Approximation Hardness of Dominating Set Problems in Bounded Degree Graphs ⋆

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Abstract

We study approximation hardness of the Minimum Dominating Set problem and its variants in undirected and directed graphs. Using a similar result obtained by Trevisan for Minimum Set Cover we prove the first explicit approximation lower bounds for various kinds of domination problems (connected, total, independent) in bounded degree graphs. Asymptotically, for degree bound approaching infinity, these bounds almost match the known upper bounds. The results are applied to improve the lower bounds for other related problems such as Maximum Induced Matching and Maximum Leaf Spanning Tree.

1 Introduction

A *dominating set* in a graph is a set of vertices such that every vertex in the graph is either in the set or adjacent to a vertex in it. The Minimum Dominating Set problem (shortly, Min-DS) asks for a dominating set of minimum size. The variants of dominating set problems seek for a minimum dominating set with some additional properties, e.g., to be independent, or to induce a connected graph. These problems arise in a number of distributed network applications, where the problem is to locate the smallest number

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of centers in networks such that every vertex is nearby at least one center. Furthermore, the approximation hardness results for dominating set problems can be applied to achieve some inapproximability results for other problems.

Preliminaries and definitions

Let G be a simple graph. A set I of vertices is called *independent* if no two vertices from I are adjacent by an edge in G. A dominating set D in a graph G is an *independent dominating set* if the subgraph G_D of G induced by D has no edges; D is a *total dominating set* if G_D has no isolated vertices; and D is a *connected dominating set* if G_D is a connected graph. The corresponding domination problems Minimum Independent Dominating Set (Min-IDS), Minimum Total Dominating Set (Min-TDS), and Minimum Connected Dominating Set (Min-CDS) ask for an independent, total, and connected dominating set of minimum size, respectively. When a graph problem is restricted to the class of graphs with maximum degree at most B, called also as B-bounded graphs, we use the acronym B in the notation, e.g., B-Min-DS. Let $ds(G)$ stand for the minimum cardinality of a dominating set in G. Similarly, let $ids(G)$, $tds(G)$, and $cds(G)$, stand for the corresponding minima for Min-IDS, Min-TDS, and Min-CDS for G, respectively. For definiteness, the corresponding optimal value is set to infinity if no feasible solution exists for G. That means, $tds(G) < \infty$ iff G has no isolated vertices, and $cds(G) < \infty$ iff G is connected. It is easy to see that $ds(G) \leq ids(G)$, $ds(G) \leq tds(G)$, and $ds(G) \leq cds(G)$. Moreover, $tds(G) \leq cds(G)$ unless $ds(G) = 1$.

In fact, dominating set problems are closely tied to the well-known Minimum Set Cover problem (shortly, Min-SC). Let a set system $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ be given, where $\mathcal{U}$ is a universe and $\mathcal{S}$ is a collection of (nonempty) subsets of $\mathcal{U}$ such that $\cup\mathcal{S} := \cup\{S : S \in \mathcal{S}\} = \mathcal{U}$. Any subcollection $\mathcal{S}' \subseteq \mathcal{S}$ such that $\cup\mathcal{S}' = \mathcal{U}$ is termed a *set cover*. The Minimum Set Cover problem asks for a set cover of minimum cardinality whose size is denoted by $sc(\mathcal{G})$.

An instance $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ of the Minimum Set Cover problem can be viewed as a hypergraph $\mathcal{G}$ with vertices $\mathcal{U}$ and hyperedges $\mathcal{S}$. For an element $x \in \mathcal{U}$ let $\deg(x)$ denote the number of sets in $\mathcal{S}$ containing x and $\deg(\mathcal{G}) := \max_{x \in \mathcal{U}} \deg(x)$ be degree of the instance $\mathcal{G}$. Let $\Delta(\mathcal{G})$ denote size of the largest set in $\mathcal{S}$. The restriction of the set cover problem to instances $\mathcal{G}$ with bounded both parameters $\Delta(\mathcal{G}) \leq k$ and $\deg(\mathcal{G}) \leq d$ will be denoted by (k, d)-Min-SC. Hence, (k, ∞)-Min-SC in this notation corresponds to the well studied problem k-Min-SC, in which instances $\mathcal{G}$ are restricted to those with size of the largest set bounded by k.

For a hypergraph $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ define the *dual hypergraph* $\widetilde{\mathcal{G}} = (\mathcal{S}, \mathcal{U}_{\mathcal{G}})$ such that vertices in $\widetilde{\mathcal{G}}$ are hyperedges of $\mathcal{G}$, and hyperedges $\mathcal{U}_{\mathcal{G}} = \{x_{\mathcal{G}} : x \in \mathcal{U}\}$ of $\widetilde{\mathcal{G}}$ correspond to vertices of $\mathcal{G}$ in the following sense: given $x \in \mathcal{U}$, $x_{\mathcal{G}}$ contains all $S \in \mathcal{S}$ such that $x \in S$. (As we assume $\cup\mathcal{S} = \mathcal{U}$, every $x_{\mathcal{G}}$ is nonempty.) In the context of hypergraphs, the Minimum Set Cover problem is the Minimum Vertex Cover problem (shortly, Min-VC) for the dual hypergraph. Recall that for a hypergraph $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ a *vertex cover* of $\mathcal{G}$ is a subset $C \subseteq \mathcal{U}$ such that each hyperedge e in $\mathcal{S}$ intersects C, i.e., $e \cap C \neq \emptyset$. Clearly, deg and Δ are dual notions in the hypergraph duality. In fact, the (k, d)-Min-SC problem is the same as (d, k)-Min-VC, but in the dual formulation.

We say that an algorithm $\mathcal{A}$ is a c-approximation algorithms for maximization (resp. minimization) problem Π for a constant $c \geq 1$ if, for every instance I of Π whose optimal solution has value $\mathrm{OPT}(I)$, the output of $\mathcal{A}$ on I satisfies $\frac{1}{c}\mathrm{OPT}(I) \leq \mathcal{A}(I) \leq \mathrm{OPT}(I)$ (resp. $\mathrm{OPT}(I) \leq \mathcal{A}(I) \leq c\mathrm{OPT}(I)$). (More generally, one allows c to be a function of an input instance I.) Any such c is called approximation ratio of approximation algorithm $\mathcal{A}$. For any NP-hard optimization problem Π one can define approximation *thresholds* t_{P} and t_{NP} of its constant factor approximability as follows

$$\begin{aligned} t_{\mathrm{P}} &= \inf\{c > 1 : \text{there is a polynomial } c\text{-approximation algorithm for } \Pi\}, \text{ and} \\ t_{\mathrm{NP}} &= \sup\{c \geq 1 : \text{achieving approximation ratio } c \text{ for } \Pi \text{ is NP-hard}\}. \end{aligned}$$

For definiteness, $\inf \emptyset := \infty$. Hence $t_{\mathrm{P}} < \infty$ iff Π is in APX. Further, $t_{\mathrm{P}} = 1$ iff Π has a PTAS. Clearly $t_{\mathrm{NP}} \leq t_{\mathrm{P}}$ unless P = NP. For further optimization terminology we refer the reader to Ausiello et. al. [2].

The Minimum Set Cover problem can be approximated by a natural greedy algorithm that iteratively adds a set that covers the most number of yet uncovered elements. It provides an $\mathcal{H}_{\Delta}$-approximation, where $\mathcal{H}_i := 1 + \frac{1}{2} + \cdots + \frac{1}{i}$ is the i-th harmonic number. (Recall that $\ln i + \gamma < \mathcal{H}_i < \ln i + \frac{1}{2i} + \gamma$, where $\gamma \approx 0.5772156649$ is the Euler constant.) This factor has been improved by Duh and Fürer [8] to $\mathcal{H}_{\Delta} - \frac{1}{2}$. Additionally, Feige [9] has shown that the approximation ratio of $\ln n$ achieved by the greedy algorithm for the Minimum Set Cover problem is the best possible (as a function of $n := |\mathcal{U}|$, up to a lower order additive term) unless the class NP has slightly superpolynomial-time algorithms (namely, $\mathrm{NP} \subseteq \mathrm{DTIME}(n^{O(\log\log n)})$).

Relation of Dominating Set Problems to Minimum Set Cover

It is easy to see, that the Minimum Dominating Set problem in general graphs has the same approximation hardness as the Minimum Set Cover problem. Using the standard reductions similar hardness results can be proved also for other domination problems and even in some restricted graph classes.

DS-SC reduction. Let $G = (V, E)$ be a graph and for each vertex $v \in V$ denote by N_v the set of all neighbors of v. Each vertex $v \in V$ will correspond to an element of $\mathcal{U}$, and the collection $\mathcal{S}$ will consist of the sets $N_v \cup \{v\}$ for each vertex $v \in V$ (resp., only N_v for the Minimum Total Dominating Set problem).

The DS-SC reduction exactly preserves feasibility of solutions: every dominating set in G (resp., total dominating set for a graph without isolated vertices) corresponds to a set cover of the same size in the set system $(\mathcal{U}, \mathcal{S})$, and vice versa.

Hence, using results for Minimum Set Cover [8], we get $(\mathcal{H}_{(\deg(G)+1)} - \frac{1}{2})$-approximation algorithm for Minimum Dominating Set and $(\mathcal{H}_{\deg(G)} - \frac{1}{2})$-approximation algorithm for Minimum Total Dominating Set, where $\deg(G)$ denotes the maximum degree of G. For the Minimum Connected Dominating Set problem $(\mathcal{H}_{\deg(G)} + 2)$-approximation algorithm is known ([11]).

Now we recall two reductions in the opposite direction that we use to obtain inapproximability results for dominating set problems. Recall that a *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set; a *chordal* graph is a graph which contains no cycle with at least four vertices as an induced subgraph.

Definition 1 *For an instance $(\mathcal{U}, \mathcal{S})$ of the* Minimum Set Cover *problem, the $(\mathcal{U}, \mathcal{S})$-bipartite graph is a bipartite graph with bipartition $(\mathcal{U}, \mathcal{S})$ connecting each set $S \in \mathcal{S}$ by an edge to each of its elements $x \in S$.*

Split SC-DS reduction. Given an instance $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ of Minimum Set Cover, create first a $(\mathcal{U}, \mathcal{S})$-bipartite graph and then make a clique of all vertices of $\mathcal{S}$.

Any set cover in $(\mathcal{U}, \mathcal{S})$ corresponds in the resulting split graph G to a dominating set (contained in $\mathcal{S}$) of the same size. It is not difficult to see that a dominating set of minimum size in G is achieved also among dominating sets which contains only vertices from $\mathcal{S}$: any dominating set D in G can be efficiently transformed to the one, say D', with $|D'| \leq |D|$ and $D' \subseteq \mathcal{S}$.

Since a dominating set contained in $\mathcal{S}$ induces a clique, problems Minimum Dominating Set, Minimum Total Dominating Set, and Minimum Connected Dominating Set have the same complexity in graphs constructed using the split SC-DS reduction.

Bipartite SC-DS reduction. Given an instance $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ of Minimum

Set Cover, create first a $(\mathcal{U}, \mathcal{S})$-bipartite graph. Then add two new vertices y and y', and connect the vertex y to each $S \in \mathcal{S}$ and to y'.

For the resulting bipartite graph G one can now confine to dominating sets consisting of y and a subset of $\mathcal{S}$ corresponding to a set cover, hence we have $ds(G) = cds(G) = tds(G) = sc(\mathcal{G}) + 1$.

In order to transfer Feige's ([9]) approximation lower bound of $(1 - \varepsilon) \ln |\mathcal{U}|$ from Minimum Set Cover to the lower bound $(1-\varepsilon) \ln n$ for dominating set problems using split and bipartite SC-DS reductions, we need such hardness result on instances of set cover satisfying $\ln(|\mathcal{U}| + |\mathcal{S}|) \approx \ln(|\mathcal{U}|)$. It turns out that this is indeed true analyzing of Feige's construction. In this way one can obtain the logarithmic lower bound for Minimum Dominating Set, Minimum Total Dominating Set, and Minimum Connected Dominating Set even in split and bipartite graphs.

Hence we can summarize the previous as

Theorem 1 Minimum Dominating Set, Minimum Total Dominating Set, *and* Minimum Connected Dominating Set *cannot be approximated to within a factor of* $(1 - \varepsilon) \ln n$ *in polynomial time for any constant* $\varepsilon > 0$ *unless* NP $\subseteq$ DTIME$(n^{O(\log \log n)})$. *The same results hold also in bipartite and split graphs (hence in chordal graphs, and in complements of chordal graphs as well).*

The Minimum Independent Dominating Set problem is NP-hard, and it appears to be very difficult to approximate owing to non-monotonicity of independent dominating sets, or equivalently, maximal (inclusionwise) independent sets. In fact, no method to approximate *ids* within a factor better than trivial one $O(n)$ appears to be known. Halldórsson [12] proved that Minimum Independent Dominating Set cannot be approximated in polynomial time within a factor of $n^{1-\varepsilon}$ for any $\varepsilon > 0$, unless P = NP. This problem has the strongest known approximation hardness results among unweighted NP-hard problems under various complexity-theoretic assumptions.

Main results

In this paper we investigate the approximability of the dominating set problem and its several variants in bounded degree graphs of large and small degree and directed graphs. We apply these results to other graph optimization problems to improve known or to obtain the first explicit inapproximability results for them.

Problem (asympt.)	B-Min-DS	B-Min-CDS	B-Min-TDS	B-Min-IDS
Lower bound	$\ln B - C \ln\ln B$	$\ln B - C \ln\ln B$	$\ln B - C \ln\ln B$	δB
Upper bound	$\mathcal{H}_{B+1} - \frac{1}{2}$	$\mathcal{H}_B + 2$	$\mathcal{H}_B - \frac{1}{2}$	$B - \frac{B-1}{B^2+1}$

Table 1

Problem	3-Min-DS	4-Min-DS	5-Min-DS	3-Min-IDS	4-Min-IDS	5-Min-IDS
Lower bound	$\frac{391}{390}$*	$\frac{100}{99}$	$\frac{53}{52}$	$\frac{681}{680}$	$\frac{294}{293}$*	$\frac{152}{151}$*
Upper bound	$\frac{19}{12}$	$\frac{107}{60}$	$\frac{117}{60}$	2	$\frac{65}{17}$	$\frac{63}{13}$

Table 2

In B-bounded graphs we prove asymptotically tight lower bounds of $\ln B$ (up to lower order terms) for Minimum Dominating Set, Minimum Total Dominating Set, and Minimum Connected Dominating Set also in bipartite graphs (Section 2). As in general graphs, the Minimum Independent Dominating Set problem completely differs from other studied variants of dominating set problems. We present a lower bound for Minimum Independent Dominating Set in B-bounded graphs that increases linearly with B, similarly as an upper bound. Table 1 summarizes the current state of the research for dominating set problems in the case when the degree bound B increases. All lower bounds are new contributions of this paper and hold even in bipartite graphs, upper bounds are due to [1], [8], [11].

In Section 3 we introduce various kinds of reductions to achieve lower bounds for Minimum Dominating Set and Minimum Independent Dominating Set in graphs of very small maximum degree B. All these lower bounds are summarized in Table 2 (* means that the lower bound is achieved also in bipartite graphs), upper bounds follow from [1], [8]. To the best of our knowledge no explicit approximation hardness results were known in these cases prior this work.

Section 4 deals with domination problems in directed graphs. We show that in directed graphs with indegree bounded by a constant $B \geq 2$ the directed version of Minimum Dominating Set has simple $(B+1)$-approximation algorithm, but it is NP-hard to approximate within any constant smaller than $B-1$ for $B \geq 3$ (resp. 1.36 for $B = 2$). In directed graphs with outdegree bounded by a constant $B \geq 2$ we prove almost tight approximation lower bound of $\ln B$ for directed version of the Minimum Dominating Set problem. We also point out that the problem to decide of whether there exists a feasible solution for the Minimum Independent Dominating Set problem in directed graphs is NP-complete even for graphs with small degree bound.

In Section 5 we apply inapproximability results obtained for domination and covering problems to improve on approximation hardness results of some graph optimization problems. We improve the previous lower bound for the MAXIMUM INDUCED MATCHING problems in graphs of maximum degree 3 to $\frac{294}{293}$, and to $\frac{967}{966}$ in graphs that are additionally bipartite (previous bounds of [7] were $\frac{475}{474}$ and $\frac{6660}{6659}$, respectively). Additionally, our lower bound for MAXIMUM INDUCED MATCHING in B-regular graphs (B large) almost matches known linear upper bound in B-bounded graphs (only APX-completeness was previously known with a lower bound very close to 1, even for large B). We also establish the first explicit lower bound $\frac{245}{244}$ for the MAXIMUM LEAF SPANNING TREE problem, even in bipartite graphs with all vertices but one of degree at most 5.

2 Case of Graphs with Large Degree Bound

In this section we consider asymptotical approximation thresholds for domination problems in graphs of maximum degree bounded by a large constant B. From known approximation algorithms mentioned in Section 2 we can obtain the following results: $t_{\mathrm{P}}(B\text{-MIN-DS}) \leq \mathcal{H}_{B+1} - \frac{1}{2}$, $t_{\mathrm{P}}(B\text{-MIN-TDS}) \leq \mathcal{H}_B - \frac{1}{2}$, and $t_{\mathrm{P}}(B\text{-MIN-CDS}) \leq \mathcal{H}_B + 2$. In what follows we prove asymptotically tight lower bounds of $\ln B$ (up to lower order terms) for all three mentioned problems.

2.1 Minimum Dominating Set in B-Bounded Graphs

Trevisan [17] in the analysis of Feige's construction proved the following inapproximability result for the MINIMUM SET COVER problem restricted to instances with sets of size at most B.

Theorem 2 (Trevisan) *There are absolute constants $C > 0$ and $B_0 \geq 3$ such that for every $B \geq B_0$ it is* NP*-hard to approximate the* MINIMUM SET COVER *problem restricted to instances with sets of size at most B within a factor of* $\ln B - C \ln \ln B$.

In fact, in the proof of the corresponding NP-hard gap type result an instance Ψ of SAT of size n is converted to an instance $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ of B-MIN-SC. Trevisan uses two parameters l and m where $l = \theta(\ln \ln B)$ and $m = \frac{B}{\text{poly} \log B}$. The produced instances have the following properties: $|\mathcal{U}| = mn^l \text{poly} \log B$, $|\mathcal{S}| = n^l \text{poly} \log B$, $\Delta(\mathcal{G}) \leq B$, and $\deg(\mathcal{G}) \leq \text{poly} \log B$. Further, if Ψ is satisfiable then $sc(\mathcal{G}) < \alpha|\mathcal{S}|$ (for some α easily computable from n and B), but if Ψ is not satisfiable, then $sc(\mathcal{G}) > \alpha|\mathcal{S}|(\ln B - C \ln \ln B)$.

We use Trevisan's result to prove inapproximability results for the Minimum Dominating Set problem in graphs of (large) degree at most B. First, we define the gap preserving reduction from $(B-1, B)$-Min-SC to B-Min-DS.

SC-DS$_1$ reduction. Let $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ be an instance of the $(B-1, B)$-Min-SC problem. Add a set W of $\left\lceil \frac{|\mathcal{S}|}{B} \right\rceil$ new vertices and connect them to the $(\mathcal{U}, \mathcal{S})$-bipartite graph as follows: each vertex $S \in \mathcal{S}$ is connected to one vertex of W and allocate these edges to vertices of W such that degree of each vertex in W is also at most B. Let G denote the bipartite graph of degree at most B constructed in this way.

Claim 1 *The* SC-DS$_1$ *reduction has the properties* $sc(\mathcal{G}) \leq ds(G) \leq sc(\mathcal{G}) + \left\lceil \frac{|\mathcal{S}|}{B} \right\rceil$.

Proof. Given a set cover $\mathcal{S}'$ (say, with $|\mathcal{S}'| = sc(\mathcal{G})$), $\mathcal{S}' \cup W$ is clearly a dominating set, hence the second inequality easily follows. The first inequality is obvious, as any dominating set in G has to contain at least $sc(\mathcal{G})$ vertices already in $\mathcal{U} \cup \mathcal{S}$. $\square$

Using this claim we can prove the following

Theorem 3 *There are absolute constants $C > 0$ and $B_0 \geq 3$ such that for every $B \geq B_0$ it is* NP*-hard to approximate the* Minimum Dominating Set *problem in bipartite graphs of degree at most B within a factor of* $\ln B - C \ln \ln B$.

Proof. The SC-DS$_1$ reduction translates the NP-hard question for $(B-1, B)$-Min-SC to decide of whether $sc(\mathcal{G}) < \alpha|\mathcal{S}|$, or $sc(\mathcal{G}) > \beta|\mathcal{S}|$ (for some efficiently computable functions α, β) to the NP-hard question of whether $ds(G) < (\alpha + \frac{1}{B})|\mathcal{S}|$, or $ds(G) > \beta|\mathcal{S}|$ (assuming $\beta - \alpha > \frac{1}{B}$).

It is easy to check that the SC-DS$_1$ reduction from $(B-1, \operatorname{poly}\log B)$-Min-SC to B-Min-DS can decrease the approximation hardness factor of $\ln(B-1) - C \ln \ln(B-1)$ from Theorem 2 only marginally (by an additive term of $\frac{\operatorname{poly}\log B}{B}$). Hence an approximation threshold t_{NP} for B-Min-DS (with B sufficiently large) is again at least $\ln B - C \ln \ln B$, with slightly larger constant C than in Theorem 2. $\square$

2.2 Minimum Total Dominating Set and Minimum Connected Dominating Set in B-Bounded Graphs

To obtain essentially the same inapproximability results for other two problems, Minimum Total Dominating Set and Minimum Connected Dominating Set in B-bounded graphs, we modify slightly the SC-DS$_1$ reduction.

SC-DS$_2$ reduction. For an instance $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ of the $(B-1, B)$-Min-SC problem (with B sufficiently large) construct the $(\mathcal{U}, \mathcal{S})$-bipartite graph and add a set W of $\left\lceil \frac{|\mathcal{S}|}{B-2} \right\rceil$ new vertices. Connect them to vertices of $\mathcal{S}$ in the same way as in the SC-DS$_1$ reduction and add a set W' of additional vertices, with $|W'| = |W|$. The vertices of W and W' are connected to a $2|W|$-cycle with vertices of W and W' alternating in it. The result of this reduction will be a bipartite graph G of degree at most B.

Furthermore, the following claim can be proved analogously as for the SC-DS$_1$ reduction.

Claim 2 *The* SC-DS$_2$ *reduction has the following properties* $sc(\mathcal{G}) \le tds(G) \le sc(\mathcal{G}) + 2\left\lceil \frac{|\mathcal{S}|}{B-2} \right\rceil$ *and* $sc(\mathcal{G}) \le cds(G) \le sc(\mathcal{G}) + 2\left\lceil \frac{|\mathcal{S}|}{B-2} \right\rceil$.

Hence we can prove essentially the same asymptotical results as for the Minimum Dominating Set problem in graphs of degree at most B.

Theorem 4 *There are absolute constants $C > 0$ and $B_0 \ge 3$ such that for every $B \ge B_0$ it is* NP*-hard in bipartite graphs of degree at most B to approximate the problems* Minimum Total Dominating Set*, resp.* Minimum Connected Dominating Set*, within a factor of* $\ln B - C \ln\ln B$.

Proof. It can be proved in the same way as Theorem 3 using the previous claim.

2.3 Minimum Independent Dominating Set in B-Bounded Graphs

Similarly as in general case, the Minimum Independent Dominating Set problem completely differs from all others studied variants of dominating set problems in bounded degree graphs as well. In the following lemma we make simple observation that in B-bounded graphs any inclusionwise maximal independent set (i.e., an independent dominating set) approximates Minimum Independent Dominating Set within B.

Lemma 1 *Let G be a $(B+1)$-claw free graph, $B \geq 1$. Then $\dfrac{is(G)}{B} \leq ds(G) \leq ids(G) \leq is(G)$, where $is(G)$ denotes the maximum cardinality of an independent set in G.*

Proof. It suffices to show $is(G) \leq B\,ds(G)$, i.e., $|I| \leq B|D|$ for every independent set I and every dominating set D. Fix an independent set I and a dominating set D in G. Denote $Z := I \cap D$.

Each vertex $v \in I \setminus Z$ is dominated by a vertex of D, hence it has a neighbor in $D \setminus Z$. However, any $u \in D \setminus Z$ has at most B neighbors in $I \setminus Z$, hence $|I \setminus Z| \leq B|D \setminus Z|$, and $|I| \leq B|D|$ follows. In particular, if U is any inclusionwise maximal independent set in G (which can be found by simple greedy algorithm), we have $\frac{is(G)}{B} \leq ds(G) \leq ids(G) \leq |U| \leq is(G)$. Consequently, any independent dominating set in a $(B+1)$-claw free graph G approximates a minimum independent dominating set, a minimum dominating set, and a maximum independent set within B. $\square$

As any graph of maximum degree at most B is trivially $(B+1)$-claw free, Lemma 1 applies to B-bounded graphs as well. For many problems significantly better approximation ratios are known for B-bounded graphs than for $(B+1)$-claw free graphs. However, for the B-Minimum Independent Dominating Set problem only slightly better upper bounds are known asymptotically ([1]), namely $t_P \leq B - \frac{B-1}{B^2+1}$ for $B \geq 4$, $t_P \leq 2$ for $B = 3$, and in B-regular graphs $t_P \leq B - 1 - \frac{B-3}{B^2+1}$ for $B \geq 5$.

One can ask if there are polynomial time algorithms for the Minimum Independent Dominating Set problem in B-bounded graphs with approximation ratios $o(B)$ when B approaches infinity. We answer this question in the negative (unless P = NP) proving the following

Theorem 5 *There are absolute constants $\delta > 0$ and B_0 such that for every $B \geq B_0$ in graphs of degree at most B the* Minimum Independent Dominating Set *problem is* NP*-hard to approximate within δB. The same hardness result applies to bipartite graphs as well.*

Proof. We extract the core of arguments used in hardness results for Minimum Independent Dominating Set by Halldórsson [12] (and earlier by Irwing [14]), and adapt the construction to produce "hard instances" of bounded degree.

A convenient starting point is the Maximum 3-Satisfiability problem (Max-3SAT). The well known PCP Theorem implies the following NP-hard gap version: for some constant $\alpha \in (0, 1)$ it is NP-hard to distinguish between instances of Max-3SAT that are satisfiable (which we call *yes* instances) and instances in which every assignment satisfies at most a $(1-\alpha)$-fraction of clauses (which we call *no* instances). This hardness result applies also to a restricted version Max-E3SAT-E5 of Max-3SAT, in which every clause contains exactly 3 literals, every variable appears in exactly 5 clauses, and a variable does not appear in a clause more than once. Furthermore, any input formula is promised to be either satisfiable or at most a $(1-\alpha)$-fraction of its clauses is simultaneously satisfiable. (See [9] for more details.)

For any fixed B so large that $\frac{5}{3}\alpha\lfloor\frac{B-1}{5}\rfloor > 1$ we will provide a gap preserving reduction from Max-E3SAT-E5 to B-Min-IDS. Put $t := \lfloor\frac{B-1}{5}\rfloor$. Let ϕ be a Max-E3SAT-E5 instance with $3k$ variables $x_1, x_2, \ldots, x_{3k}$ and $5k$ clauses $C_1, C_2, \ldots, C_{5k}$. We will provide a graph $G_{\phi,t}$ of degree at most B with $(5t+6)k$ vertices, and with the property that

(i) $ids(G_{\phi,t}) \le 3k$, if ϕ is *yes* instance; and

(ii) $ids(G_{\phi,t}) > 5k\alpha t$, if ϕ is *no* instance.

The graph $G_{\phi,t}$ has two vertices labeled x_i and $\overline{x}_i$, for every variable x_i, and t vertices, labeled $C_{j,1}, C_{j,2}, \ldots, C_{j,t}$, for every clause C_j. The edges of $G_{\phi,t}$ are $\{x_i, \overline{x}_i\}$ for each $i = 1, 2, \ldots, 3k$, $\{x_i, C_{j,s}\}$ for all $s \in \{1, 2, \ldots, t\}$ whenever literal x_i is in a clause C_j, and $\{\overline{x}_i, C_{j,s}\}$ for all $s \in \{1, 2, \ldots, t\}$ whenever literal $\overline{x}_i$ is in a clause C_j. The maximum degree of $G_{\phi,t}$ is at most $5t+1 \le B$.

Now we prove the properties (i) and (ii).

(i) Suppose ϕ is *yes* instance and consider a particular satisfying assignment $\sigma : \{x_1, x_2, \ldots, x_{3k}\} \to \{0,1\}$. Then the vertex set $\{x_i : \sigma(x_i) = 1\} \cup \{\overline{x}_i : \sigma(x_i) = 0\}$ is an independent dominating set in $G_{\phi,t}$ of size $3k$, hence $ids(G_{\phi,t}) \le 3k$.

(ii) Let ϕ be *no* instance and consider an independent dominating set D in $G_{\phi,t}$, say with $|D| = ids(G_{\phi,t})$. Let D_1 denote the vertices of D that represent literals and let $D_2 = D \setminus D_1$ represent (repeated) clauses. For each $i \in \{1, 2, \ldots, 3k\}$ at most one of x_i, $\overline{x}_i$ belongs to D_1. Hence D_1 defines a partial assignment to variables, and if a clause C_j contains a literal from D_1, C_j is satisfied by this partial assignment. We will call such clause *good*, otherwise it will be a *bad* clause. Let the number of good clauses be $(5k)g$, and the number of bad ones be $(5k)b$, where $b + g = 1$. Moreover, $g \le 1 - \alpha$, hence $b \ge \alpha$.

For every bad clause C_j, all vertices labeled by $C_{j,1}, C_{j,2}, \ldots, C_{j,t}$ have to belong to D_2. Hence $|D| = |D_1| + |D_2| = |D_1| + (5k)bt$. Moreover, any literal in D_1 makes at most 5 clauses good, hence $5|D_1| \ge (5k)g = 5k(1-b)$, and $|D| \ge (1 + b(5t-1))k \ge (1 + \alpha(5t-1))k > 5k\alpha t$ follows. This finishes the proof of the properties (i) and (ii).

Whenever B is sufficiently large we can obtain the lower bound $\frac{5}{3}\alpha\lfloor\frac{B-1}{5}\rfloor \geq \frac{\alpha}{3}(B-5)$. Hence choosing $\delta \in (0, \frac{\alpha}{3})$, it is NP-hard to approximate the B-Min-IDS problem within δB, for B sufficiently large.

The NP-hard gap can be proven also for bipartite instances of B-Min-IDS. It is easy to see that graphs $G_{\phi,t}$ are bipartite whenever an instance ϕ of Max-SAT is *monotone* (or non-mixed), i.e., none of clauses have both negated and unnegated literals. For monotone variants of Max-SAT there are similar NP-hard gap results for highly restricted instances, as the one for Max-E3SAT-E5. For example, Håstad's result [13] on Maximum E4-Set Splitting can be transformed using simple gadget (namely, replace the constraint *split* (x_1, x_2, x_3, x_4) by two clauses $(x_1 \vee x_2 \vee x_3 \vee x_4)$ and $(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_4)$) to the following: for any $\varepsilon > 0$, it is NP-hard for monotone Max-E4SAT with at most B_ε occurrences of every variable to distinguish satisfiable instances and instances where at most a $(\frac{15}{16} + \varepsilon)$-fraction of clauses can be satisfied. Fixing, e.g., $\varepsilon = \frac{1}{32}$, we can take this restricted version of Max-SAT instead of Max-E3SAT-E5 to prove the theorem for bipartite instances. □

3 Case of Graphs with Small Degree Bound

Now we explore the complexity of dominating set problems in very small degree graphs. Graphs with degree at most 2 have simple structure and all domination problems studied above can be solved efficiently in this class. Thus we will consider the graphs of maximum degree at least 3.

3.1 Minimum Dominating Set problem

Using the standard DS-SC reduction and known approximation results for the Minimum Dominating Set problem restricted to instances with sets of size at most $(B+1)$ for small value of B [8], there is a polynomial time approximation algorithm with the performance ratio $\mathcal{H}_{B+1} - \frac{1}{2}$ for the Minimum Dominating Set problem in B-bounded graphs. It means $\frac{19}{12}$, $\frac{107}{60}$, and $\frac{117}{60}$ for $B = 3$, 4, and 5, respectively.

We cannot rely on the split and bipartite SC-DS reductions from Section 2 to obtain a lower bound on approximability for the Minimum Dominating Set problem in B-bounded degree graphs. The reason is that for any fixed B, only finitely many instances of Minimum Set Cover will transform to B-bounded instances of Minimum Dominating Set. However instead of that we can use the following simple reduction f from Minimum Vertex Cover to Minimum Dominating Set instead.

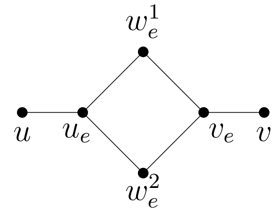

Fig. 1.

VC-DS reduction. Given a graph $G = (V, E)$ with n vertices and m edges (without isolated vertices), replace each edge $e = \{u, v\} \in E$ by a simple gadget G_e (see Fig. 1).

The constructed graph $f(G)$ has $n + 4m$ vertices and $6m$ edges. Moreover, $f(G)$ is bipartite, and if G is of maximum degree B (≥ 3) then the same is true for $f(G)$.

Claim 3 *The* VC-DS *reduction has the property $ds(f(G)) = vc(G) + m$, where $vc(G)$ denote the minimum cardinality of a vertex cover in G.*

Proof. Consider the class $\mathcal{D}$ of dominating sets in $f(G)$ that are related to some vertex cover C of G as follows: given a vertex cover C of G, one can create the corresponding dominating set D of $f(G)$ that contains C, and for each $e = \{u, v\} \in E$ it contains exactly one of vertices u_e, v_e. More precisely, if $u \notin C$ we take u_e, and for an edge $e = \{u, v\}$ with both vertices u, v in C the choice of either u_e or v_e can be made arbitrarily. Easily, D is a dominating set in $f(G)$ and its cardinality is $|C| + m$. Taking C optimally, i.e., with $|C| = vc(G)$ we get $ds(f(G)) \leq vc(G) + m$.

To show the opposite inequality, consider any dominating set D of $f(G)$ and the goal is to prove that $|D| \geq vc(G) + m$. We will show that D can be transformed without increasing its size into another dominating set D' of $f(G)$ such that $D' \in \mathcal{D}$. Consider any $e = \{u, v\} \in E$. Observe first that $D_e := D \cap \{u_e, v_e, w_e^1, w_e^2\} \neq \emptyset$. If $u \in D$, (resp., $v \in D$) replace D_e in D by v_e (resp., u_e); if both u and v are in D, the choice of either v_e or u_e can be made arbitrarily. If neither $u \in D$ nor $v \in D$, then clearly $|D_e| \geq 2$, and we can replace D_e by either $\{u, v_e\}$ or $\{v, u_e\}$. Having this done for each $e = \{u, v\} \in E$ one after another, we will obtain a dominating set D' with $|D'| \leq |D|$ such that $C := D' \cap V$ is a vertex cover and $|D'| = |C| + m$. Hence $|D| \geq |D'| = |C| + m \geq vc(G) + m$, that completes the proof. $\square$

Hence we have the following

Theorem 6 *It is* NP-*hard to approximate the* Minimum Dominating Set *problem in bipartite graphs of degree at most* 3 *within* $1 + \frac{1}{390}$.

Proof. Applying the VC-DS reduction to a 3-regular graph G with n vertices produces a bipartite graph $f(G)$ of maximum degree at most 3 with $7n$ vertices and $9n$ edges. Using NP-hard gap result for MIN-VC in 3-regular graphs [4] we obtain that it is NP-hard to decide of whether $ds(f(G))$ is greater than $2.01549586n$, or less than $2.0103305n$, hence to approximate MIN-DS in bipartite graphs of degree 3 within $\frac{391}{390}$ is NP-hard. $\square$

For larger value of B, $B \geq 4$, better inapproximability results can be achieved by the following SC-DS$_3$ reduction.

SC-DS$_3$ reduction. From an instance $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ of MINIMUM SET COVER construct firstly the $(\mathcal{U}, \mathcal{S})$-bipartite graph. Then for each $S \in \mathcal{S}$ pick one fixed representative $u_S \in S$ and add new edges to the $(\mathcal{U}, \mathcal{S})$-bipartite graph connecting S with each other $S' \in \mathcal{S}$ containing u_S (without creating multiple edges). Let G denote the resulting graph.

Claim 4 *The* SC-DS$_3$ *reduction has the property* $ds(G) = sc(\mathcal{G})$.

Proof. Firstly we prove that any set cover $\mathcal{C} \subseteq \mathcal{S}$ is a dominating set in G. Given a set cover $\mathcal{C}$, all vertices in $\mathcal{U}$ (and in $\mathcal{C}$ itself) are dominated by $\mathcal{C}$. Consider any $S \in \mathcal{S} \setminus \mathcal{C}$ and let u_S be its fixed representative. As $\mathcal{C}$ is a set cover, u_S is contained in some $S' \in \mathcal{C}$. According the definition there is an edge connecting S and S' in G and hence S is dominated as well. In particular, $ds(G) \leq sc(\mathcal{G})$.

If $D \subseteq \mathcal{U} \cup \mathcal{S}$ dominates the set $\mathcal{U}$ we can conclude that $|D| \geq sc(\mathcal{G})$ in the same way as in the previous SC-DS reductions in Section 2. Hence, if D is a minimum dominating set in G we get $ds(G) \geq sc(\mathcal{G})$ as well, and the equality follows. $\square$

The SC-DS$_3$ reduction can be used as a gap preserving reduction from the MINIMUM VERTEX COVER problem in $(B-1)$-bounded graphs with a perfect matching to the MINIMUM DOMINATING SET problem in B-bounded graphs. In this way we can obtain the following

Theorem 7 *The* MINIMUM DOMINATING SET *problem is* NP-*hard to approximate within* $1 + \frac{1}{99}$ *in graphs of degree at most* 4, *within* $1 + \frac{1}{52}$ *in graphs of degree at most* 5, *and within* $1 + \frac{1}{50}$ *in graphs of degree at most* 6.

Proof. Let $H = (V, E)$ be an instance of $(B-1)$-MIN-VC with a fixed perfect matching M in it. Let $\widetilde{\mathcal{G}} = (E, V_H)$ be the dual hypergraph to (hyper)graph H. Due to duality, $\widetilde{\mathcal{G}}$ can be viewed as a $(B-1, 2)$-instance of MIN-SC, and $sc(\widetilde{\mathcal{G}}) = vc(H)$. The corresponding (E, V_H)-bipartite graph for $\widetilde{\mathcal{G}}$ is just division of H (for every edge put a single vertex on it), if one identifies each

$v \in V$ with the corresponding set v_H containing all edges incident with v in H. Now we consider the SC-DS$_3$ reduction and for each set S (corresponding to $v \in V$) we take as u_S exactly that edge adjacent to v in H that belongs to M. Hence the resulting graph G can be obtained from a division of H by adding edges of M. Therefore, G is of degree at most B and, due to the previous claim, $ds(G) = sc(\widetilde{\mathcal{G}})$. Hence, $ds(G) = vc(H)$ follows.

It is easy to verify that NP-hard gap results obtained in [4] for B-Min-VC ($B = 3$, 4, and 5) apply to B-regular graphs with a perfect matching as well. (For $B = 3, 4$ it is proved in [4] that produced hard instances are B-regular and edge B-colorable, which implies the existence of a perfect matching in them.) Thus for $B \geq 4$ we obtain for B-Min-DS the same lower bound as for $(B-1)$-Min-VC. Namely, $t_{\mathrm{NP}}(4\text{-Min-DS}) > \frac{100}{99}$, $t_{\mathrm{NP}}(5\text{-Min-DS}) > \frac{53}{52}$, and $t_{\mathrm{NP}}(6\text{-Min-DS}) > \frac{51}{50}$, respectively. $\square$

Remark. From results for the Minimum Edge Dominating Set problem from [3] it also follows that for 4-regular graphs, which can be obtained as line graphs of 3-regular graphs, it is NP-hard to approximate Minimum Dominating Set within $1 + \frac{1}{390}$. Recall that for the Minimum Dominating Set problem restricted to line graphs there is a simple 2-approximation algorithm, but it is NP-hard to approximate within any constant smaller than $\frac{7}{6}$, as easily follows from results of [3].

3.2 Minimum Independent Dominating Set problem

For the Minimum Independent Dominating Set problem in small degree graphs the best upper bounds are due to [1]: $t_{\mathrm{P}}(4\text{-Min-IDS}) \leq \frac{65}{17}$, $t_{\mathrm{P}}(5\text{-Min-IDS}) \leq \frac{63}{13}$, $t_{\mathrm{P}}(6\text{-Min-IDS}) \leq \frac{217}{37}$, and $t_{\mathrm{P}}(3\text{-Min-IDS}) \leq 2$. To obtain inapproximability results in such restricted cases we use the following polynomial time reduction from the Minimum Set Cover problem.

SC-IDS reduction. Let an instance $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ of $(B-1, B)$-Min-SC be given. Start with the corresponding $(\mathcal{U}, \mathcal{S})$-bipartite graph and for each $S \in \mathcal{S}$ add two new vertices S', S'', and two edges $\{S, S'\}$, $\{S', S''\}$. The resulting graph G is bipartite of maximum degree at most B.

Claim 5 *The* SC-IDS *reduction has the properties* $ids(G) = ds(G) = sc(\mathcal{G}) + |\mathcal{S}|$.

Proof. Since $ds \leq ids$, it suffices to prove that (i) $ids(G) \leq sc(\mathcal{G}) + |\mathcal{S}|$, and (ii) $ds(G) \geq sc(\mathcal{G}) + |\mathcal{S}|$.

(i) For a given set cover $\mathcal{C} \subseteq \mathcal{S}$ consider the following set $D := \mathcal{C} \cup \{S'' : S \in \mathcal{C}\} \cup \{S' : S \in \mathcal{S} \setminus \mathcal{C}\}$ of vertices in G. Clearly, D is an independent dominating set in G of cardinality $|\mathcal{C}| + |\mathcal{S}|$ and $ids(G) \leq sc(\mathcal{G}) + |\mathcal{S}|$ follows.

(ii) Given any dominating set D in G (say, with $|D| = ds(G)$), it can be easily transformed to another dominating set D_1 with $|D_1| \leq |D|$ such that $D_1 \cap \mathcal{U} = \emptyset$, and $D_1 \cap \{S', S''\} = S'$ for each $S \in \mathcal{S}$. Then clearly $D_1 \cap \mathcal{S}$ is a set cover in $\mathcal{G}$, and $ds(G) = |D| \geq |D_1| \geq sc(\mathcal{G}) + |\mathcal{S}|$ follows. □

Using the previous claim one can obtain an NP-hard gap result for the Minimum Independent Dominating Set problem in graphs of degree at most B (and Minimum Dominating Set as well) from the one for $(B-1, B)$-Min-SC or equivalently, for the $(B, B-1)$-Min-VC problem. Due to lack of such results we use inapproximability results for $(2, B-1)$-Min-VC, it means for Minimum Vertex Cover in $(B-1)$-bounded graphs. More precisely, one can translate NP-hard gap results of [4] for Minimum Vertex Cover in $(B-1)$-bounded graphs to the ones for Minimum Independent Dominating Set in B-bounded graphs as follows.

Theorem 8 *The* Minimum Independent Dominating Set *is* NP*-hard to approximate within* $1 + \frac{1}{293}$ *in graphs of degree at most* 4*, within* $1 + \frac{1}{151}$ *in graphs of degree at most* 5*, and within* $1 + \frac{1}{145}$ *in graphs of degree at most* 6*. The same hardness results applies to bipartite graphs.*

Proof. We start from a 3-regular instance for Minimum Vertex Cover with n vertices. Using the SC-IDS reduction and results of [4] we obtain a bipartite graph G of degree at most 4 and with the NP-hard question of whether $ids(G)$ is greater than $1.51549586n$ or less than $1.5103305n$. Hence, it is NP-hard to approximate 4-Min-IDS even in bipartite graphs within $\frac{294}{293}$. Starting from a 4-regular graph with n vertices the corresponding NP-hard question for 5-Min-IDS is of whether the optimum is greater than $1.5303643725n$ or less than $1.520242915n$, hence inapproximability within $\frac{152}{151}$ follows. Analogously starting from a 5-regular graph with n vertices the corresponding NP-hard question is of whether the optimum is greater than $1.5316455696n$ or less than $1.5210970464n$, hence inapproximability within $\frac{146}{145}$ follows for 6-Min-IDS. □

To obtain a lower bound for the Minimum Independent Dominating Set problem in graphs of degree at most 3, let us consider the following reduction h from Minimum Vertex Cover to Minimum Independent Dominating Set:

VC-IDS reduction. Given a graph $G = (V, E)$ with n vertices and m edges (without isolated vertices), replace each edge $e = \{u, v\} \in E$ by a simple edge

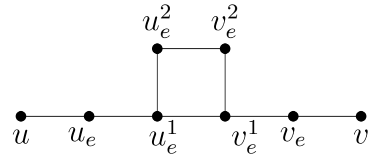

Fig. 2.

gadget G_e (see Figure 2).

The graph $h(G)$ constructed in this way has $n + 6m$ vertices and $8m$ edges. Moreover, if G is of maximum degree at most B (≥ 3) then the same is true for $h(G)$.

Claim 6 *The* VC-IDS *reduction has the property* $ids(h(G)) = vc(G) + 2m$.

Proof. (i) Given a vertex cover C of G (say, with $|C| = vc(G)$), one can create the corresponding independent dominating set D in $h(G)$ of cardinality $|C| + 2m$ as follows: for $e = \{u, v\}$ with $u \notin C$ (that implies $v \in C$) we take exactly u_e and v_e^2 to D from the gadget G_e; for $e = \{u, v\}$ with both $u, v \in C$ we take u_e^1 and v_e^2. This shows that $ids(h(G)) \leq vc(G) + 2m$.

(ii) To show the opposite inequality, consider an independent dominating set D in $h(G)$ (say, with $|D| = ids(h(G))$. The goal is to prove that D can be transformed without increasing its size into another dominating set D' in $h(G)$ such that in each G_e ($e \in E$) D' is one of the forms as in (i). Fix $e = \{u, v\} \in E$. If $D \cap \{u, v\} \neq \emptyset$ then it is easy to see that $|D \cap \{u_e, u_e^1, u_e^2, v_e, v_e^1, v_e^2\}| \geq 2$, and if $D \cap \{u, v\} = \emptyset$ then $|D \cap \{u_e, u_e^1, u_e^2, v_e, v_e^1, v_e^2\}| \geq 3$. Hence one can easily modify D to a dominating set D' with $ids(h(G)) = |D| \geq |D'| \geq vc(G) + 2m$. $\square$

Therefore we can prove

Theorem 9 *It is* NP*-hard to approximate the* Minimum Independent Dominating Set *problem in graphs of degree at most* 3 *within* $1 + \frac{1}{680}$.

Proof. Applying the VC-IDS reduction to a 3-regular instance G of Min-VC (with n vertices) and using NP-hard gap result for it [3], we obtain that it is NP-hard to decide of whether $ids(h(G))$ is greater than $3.51549586n$, or less than $3.5103305n$. Hence to approximate 3-Min-IDS within $\frac{681}{680}$ is NP-hard. $\square$

4 Minimum Dominating Set in Directed Graphs

In a directed graph $G = (V, \overrightarrow{E})$ a set $D \subseteq V$ is a *dominating set* if for each $v \in V \setminus D$ there is $u \in D$ such that $\overrightarrow{uv} \in \overrightarrow{E}$. For a vertex $v \in V$ denote by $N_v^+ := \{v\} \cup \{u \in V : \overrightarrow{vu} \in \overrightarrow{E}\}$ the set of its neighbors. Then $|N_v^+| = 1 + d_{\text{out}}(v)$ and $|\{u \in V : v \in N_u^+\}| = 1 + d_{\text{in}}(v)$, where $d_{\text{out}}(v)$, resp. $d_{\text{in}}(v)$, denotes outdegree, resp. indegree, of v in G.

Similarly as in undirected case, the Minimum Dominating Set problem in directed graph is special case of the Minimum Set Cover problem due to the following simple reduction:

Directed DS-SC reduction. For a directed graph $G = (V, \overrightarrow{E})$ we define an instance $(\mathcal{U}, \mathcal{S})$ of Min-SC as $\mathcal{U} := V$ and $\mathcal{S} := \{N_v^+ : v \in V\}$. For such instance $(\mathcal{U}, \mathcal{S})$ set covers are in one-to-one correspondence with dominating sets in G.

4.1 Minimum Dominating Set *in Graphs with Bounded Indegree*

Due to the directed DS-SC reduction, instances of Minimum Dominating Set with indegree bounded by a constant B can be viewed as instances of Minimum Set Cover with degree at most $B + 1$. Hence the problem has a simple $(B+1)$-approximation algorithm in this case. Furthermore, case $B = 1$ can be easily solved exactly. Asymptotically, we can obtain almost matching lower bound as follows from the following theorem.

Theorem 10 *It is* NP*-hard to approximate the* Minimum Dominating Set *problem in directed graphs with indegree bounded by a constant B within any constant smaller than $B - 1$ for $B \geq 3$, and within* 1.36 *for $B = 2$. On the other hand, the problem has a simple $(B + 1)$-approximation algorithm for $B \geq 2$.*

Proof. Consider the following reduction from restricted instances of Min-SC to directed instances of Min-DS: for an instance $\mathcal{G} = (\mathcal{U}, \mathcal{S})$ with $\deg(\mathcal{G}) \leq B$ construct a graph G with the vertex set $V = \mathcal{U} \cup \mathcal{S} \cup \{S_0\}$, where S_0 is a new vertex. Add edges $\overrightarrow{S_0 S}$ in G for each $S \in \mathcal{S}$, and an edge $\overrightarrow{Sx}$ for each $S \in \mathcal{S}$ and each $x \in S$. The directed graph $G = (V, \overrightarrow{E})$ created in this way has indegree bounded by B. Obviously, there are minimum dominating sets in G consisting of S_0 and $\mathcal{C} \subseteq \mathcal{S}$, where $\mathcal{C}$ is a minimum set cover in $(\mathcal{U}, \mathcal{S})$. Hence this reduction preserves NP-hard gap results for (∞, B)-Min-SC, i.e., Min-SC restricted to instances $\mathcal{G}$ with $\deg(\mathcal{G}) \leq B$. Recall that this is equivalent to the hypergraph (B,∞)-MIN-VC problem for which Dinur et al. ([5]) gave nearly tight lower bound $(B-1)$ on approximability in B-uniform hypergraphs, $B \geq 3$. For $B = 2$ the lower bound 1.36 follows from currently the best approximation hardness result for the MIN-VC problem on graphs [6]. $\square$

4.2 MINIMUM DOMINATING SET *in Graphs with Bounded Outdegree*

Instances of the MINIMUM DOMINATING SET problem with outdegree bounded by a constant B can be viewed as instances of set cover with sets of size at most $B+1$. Hence the problem is polynomially solvable for $B = 1$. For $B \geq 2$ a polynomial time approximation algorithm with the ratio $\mathcal{H}_{B+1} - \frac{1}{2} < \ln B + O(1)$ is known [8].

To obtain a lower bound, replace in undirected B-bounded instances of MINIMUM DOMINATING SET every edge $\{u, v\}$ by two directed edges $\overrightarrow{uv}$, $\overrightarrow{vu}$. It can be seen that instances have both, outdegree and indegree, bounded by a constant B and the reduction preserves dominating sets. Hence, the MINIMUM DOMINATING SET problem in directed case is at least as hard as in undirected and applying Theorem 3 we can obtain

Theorem 11 *There are absolute constants $C > 0$ and $B_0 \geq 3$ such that for every $B \geq B_0$ it is* NP*-hard to approximate the* MINIMUM DOMINATING SET *problem in directed graphs with outdegree bounded by a constant B within $\ln B - C \ln\ln B$. However, there exists $(\mathcal{H}_{B+1} - \frac{1}{2})$-approximation algorithm for the problem for any $B \geq 2$.*

4.3 Other Dominating Set Problems in Directed Graphs

The variants of the MINIMUM DOMINATING SET problem, namely MINIMUM TOTAL DOMINATING SET, MINIMUM CONNECTED DOMINATING SET, and MINIMUM INDEPENDENT DOMINATING SET can be formulated for directed graphs as well. For connected domination problems, in particular, there are many interesting questions left open.

Let us point out that the MINIMUM INDEPENDENT DOMINATING SET problem in directed graphs is very different from its undirected counterpart. The problem to decide of whether there exists a feasible solution (i.e., an independent dominating set) in a given directed graph is NP-complete, even in bounded degree graphs. To see that, consider the following reduction from MAX-3SAT-5: given an instance ϕ, create a graph G_ϕ with two vertices labeled by x and $\overline{x}$, for every variable x, and three vertices labeled by c, c'', and c'', for every clause C. Edges are chosen so that every pair x, $\overline{x}$ is a 2-cycle , every triple c, c', c'' is a directed 3-cycle , and there is an edge $\overrightarrow{lc}$ whenever literal l is in a clause C. One can easily check that G_ϕ has an independent dominating set if and only if ϕ is satisfiable. Moreover, G_ϕ has fulldegree bounded by 7.

5 Application to Other Problems

In this section we apply the inapproximability results for domination and covering problems to improve on approximation hardness results of some other graph optimization problems.

5.1 Maximum Induced Matching problem

Definition 2 *A* matching *in a graph $G = (V, E)$ is a subset of edges E with no shared endvertices. A matching M is* induced *if for each edge $e = \{u, v\} \in E$, $u, v \in V(M)$ implies $e \in M$. The objective of the* Maximum Induced Matching *problem (*Max-IM*) is to find a maximum induced matching in G, let $im(G)$ denote its cardinality.*

The problem is known to be NP-complete even in bipartite graphs of degree at most 3 and the current state of the art can be found in [7], [15], and [18]. For the Maximum Induced Matching problem in B-bounded graphs, $B \geq 3$, any inclusionwise maximal induced matching approximates the optimal solution within $2(B-1)$ and in B-regular graphs within $B - \frac{(B-1)}{(2B-1)}$ ([18]). This was improved to an asymptotic ratio $B-1$ in B-regular graphs in [7], where also the proof of APX-completeness of Maximum Induced Matching in B-regular graphs is given.

In what follows we present a lower bound for the Maximum Induced Matching problem in B-regular graphs (for large B) that approaches infinity with B and almost matches linear upper bound.

Theorem 12 *The* Maximum Induced Matching *problem is* NP*-hard to approximate within $1 + \frac{1}{293}$ in graphs of degree at most 3, and within $1 + \frac{1}{966}$ in graphs that are additionally bipartite. Further,* Maximum Induced Matching *is* NP*-hard to approximate within $1 + \frac{1}{94}$ in graphs of degree at most 4, within $1+\frac{1}{47}$ in graphs of degree at most 5, and within $1+\frac{1}{45}$ in graphs of degree at most 6. Asymptotically, it is* NP*-hard to approximate* Maximum Induced Matching *in B-bounded graphs within a factor $\frac{B}{2^{O(\sqrt{\ln B})}}$ and this* *asymptotical lower bound applies to B-regular graphs as well.*

Proof. Firstly, one can easily check that the lower bound of $\frac{B}{2^{O(\sqrt{\ln B})}}$ given by Trevisan [17] for B-Max-IS applies to B-regular graphs as well. Now consider the following transformation g for a $(B-1)$-regular graph $G=(V,E)$: take another copy $G'=(V',E')$ of the same graph G (with $v'\in V'$ corresponding to $v\in V$), and make every pair $\{v,v'\}$ adjacent. The resulting graph is B-regular and it is easy to observe that $is(G)\le im(g(G))\le 2is(G)$. Hence a lower bound on approximability for B-Max-IM in B-regular graphs is at least $\frac{1}{2}$ of Trevisan's one, it means again of the form $\frac{B}{2^{O(\sqrt{\ln B})}}$.

For all $B\ge 4$ we can use the following simple reduction f from $(B-1)$-Max-IS to B-Max-IM: $f(G)$ is constructed from a graph G adding a pending $\{v,v'\}$ at each vertex v of G. Obviously, $im(f(G))=is(G)$ and hence NP-hard gap results for $(B-1)$-Max-IS directly translates to the one for B-Max-IM. In particular, t_{NP}(4-Max-IM) $>\frac{95}{94}$, t_{NP}(5-Max-IM) $>\frac{48}{47}$, and t_{NP}(6-Max-IM) $>\frac{46}{45}$.

The problem to obtain any decent lower bound for 3-Max-IM is more difficult. One can observe (see, e.g., [15]) that for any graph $G=(V,E)$ its subdivision G^0 (G^0 is obtained from G replacing every edge $\{u,v\}$ with a path u, w, v through a new vertex w) satisfies $im(G^0)=|V|-ds(G)$. Using NP-hard gap result for 3-Min-DS from Theorem 6, we obtain instances G^0 of maximum degree at most 3 with $16n$ vertices, $18n$ edges with the NP-hard question to decide of whether $im(G^0)$ is greater than $4.9896695n$, or less than $4.9845042n$. Hence to approximate 3-Max-IM even in subdivision (and, in particular, bipartite) graphs within $\frac{967}{966}$ is NP-hard. It improves the previous lower bound $\frac{6660}{6659}$ for the 3-Max-IM problem in bipartite graphs from [7]. Using the reduction from Max-IS to Max-IM presented in [7], we can improve also a lower bound $\frac{475}{474}$ for 3-Max-IM in general graphs. From a 3-regular instance G of 3-Max-IS with n vertices, in the combination with NP-hard gap results for them ([4]), we produce an instance G' of 3-Max-IM (with $5n$ vertices, $\frac{11}{2}n$ edges and with $im(G')=n+is(G)$) with the NP-hard question to decide of whether im is greater than $1.51549586n$ or less than $1.5103305n$. Hence it is NP-hard to approximate 3-Max-IM within $\frac{294}{293}$. □

5.2 Maximum Leaf Spanning Tree Problem

The goal of the Maximum Leaf Spanning Tree problem (Max-LST) is for an input (connected) graph to find a spanning tree with the maximum number of leaves. The problem is approximable within 3 [16] and known to be APX-complete [10].

If $G = (V, E)$ is a connected graph with $|V| \geq 3$ then it is easy to see that $|V| - cds(G)$ is the maximum number of leaves in a spanning tree of G. This simple observation allows us to obtain the first explicit inapproximability results for the MAXIMUM LEAF SPANNING TREE problem.

Theorem 13 *It is* NP*-hard to approximate (even in bipartite graphs with all vertices but one of degree at most* 5*) the* MAXIMUM LEAF SPANNING TREE *problem within* $1 + \frac{1}{244}$.

Proof. The NP-hard gap result for MIN-VC in 4-regular graphs [4] implies the same NP-hard gap for the $(4, 2)$-MIN-SC problem due to the duality of both problems. Hence it is NP-hard to decide if the optimum for $(4, 2)$-MIN-SC is greater than $0.5303643725n$ or smaller than $0.520242915n$, where n is the number of vertices for dual 4-regular graph. Applying the bipartite SC-DS reduction from Introduction for such hard instances of $(4, 2)$-MIN-SC we obtain a bipartite graph with $3n + 2$ vertices, all but one of degree at most 5, and with the NP-hard question for MAX-LST to decide of whether the optimum is less than $2.469635627n + 1$, or greater than $2.479757085n + 1$. Hence inapproximability within $\frac{245}{244}$ follows. $\square$

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