

Cosmological magnetic fields from nonlinear effects

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In the standard cosmological model, magnetic fields and vorticity are generated during the radiation era via second-order density perturbations. In order to clarify the complicated physics of this second-order magnetogenesis, we use a covariant approach and present the electromagneto-dynamical equations in the nonlinear regime. We use the tight-coupling approximation to analyze Thomson and Coulomb scattering. At the zero-order limit of exact tight-coupling, we show that the vorticity is zero and no magnetogenesis takes place at any nonlinear order. We show that magnetogenesis also fails at all orders if either protons or electrons have the same velocity as the radiation, and momentum transfer is neglected. Then we prove a key no-go result: at first-order in the tight-coupling approximation, magnetic fields and vorticity still cannot be generated even via nonlinear effects. The tight-coupling approximation must be broken at first order, for the generation of vorticity and magnetic fields, and we derive a closed set of nonlinear evolution equations that governs this generation. We estimate that the amplitude of the magnetic field at recombination on the horizon scale is $\sim 10^{-27}$ G.

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I. INTRODUCTION

The origin of cosmological magnetic fields is an important problem in cosmology [1]. Many mechanisms for primordial magnetogenesis (i.e., creation before structure formation) have been proposed. In order to generate fields on large scales, inflationary mechanisms are the best candidates, but they require uncertain modifications to standard physics in order to break the conformal invariance of Maxwell fields [2].

The generation of cosmological magnetic fields via plasma interactions during the radiation era, originally suggested by Harrison [3], is based on conventional physics and does not require any new postulates. The essential ingredients in this mechanism are nonzero vorticity and Thomson scattering between photons and charged particles. Because momentum transfer is more effective between photons and electrons than between photons and protons due to the mass difference, Thomson scattering induces differences in the velocity and the distribution of protons and electrons. These differences induce local electric currents and net charge density, and the electric field in turn generates a magnetic field. This process initiates after electron-positron annihilation and ends when there are insufficient free electrons, i.e., it operates over the temperature range $T_{\text{rec}} \lesssim T \lesssim m_e$.

In the standard cosmological model, there is no vorticity at first order, and the generalized Harrison mechanism is much more complicated. We can follow the basic argument in a simple Newtonian formalism. The evolution of the magnetic field is described by the induction equation,

$$\dot{\vec{B}} = -\vec{\nabla} \times \vec{E}. \quad (1)$$

Analysis of the momentum transfer in scattering leads to

the generalized Ohm's law,

$$\vec{E} = \eta \vec{J} + \vec{S}, \quad (2)$$

where η is the plasma resistivity and \vec{S} is the contribution from Thomson scattering. Since $\eta \ll H^{-1}$, we can neglect this term to obtain

$$\dot{\vec{B}} = -\vec{\nabla} \times \vec{S}. \quad (3)$$

As we will see below, $\vec{S} \sim n\vec{v}$, where n is the number density of charged particles and v is the velocity difference between radiation and charged particles. Equation (3) shows that there are two sources for magnetogenesis – vorticity, $\text{curl } \vec{v}$, and the vector product of density gradient and velocity, $\vec{\nabla} n \times \vec{v}$.

In the standard cosmology, where perturbations are generated from inflation, there are no vector modes at first-order, and therefore the vorticity vanishes at first order. The density-velocity term is a product of first-order scalar perturbations and therefore also vanishes at first order. Thus in the standard model a perturbative analysis of magnetogenesis during the radiation era must start at second order. (Exceptions can arise if there are sources of first-order vector perturbations, such as cosmic strings [4], or fine-tuned anisotropies in collisionless neutrinos [5].)

Matarrese et al. [6] analysed how vorticity and magnetic fields can be generated from second order cosmological perturbations. Subsequent work has also used second-order perturbations [7–11], but has neglected the vorticity and metric vector perturbations, and focused on the density-velocity terms, i.e., the product of first-order scalar terms. (For other work on magnetogenesis during the radiation era, see Refs. [12].) The different approaches lead to estimates in the range [6, 7, 10, 11]:

$$B \sim 10^{-24} - 10^{-27} \text{ G}, \quad (4)$$

at recombination on 100 Mpc scales. This is a very weak field, but it provides a seed which is amplified via the dynamo mechanism. It is possible that the dynamo amplification can reach the current observed value of about 10^{-6}G on galaxy scales [13].

The simplistic Newtonian description given above allows us to identify the key physical effects, but the real situation is much more complicated. The second-order perturbative treatments are a necessary foundation for computing the power spectrum of the magnetic field. However, it is also useful to adopt a covariant approach that directly generalises the Newtonian treatment to cosmology [14, 15]. This allows us to develop a direct physical understanding of the magnetogenesis process, and also to deal with the problem in the fully nonlinear regime.

The greatest complexity arises from the dynamics of momentum transfer via scattering. We use the tight coupling approximation [16], which is based on the fact that the scattering time τ is much less than the cosmic expansion time H^{-1} ,

$$H\tau \ll 1, \quad (5)$$

so that photons and charged particles are closely bound. In the limit, i.e., at the zero-order of exact tight-coupling, we have $\tau = 0$ and $v = 0$, so that all particles share the same velocity and behave as a single fluid, and no magnetogenesis takes place. Beyond the zero-order of tight coupling, there is a nonzero τ and velocity difference v , and the tight coupling approximation is an expansion in $H\tau$. Note that the tight coupling approximation is independent of the cosmological perturbative approximation. Zero-order in tight coupling is not to be confused with zero-order in cosmological perturbations; the cosmological variables can be at any nonlinear order, but no magnetogenesis is possible.

First, we give a basic result: in the limit of exact tight coupling, and neglecting anisotropic stresses, vorticity vanishes and magnetic fields cannot be generated at any nonlinear perturbative order. If the exact tight coupling limit is partially relaxed by assuming that either the protons or the electrons have the same velocity as the radiation, but neglecting momentum transfer, then vorticity and magnetic fields still cannot be generated at any nonlinear order. Then we derive the evolution equation for magnetic fields and vorticity beyond the zero-order of the tight coupling approximation. We show that *there is no magnetogenesis at the first order in the tight-coupling approximation, and magnetogenesis takes place at second order*. While it has been suggested that magnetogenesis is possible via the breaking of the tight coupling limit [17], this is the first study giving explicitly the condition for such a mechanism to work.

The rest of the paper is organised as follows. In the next section we derive the nonlinear equations for magnetic fields and vorticity in a covariant formalism. In section III, we show that vorticity cannot be generated even through nonlinear effects in the tight coupling limit.

In section IV we consider the first and second order of the tight coupling approximation. Finally we summarise our work in section V.

II. COSMOLOGICAL ELECTROMAGNETO-DYNAMICS

The Faraday tensor can be split into electric and magnetic fields as measured by a congruence of fundamental observers u^a (with $u_a u^a = -1$):

$$F_{ab} = 2u_{[a}E_{b]} + \varepsilon_{abc}B^c, \quad (6)$$

where $E_a u^a = B_a u^a = 0$. The spatial alternating tensor is $\varepsilon_{abc} = \eta_{abcd}u^d$, where η_{abcd} is the spacetime alternating tensor, using the convention $\eta_{0123} = -\sqrt{-g}$. The tensor indices represent an arbitrary coordinate or tetrad frame; at any event one can choose local inertial coordinates such that $u^a = (1, \vec{0})$, $E^0 = 0 = B^0$.

The induced metric in the observer's comoving rest space is

$$h_{ab} = g_{ab} + u_a u_b, \quad (7)$$

and it defines a covariant spatial derivative D_a . Generalizing the Newtonian case, we define kinematical quantities of the u^a congruence via its covariant derivative:

$$\nabla_b u_a = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \varepsilon_{abc}\omega^c - \dot{u}_a u_b. \quad (8)$$

Here $\theta = \nabla_a u^a$ is the volume expansion rate, $\sigma_{ab} = h_a^c h_b^d \nabla_{(c} u_{d)}$ is the shear, $\omega_a = -\frac{1}{2}\varepsilon_{abc}\nabla^b u^c$ is the vorticity, and $\dot{u}^a = u^b \nabla_b u^a$ is the acceleration.

Since we are working mainly in the radiation era, it is useful to choose u^a as the radiation four-velocity in the energy frame, i.e., with no energy flux:

$$u^a = u_\gamma^a, \quad q_\gamma^a := -T_\gamma^a u_\gamma^b - \rho_\gamma u_\gamma^a = 0. \quad (9)$$

The four-velocities of charged particles, $I = p, e$, are

$$u_I^a = \gamma_I (u^a + V_I^a), \quad u_a V_I^a = 0, \quad \gamma_I = (1 - V_{Ia} V_I^a)^{-1/2}, \quad (10)$$

where we also choose the energy frames, so that $q_I^a := -T_{Ib}^a u_I^b - \rho_I u_I^a = 0$.

The Maxwell equations $\nabla_{[a} F_{bc]} = 0$ and $\nabla_b F^{ab} = j^a$ can be split in a 1+3-covariant way (relative to u^a) as [14]

$$D_a B^a = 2E_a \omega^a, \quad (11)$$

$$D_a E^a = -2B_a \omega^a + \mu, \quad (12)$$

$$\begin{aligned} \dot{B}_a = & -\frac{2}{3}\theta B_a + (\sigma_{ab} + \varepsilon_{abc}\omega^c)B^b \\ & - \text{curl } E_a - \varepsilon_{abc}\dot{u}^b E^c, \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{E}_a = & -\frac{2}{3}\theta E_a + (\sigma_{ab} + \varepsilon_{abc}\omega^c)E^b \\ & + \text{curl } B_a + \varepsilon_{abc}\dot{u}^b B^c - J_a. \end{aligned} \quad (14)$$

Here $\mu = -j_a u^a$ is the charge density and $J_a = h_{ab} j^b$ is the current. For the radiation era plasma with $T \lesssim m_e$,

$$j^a = j_p^a + j_e^a, \quad j_I^a = e_I n_I u_I^a, \quad (15)$$

$$\mu = e(\gamma_p n_p - \gamma_e n_e), \quad (16)$$

$$J^a = e(\gamma_p n_p V_p^a - \gamma_e n_e V_e^a), \quad (17)$$

where $e_I = \pm e$, and n_I are the number densities. We can write

$$n_I = n(1 + \Delta_I), \quad (18)$$

where n is the density of charged particles in the tight coupling limit.

The four-current satisfies local charge conservation, $\nabla_a j^a = 0$, which implies

$$\dot{\mu} + \theta\mu = -D_a J^a - \dot{u}_a J^a. \quad (19)$$

In order to close the Maxwell equations, we need to specify j^a , and this is done via the equations of motion for photons and charged particles.

In the Maxwell equations we use the covariant curl,

$$\text{curl } S_a := \varepsilon_{abc} D^b S^c, \quad (20)$$

and the overdot is the covariant time derivative along the radiation 4-velocity u^a , projected into the rest-space:

$$\dot{S}_a := h_a^b u^c \nabla_c S_b. \quad (21)$$

(This is more convenient than the usual definition, which is not projected.) At any spacetime event one can choose inertial coordinates so that $D_a f = (0, \vec{\nabla} f) \dot{S}_a = (0, \partial_t \vec{S}), \text{curl } S_a = (0, \vec{\nabla} \times \vec{S})$. Two important identities are [15]

$$D_a \dot{f} = (D_a f) \dot{} - \dot{f} \dot{u}_a + \left(\frac{1}{3} \theta h_{ab} + \sigma_{ab} - \varepsilon_{abc} \omega^c \right) D^b f, \quad (22)$$

$$\text{curl } D_a f = -2 \dot{f} \omega_a. \quad (23)$$

The vorticity propagation equation is independent of the field equations, and is given covariantly by [15]

$$\dot{\omega}_a = -\frac{2}{3} \theta \omega_a + \sigma_{ab} \omega^b - \frac{1}{2} \text{curl } \dot{u}_a. \quad (24)$$

The conservation law for the electromagnetic energy-momentum tensor $T_F^{ab} = F^a_c F^{bc} - \frac{1}{4} F^{cd} F_{cd} g^{ab}$ follows from the Maxwell equations:

$$\nabla_b T_F^{ab} = -F^{ab} j_b. \quad (25)$$

Photons and charged particles obey the balance equations

$$\nabla_b T_\gamma^{ab} = K_\gamma^a, \quad \nabla_b T_I^{ab} = K_I^a + e_I n_I F^a_b u_I^b, \quad (26)$$

where the K^a four-vectors are the rates of energy-momentum density transfer to the species. By

Eqs. (15) and (25), the conservation of the total energy-momentum, $\nabla_b (T_\gamma^{ab} + \sum_I T_I^{ab} + T_F^{ab}) = 0$, implies that $K_\gamma^a + K_p^a + K_e^a = 0$.

The photon energy and momentum balance equations in the general nonlinear case are [15]

$$\dot{\rho}_\gamma + \frac{4}{3} \rho_\gamma \theta = -\sigma_{ab} \pi_\gamma^{ab} + U_\gamma, \quad (27)$$

$$\frac{4}{3} \rho_\gamma \dot{u}^a + \frac{1}{3} D^a \rho_\gamma = -D_b \pi_\gamma^{ab} - \dot{u}_b \pi_\gamma^{ab} + M_\gamma^a, \quad (28)$$

where π_γ^{ab} is the anisotropic stress, with $\pi_\gamma^{ab} u_a = 0 = \pi_\gamma^{ab} h_{ab}$. Here $U_\gamma = -u_a K_\gamma^a$ and $M_\gamma^a = h^a_b K_\gamma^b$ are the rates of energy and momentum density transfer to photons from Thomson scattering. From now on, we take $\pi_\gamma^{ab} = 0$; the role of photon anisotropic stress in magnetogenesis has been investigated by Takahashi et al. [8, 9, 11]. For electrons and protons, it is reasonable to neglect pressure and anisotropic stresses. Then the energy conservation equations are

$$u_I^b \nabla_b \rho_I + \theta_I \rho_I = U_I, \quad (29)$$

where $\rho_I = m_I n_I$ and $U_I = -u_{Ia} K_I^a$ is the rate of energy density transfer due to Thomson and Coulomb scattering. The momentum balance equations are

$$\rho_I u_I^b \nabla_b u_I^a = M_I^a + e_I n_I F^a_b u_I^b, \quad (30)$$

where $M_I^a = h^a_b K_I^b$ is the rate of momentum density transfer due to Thomson and Coulomb scattering.

As shown in Maartens et al. [15], the Thomson energy transfer rates, U_γ and U_I , start from $O(V_I^2)$ and $O(V_I^3)$, respectively, while the Thomson momentum transfer starts from linear order in V_I . As we will explain later, all $O(V_I^2)$ terms, except those from Thomson scattering, can be neglected in order to derive our evolution equation for magnetic fields. Writing down the necessary terms explicitly, we have

$$U_\gamma = \sum_I \mathcal{C}_{\gamma I} V_I^2 + O(\mathcal{C}_{\gamma I} V_I^3), \quad (31)$$

$$U_I = O(\mathcal{C}_{\gamma I} V_I^3), \quad (32)$$

$$\begin{aligned} M_\gamma^a &= -\sum_I \mathcal{C}_{\gamma I} (u^b - u_I^b) h^a_b \\ &= \sum_I \mathcal{C}_{\gamma I} V_I^a + O(\mathcal{C}_{\gamma I} V_I^3), \end{aligned} \quad (33)$$

$$\begin{aligned} M_I^a &= -\mathcal{C}_{\gamma I} (u_I^b - u^b) h_{Ib}^a - \mathcal{C}_{IJ} (u_I^b - u_J^b) h_{Ib}^a \\ &= -\mathcal{C}_{\gamma I} (V_I^a + V_I^2 u^a) - \mathcal{C}_{IJ} (V_I^a - V_J^a) \\ &\quad + O(\mathcal{C}_{\gamma I} V_I^3), \end{aligned} \quad (34)$$

where $\mathcal{C}_{\gamma I}, \mathcal{C}_{IJ}$ are the Thomson and Coulomb collision coefficients. The energy and momentum balance equa-

tions reduce to

$$\dot{\rho}_\gamma + \frac{4}{3}\rho_\gamma\theta = \sum_I \mathcal{C}_{\gamma I} V_I^2 + O(\mathcal{C}_{\gamma I} V_I^3), \quad (35)$$

$$\begin{aligned} \dot{n}_I + n_I(\theta + D_a V_I^a + \dot{u}_a V_I^a) + V_I^a D_a n_I \\ = O(\mathcal{C}_{\gamma I} V_I^3) + O(V_I^2), \end{aligned} \quad (36)$$

$$\frac{4}{3}\rho_\gamma \dot{u}^a + \frac{1}{3}D^a \rho_\gamma = \sum_I \mathcal{C}_{\gamma I} V_I^a + O(\mathcal{C}_{\gamma I} V_I^3), \quad (37)$$

$$\begin{aligned} m_I n_I (\dot{u}^a + u^b \nabla_b V_I^a + V_I^b \nabla_b u^a) \\ = e_I n_I (E^a + F^a{}_b V_I^b) - \mathcal{C}_{\gamma I} V_I^a - \mathcal{C}_{IJ} (V_I^a - V_J^a) \\ + O(\mathcal{C}_{\gamma I} V_I^3). \end{aligned} \quad (38)$$

The vorticities of charged particles are related to the radiation vorticity ω^a at $O(V_I)$ by [15]

$$\omega_I^a = \omega^a - \frac{1}{2}\text{curl } V_I^a + \frac{1}{2}\varepsilon^a{}_{bc}\dot{u}^b V_I^c + \omega_b V_I^b u^a. \quad (39)$$

Taking the tight coupling limit in Eq. (36) we recover number conservation,

$$\dot{n} + \theta n = 0, \quad (40)$$

and with Eq. (35), this leads to $u^a \nabla_a [\ln(n/\rho_\gamma^{3/4})] = 0$. We define the entropy

$$s_a := D_a \left[\ln \left(\frac{n}{\rho_\gamma^{3/4}} \right) \right]. \quad (41)$$

Using the identity (22), we arrive at

$$\dot{s}_a + \frac{1}{3}\theta s_a + (\sigma_{ba} + \varepsilon_{bac}\omega^c)s^b = 0. \quad (42)$$

In what follows we assume the adiabatic condition $s_a = 0$, i.e.,

$$\frac{D_a n}{n} = \frac{3}{4} \frac{D_a \rho_\gamma}{\rho_\gamma}, \quad (43)$$

which is consistent with Eq. (42).

III. TIGHT COUPLING LIMIT

In the exact tight coupling limit, the velocities of photons, protons and electrons are equal, $V_I^a = 0$, and the momentum transfer terms vanish. Intuitively we expect that vorticity and magnetic fields cannot be generated from zero. In fact, we can prove a stronger result, that vorticity is zero at all nonlinear orders. Equation (38) reduces to $m_I \dot{u}^a = e_I E^a$, which implies $E_a = 0 = \dot{u}_a$. Then the photon momentum balance equation (37) reduces to $D_a \rho_\gamma = 0$, and since $\dot{\rho}_\gamma \neq 0$ by Eq. (35), the identity (23) implies

$$\omega_a = 0. \quad (44)$$

The induction equation (13) reduces to

$$\dot{B}_a = -\frac{2}{3}\theta B_a + \sigma_{ab} B^b, \quad (45)$$

so there is no source term and no magnetogenesis. Equations (44) and (45) hold in the fully nonlinear regime.

Next we consider what happens if exact tight coupling is weakened by neglecting scattering terms, and neglecting the velocity difference between protons and photons, i.e., $V_p^a = 0$, but allowing $V_e^a \neq 0$. This is effectively the assumption made in Ref. [6], and here we reconsider the problem in the covariant formalism. The proton equation of motion (38) becomes $m_p \dot{u}^a = e E^a$, and taking the curl gives

$$\text{curl } \dot{u}_a = \frac{e}{m_p} \text{curl } E_a. \quad (46)$$

The curl of the photon momentum equation (37), using the photon energy equation (35) and the identity (23), gives

$$\text{curl } \dot{u}_a = -\frac{2}{3}\theta \omega_a. \quad (47)$$

Using these equations, the induction equation (13) and the vorticity propagation equation (24) become

$$\dot{B}_a = -\frac{2}{3}\theta B_a + (\sigma_{ab} + \varepsilon_{abc}\omega^c)B^b + \frac{2m_p}{3e}\theta \omega_a, \quad (48)$$

$$\dot{\omega}_a = -\frac{1}{3}\theta \omega_a + \sigma_{ab}\omega^b. \quad (49)$$

Thus vorticity and the magnetic field are *conserved* and no magnetogenesis is possible. This gives our first no-go result:

If (i) the scattering terms are neglected and the proton-photon velocity difference is neglected, (ii) anisotropic stress is neglected, (iii) the initial magnetic field and vorticity are zero, and (iv) the energy-momentum conservation equations hold, then vorticity and magnetic fields cannot be generated, at any perturbative order.

The same result holds if we assume the electron velocity equals the radiation velocity but instead $V_p^a \neq 0$. It is not clear how the results in Ref. [6] relate to our no-go result.

IV. TIGHT COUPLING APPROXIMATION

Nonlinear magnetogenesis is ruled out in the tight coupling limit of zero collision time, $\tau = 0$. Beyond the zero-order of tight coupling, there is a nonzero τ and a nonzero velocity difference v , which is governed by the momentum balance equations (37) and (38). Schematically, these are of the form

$$\dot{v} = \frac{v}{\tau} + A, \quad (50)$$

where A represents terms other than scattering terms. Since $\dot{v} \sim Hv$, we have $\dot{v} \ll v/\tau$. We expand in terms of the tight coupling parameter τH :

$$v = v_{(1)} + v_{(2)} + \dots, \quad A = A_{(0)} + A_{(1)} + \dots, \quad (51)$$

and we use $\text{TCA}(n)$ to denote n -th order in the tight coupling approximation. Then

$$\text{TCA}(1): \quad 0 = \frac{v_{(1)}}{\tau} + A_{(0)}, \quad (52)$$

$$\text{TCA}(2): \quad \dot{v}_{(1)} = \frac{v_{(2)}}{\tau} + A_{(1)}. \quad (53)$$

The TCA is complicated by the presence of Coulomb scattering, so that strictly we need to perform TCA expansions in both Thomson and Coulomb small parameters. However, as we will argue, it is reasonable to neglect the Coulomb collision time, i.e., to assume tight coupling of protons and electrons.

The collision coefficients in Eqs. (37) and (38) are

$$\mathcal{C}_{\gamma e} = \frac{4}{3} \sigma_T \rho_\gamma n_e, \quad (54)$$

$$\mathcal{C}_{\gamma p} = \beta^2 \frac{n_p}{n_e} \mathcal{C}_{\gamma e}, \quad (55)$$

$$\mathcal{C}_{pe} = e^2 n_e n_p \eta, \quad (56)$$

where σ_T is the Thomson cross-section and

$$\beta := \frac{m_e}{m_p}. \quad (57)$$

Here η is the resistivity of the cosmic plasma,

$$\eta = \frac{4\pi e^2}{m_e} \left(\frac{m_e}{T} \right)^{3/2} \ln \Lambda \sim 10^{-13} \left(\frac{T}{\text{eV}} \right)^{-3/2} \text{ s}, \quad (58)$$

where $\ln \Lambda$ is the Coulomb logarithm, $\Lambda \sim T^{3/2} e^{-3} n^{-1/2}$ and $n \sim 10^{-10} T^3$. The \mathcal{C} 's define key timescales, together with the Hubble timescale:

$$\tau_{\gamma e} := \frac{m_e n_e}{\mathcal{C}_{\gamma e}} \sim 10^5 \left(\frac{T}{\text{eV}} \right)^{-4} \text{ s}, \quad (59)$$

$$\tau_{\gamma p} := \frac{m_p n_p}{\mathcal{C}_{\gamma p}} = \beta^{-3} \tau_{\gamma e} \sim 10^{15} \left(\frac{T}{\text{eV}} \right)^{-4} \text{ s}, \quad (60)$$

$$\tau_{ep} := \frac{m_e n_e}{\mathcal{C}_{pe}} \sim 10^{-4} \left(\frac{T}{\text{eV}} \right)^{-3/2} \text{ s}, \quad (61)$$

$$H^{-1} \sim 10^{12} \left(\frac{T}{\text{eV}} \right)^{-2} \text{ s}. \quad (62)$$

Thus

$$H\tau_{\gamma p} \sim 10^3 \left(\frac{T}{\text{eV}} \right)^{-2}, \quad (63)$$

$$H\tau_{\gamma e} \sim 10^{-7} \left(\frac{T}{\text{eV}} \right)^{-2}, \quad (64)$$

$$H\tau_{ep} \sim 10^{-16} \left(\frac{T}{\text{eV}} \right)^{1/2}. \quad (65)$$

As one can see, Thomson scattering between photons and electrons and Coulomb scattering between protons and electrons are very effective on cosmological timescales, so that they are tightly coupled before recombination. Although Thomson scattering between photons and protons is less effective at low temperatures, protons also closely follow photons through their Coulomb coupling to electrons.

From Eqs. (64) and (65), we see that,

$$\frac{\tau_{\gamma e}}{\tau_{ep}} \sim 10^9 \left(\frac{T}{\text{eV}} \right)^{-5/2}. \quad (66)$$

Therefore, at lower temperatures, $T \lesssim 1$ keV, Coulomb scattering is more effective than Thomson scattering, so that protons and electrons are more tightly coupled than photons and charged particles are. This suggests that we can safely neglect Coulomb scattering, i.e., take $\tau_{ep} = 0 = V_p^a - V_e^a = \Delta_p - \Delta_e$. We define the centre of mass velocity V^a and number density deviation Δ :

$$(m_p n_p + m_e n_e) V^a = m_p n_p V_p^a + m_e n_e V_e^a, \quad (67)$$

$$(m_p + m_e) \Delta = m_p \Delta_p + m_e \Delta_e. \quad (68)$$

Then the peculiar velocities decompose as

$$V_I^a = V^a + v_I^a \approx V^a, \quad (69)$$

where v_I^a are the deviations of proton and electron velocity from their centre of mass velocity, with $m_p n_p v_p^a + m_e n_e v_e^a = 0$. The number density deviations decompose as

$$\Delta_I = \Delta + \delta_I \approx \Delta, \quad (70)$$

where δ_I are the deviations of proton and electron number density from their centre of mass density, with $m_p \delta_p + m_e \delta_e = 0$. The approximations in Eqs. (69) and (70) are based on neglecting terms of order τ_{ep} :

$$H\tau_{\gamma e} \sim |\Delta| \sim |V^a| \gg H\tau_{ep} \sim |\delta_I| \sim |v_I^a|. \quad (71)$$

In fact, protons and electrons are coupled not only by Coulomb scattering but also by the electric field, so that $|\delta_I|, |v_I^a|$ are further suppressed by a factor [18],

$$H\eta \sim 10^{-27} \left(\frac{T}{\text{eV}} \right)^{1/2}. \quad (72)$$

Thus we can safely apply the approximation at any temperatures we consider here, $m_e \gtrsim T \gtrsim T_{\text{rec}}$, and we can solve equations perturbatively with respect to the small parameter $H\tau_{\gamma e}$.

In Eqs. (36)–(38), we can set $V_I^a = V^a$. We can see from Eqs. (52) and (53) that, upto TCA(2) we consider here, $O(\mathcal{C}_{\gamma I} V_I^3)$ and $O(V_I^2)$ terms can be neglected and the momentum equations become,

$$\frac{4}{3} \rho_\gamma \dot{u}^a + \frac{1}{3} D^a \rho_\gamma = \frac{(\beta^3 m_p n_p + m_e n_e)}{\tau} V^a, \quad (73)$$

$$m_I (\dot{u}^a + u^b \nabla_b V^a + V^b \nabla_b u^a) = e_I (E^a + F^a{}_b V^b) - \frac{m_I}{\tau} V^a, \quad (74)$$

where we have defined

$$\tau := \tau_{\gamma e}, \quad \tau_I := \tau_{\gamma I} = (\beta^{-3}\tau, \tau). \quad (75)$$

On the other hand, as we will see in section IV B, we need only TCA(1) for the charged particle conservation equation (36) to derive the evolution equation:

$$\dot{\Delta} + D_a V^a + \dot{u}_a V^a + V^a \frac{D_a n}{n} = 0. \quad (76)$$

A. First order – TCA(1)

At first-order TCA, we follow Eq. (52) and keep only the first-order V^a and the zero-order of other terms.

By summing the photon and charged particle equations (73) and (74), we obtain

$$\dot{u}^a = -\frac{1}{4\rho_\gamma(1+R)} D^a \rho_\gamma, \quad (77)$$

where

$$R := \frac{3\rho_b}{4\rho_\gamma}, \quad \rho_b := (m_p + m_e)n. \quad (78)$$

The charged particle equations (74) imply

$$0 = (1+\beta) \frac{e}{m_e} E^a + \frac{(1-\beta^3)}{\tau} V_{(1)}^a. \quad (79)$$

The photon and charged particle equations (73) and (74) also lead to

$$0 = \frac{1}{4\rho_\gamma} D^a \rho_\gamma - \left(\frac{1+\beta^2}{1+\beta} \right) \frac{\beta(1+R)}{\tau} V_{(1)}^a. \quad (80)$$

It follows that

$$\begin{aligned} E^a &= -\frac{m_e}{e} \frac{(1-\beta^3)}{(1+\beta)} \frac{1}{\tau} V_{(1)}^a \\ &= -\frac{m_p}{e} \frac{(1-\beta^3)}{(1+\beta^2)} \frac{1}{4\rho_\gamma(1+R)} D^a \rho_\gamma. \end{aligned} \quad (81)$$

Using Eqs. (77) and (81) in the induction equation (13), we find the evolution equation for the magnetic field at TCA(1):

$$\begin{aligned} \dot{B}_a &= -\frac{2}{3}\theta B_a + (\sigma_{ab} + \varepsilon_{abc}\omega^c) B^b \\ &\quad - \left[\frac{m_p}{e} \frac{(1-\beta^3)}{(1+\beta^2)} \frac{\dot{\rho}_\gamma}{2\rho_\gamma(1+R)} \right] \omega_a, \end{aligned} \quad (82)$$

where we used the adiabatic condition (43). The magnetic field is sourced by the vorticity of photons. However, from Eqs. (43) and (77), we see that $\text{curl } \dot{u}^a = [\dot{\rho}_\gamma/2\rho_\gamma(1+R)]\omega^a$, and the vorticity propagation equation (24) becomes

$$\dot{\omega}_a = \left[\frac{1+2R}{4(1+R)} \frac{\dot{\rho}_\gamma}{\rho_\gamma} \right] \omega_a + \sigma_{ab}\omega^b. \quad (83)$$

This shows that there is no source term for the vorticity, and we have our second no-go result:

No generation of vorticity or magnetic fields is possible at first order in the tight coupling approximation.

Note that at TCA(1), terms of the form $\vec{\nabla} n \times \vec{v}$ do not arise. These terms come in at TCA(2). Magnetogenesis via the $\vec{\nabla} \times \vec{v}$ vorticity term is not possible without breaking the tight coupling approximation at first order; it is not clear how the results of Ref. [6] conform to this no-go result.

B. Second order – TCA(2)

Now we proceed to the second-order TCA version of the nonlinear evolution equations for the magnetic field and vorticity.

The charged particle equations (74) imply

$$E^a + F^a{}_b V^b = -\frac{m_e}{e} \frac{(1-\beta^3)}{(1+\beta)} \frac{1}{\tau} V^a. \quad (84)$$

Substituting into the induction equation (13), we have

$$\begin{aligned} \dot{B}_a &= -\frac{2}{3}\theta B_a + (\sigma_{ab} + \varepsilon_{abc}\omega^c) B^b \\ &\quad + \frac{m_e}{e} \frac{(1-\beta^3)}{(1+\beta)} \frac{1}{\tau} \left[\text{curl } V^a - \frac{3}{4}\varepsilon^{abc} V_b \frac{D_c \rho_\gamma}{\rho_\gamma} \right], \end{aligned} \quad (85)$$

where $F^a{}_b V^b$ in Eq. (84) was dropped because it is always negligible compared to \dot{B}^a .

We need to solve for V^a at TCA(2). The photon and charged particle equations (73) and (74) give

$$\begin{aligned} u^b \nabla_b V^a + V^b \nabla_b u^a - \frac{1}{4} \frac{D^a \rho_\gamma}{\rho_\gamma} \\ = -\frac{\beta(1+\beta^2)}{(1+\beta)} \frac{(1+R+R\Delta)}{\tau} V^a. \end{aligned} \quad (86)$$

This can be split into TCA(1) and TCA(2) equations:

$$\begin{aligned} -\frac{1}{4} \frac{D^a \rho_\gamma}{\rho_\gamma} &= -\frac{\beta(1+\beta^2)}{(1+\beta)} \frac{(1+R)}{\tau} V_{(1)}^a, \\ u^b \nabla_b V_{(1)}^a + V_{(1)}^b \nabla_b u^a \\ &= -\frac{\beta(1+\beta^2)}{(1+\beta)} \frac{1}{\tau} \left[(1+R) V_{(2)}^a + R\Delta_{(1)} V_{(1)}^a \right], \end{aligned} \quad (87)$$

where the first equation is equivalent to Eq. (80). We can solve these equations for V^a :

$$\begin{aligned} V_{(1)}^a &= \frac{1}{4(1+R)} \frac{(1+\beta)}{\beta(1+\beta^2)} \frac{D^a \rho_\gamma}{\rho_\gamma}, \\ V_{(2)}^a &= -\frac{R}{4(1+R)^2} \frac{(1+\beta)}{\beta(1+\beta^2)} \tau \Delta_{(1)} \frac{D^a \rho_\gamma}{\rho_\gamma} \\ &\quad - \frac{1}{(1+R)} \frac{(1+\beta)}{\beta(1+\beta^2)} \tau \left[u^b \nabla_b V_{(1)}^a + V_{(1)}^b \nabla_b u^a \right], \end{aligned} \quad (89)$$

$$- \frac{1}{(1+R)} \frac{(1+\beta)}{\beta(1+\beta^2)} \tau \left[u^b \nabla_b V_{(1)}^a + V_{(1)}^b \nabla_b u^a \right] \quad (90)$$

The last term on the right of Eq. (90) is determined in terms of $D^a \rho_\gamma$ from Eq. (89), and the vorticity occurs explicitly since $V_{(1)}^b \nabla_b u^a \propto D^b \rho_\gamma [\frac{1}{3} \theta h^a_b + \sigma^a_b + \varepsilon^a_{bc} \omega^c]$.

Now we can compute the crucial term in Eq. (85):

$$\begin{aligned} \text{curl } V^a &= \frac{3}{4} \varepsilon^{abc} V_b \frac{D_c \rho_\gamma}{\rho_\gamma} \\ &= -\frac{1}{2(1+R)} \frac{(1+\beta)}{\beta(1+\beta^2)} \tau \frac{\dot{\rho}_\gamma}{\rho_\gamma} \omega^a \\ &\quad - \frac{R}{4(1+R)^2} \frac{(1+\beta)}{\beta(1+\beta^2)} \tau \varepsilon^{abc} D_b \Delta_{(1)} \frac{D_c \rho_\gamma}{\rho_\gamma}. \end{aligned} \quad (91)$$

Finally the evolution equation (85) for the magnetic field can be written, up to TCA(2), as

$$\begin{aligned} \dot{B}^a &= -\frac{2}{3} \theta B^a + \sigma^a_b B^b \\ &\quad - \left[\frac{m_p}{e} \frac{R}{4(1+R)^2} \frac{(1-\beta^3)}{(1+\beta^2)} \right] \varepsilon^{abc} D_b \Delta_{(1)} \frac{D_c \rho_\gamma}{\rho_\gamma} \\ &\quad - \left[\frac{m_p}{e} \frac{1}{2(1+R)} \frac{(1-\beta^3)}{(1+\beta^2)} \frac{\dot{\rho}_\gamma}{\rho_\gamma} \right] \omega^a. \end{aligned} \quad (92)$$

The vorticity evolution equation (24) can be rewritten, using Eqs. (73), (89) and (90), as

$$\begin{aligned} \dot{\omega}^a &= \left[\frac{(1+2R)}{4(1+R)} \frac{\dot{\rho}_\gamma}{\rho_\gamma} \right] \omega^a + \sigma^a_b \omega^b \\ &\quad - \left[\frac{R}{8(1+R)^2} \right] \varepsilon^{abc} D_b \Delta_{(1)} \frac{D_c \rho_\gamma}{\rho_\gamma}. \end{aligned} \quad (93)$$

The evolution of the baryonic number density deviation is governed by Eq. (76), which becomes, up to TCA(1),

$$\dot{\Delta}_{(1)} = -\frac{3}{4} V_{(1)}^a \frac{D_a \rho_\gamma}{\rho_\gamma} - D_a V_{(1)}^a - \dot{u}_a V_{(1)}^a, \quad (94)$$

where $V_{(1)}^a$ is given by Eq. (89).

Equations (92), (93) and (94), for given $D^a \rho_\gamma / \rho_\gamma$, form a complete set of equations which describe the evolution of $\Delta_{(1)}$, B^a and ω^a . We can see that *both magnetic field and vorticity are generated at the second order in the tight coupling approximation.*

V. SUMMARY

In this paper we have derived the evolution equations for cosmological magnetic fields and vorticity using the 1+3-covariant formalism. The covariant approach allows us to construct a set of equations describing the *fully nonlinear* evolution of cosmic inhomogeneities. We have performed a tight coupling expansion for Thomson and Coulomb interactions to make the key physical processes transparent. It should be emphasized that we have not expanded inhomogeneous quantities with respect to cosmological perturbations, and therefore our results are valid at any order in cosmological perturbations. Thus,

the present analysis is complementary to previous studies based on cosmological perturbation theory.

Our first no-go result is that magnetic fields and vorticity cannot be generated in the tight coupling limit or its weak extension without anisotropic stresses. Then we have considered leading and next-to-leading order effects in the tight coupling approximation. We have found that magnetic fields and vorticity are not generated at first order in the tight coupling approximation [TCA(1)]. The second order tight coupling approximation [TCA(2)] is necessary for generating both of them, and we have derived a closed set of nonlinear evolution equations at TCA(2). It is worth noting that we have not invoked the Einstein equations, so that our result does not rely on any specific theory of gravity.

The magnitude of the generated magnetic field can be roughly estimated as follows. From Eqs. (89) and (94), we have $\Delta_{(1)} \sim (\tau k^2 \delta) / (\beta H a^2)$ where $\delta \sim 10^{-5}$ is the density perturbation, k is the wave number and a is the scale factor. Using Eqs. (92) and (93), we see that the contributions to B^a from the vorticity term and the gradient term are of the same order of magnitude, and we obtain

$$\langle |B| \rangle \sim \sqrt{B_a B^a} \sim \frac{m_p R \tau}{e \beta H^2} \left(\frac{k}{a} \right)^4 \delta^2 \sim 10^{-27} \text{G}, \quad (95)$$

where we evaluated the amplitude at the horizon scale at recombination, $H \sim k \sim 1/(100 \text{Mpc})$ and $a \sim 10^{-3}$. This is in the range of previous estimates [6, 7, 10, 11], as in Eq. (4).

The anisotropic stress of photons is neglected in the present analysis. However, as reported by Ichiki et al. [9, 11], this is important for magnetogenesis on small scales ($\lesssim 1 \text{Mpc}$) and in the earlier universe. It is expected that the anisotropic stress is important also for the generation of vorticity on the same scales and in the same era. We will discuss the effect of the anisotropic stresses in future work.

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