$

where $\frac{5}{16}$ appears $4n+2$ times. We note that

$$\sqrt{1+4\beta_0} = 2\sqrt{n^2-2n-1} \ \notin \mathbb{Q} \ \forall \ n \in \mathbb{N}.$$

To show this we note that $\sqrt{1+4\beta_0} \in \mathbb{Q} \Rightarrow n^2 - 2n - 1 = m^2$ for some $m \in \mathbb{N}$. Since $n > m \Rightarrow m = n - \eta$ for $\eta \in \mathbb{N}$ which leads to a quadratic in η whose roots are $(2n + \frac{1}{2}, -\frac{1}{2}) \notin \mathbb{N}$. Therefore we can rule out cases I and III of the Kovacic algorithm as in Section 3.

However, in analysing case II, the E_j sets as in (2) are

$$E_0 = \{2\}, \quad E_{1\ldots(4n+2)} = \{2, 5, -1\}, \quad E_\infty = \{0, 2, 4\}$$

and as such there will be a combination leading to $d = \frac{1}{2}\big(4 - (2 - (4n+2))\big) = 2n+2$ and all values below. Thus we can at best look at individual values of n, for example:

Theorem 3. *The geodesic flow on the surface $x^2y^2z = 1$ in not Louiville integrable with meromorphic first integrals.*

Proof. There are 615 combinations of the indicial exponents leading to $d = 0$; 55 leading to $d = 1, 2, 3$ and 1 leading to $d = 4, 5, 6$. Each of these need to be check as described in Section 3. Again, the procedure is tedious rather than difficult. As there are no combinations for which the necessary P exists, we can rule out case II of Kovacic's algorithm. Thus the identity component of the differential Galois group of the normal variational equation is not abelian, and therefore the geodesic flow on the surface $z = 1/(x^2 - y^2)^2$ is not integrable. □

As the number of cases which need to be checked increases rapidly for increasing n, the methods described in this paper are not appropriate for testing the integrability of surfaces of the form $x^ny^nz = 1$. Having said that, since the $n = 1$ and $n = 2$ cases are not integrable, we would conjecture that all surfaces of this form with $n \in \mathbb{N}$ are also non-integrable.

5. Conclusions

Using Morales-Ramis theory and Kovacic's algorithm we are able to rigorously prove the (meromorphic, Louiville) non-integrability of the geodesic flow on certain algebraic surfaces. This approach is very geometrical in flavour, as opposed to the topological approach followed by Kozlov [7]; nonetheless it is robust enough to deal with free parameters and perturbations as another paper by the author has shown [15]. The analysis was facilitated by two features of the surfaces considered: Monge patches allow a simple intrinsic coordinate system/parameterization to be defined, and a plane of symmetry leads to a planar geodesic along which the variational equations decouple easily. It would be of interest to consider other surfaces where these properties do not hold.

6. Bibliography

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