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Adiabatic initial conditions for perturbations in interacting dark energy models

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ABSTRACT

We present a new systematic analysis of the early radiation era solution in an interacting dark energy model to find the adiabatic initial conditions for the Boltzmann integration. In a model where the interaction is proportional to the dark matter density, adiabatic initial conditions and viable cosmologies are possible if the early-time dark energy equation of state parameter is $w_e > -4/5$. We find that when adiabaticity between cold dark matter, baryons, neutrinos and photons is demanded, the dark energy component satisfies automatically the adiabaticity condition. As supernovae Ia or baryon acoustic oscillation data require the recent-time equation of state parameter to be more negative, we consider a time-varying equation of state in our model. In a companion paper [arXiv:0907.4987] we apply the initial conditions derived here, and perform a full Monte Carlo Markov Chain likelihood analysis of this model.

Key words: cosmology:theory, cosmic microwave background, cosmological parameters, dark matter

1 INTERACTING DARK ENERGY

Dark energy and dark matter, the dominant sources in the 'standard' model for the evolution of the universe, are currently only detected via their gravitational effects. This implies an inevitable degeneracy between them. A dark sector interaction could thus be consistent with current observational constraints. We look at such a model, assuming that the dark matter and dark energy can be treated as fluids whose interaction is proportional to the dark matter density.

For interacting dark energy, the energy balance equations in the background are

$$\rho_c' = -3\mathcal{H}\rho_c + aQ_c\,, \tag{1}$$

$$\rho_{de}' = -3\mathcal{H}(1+w_{de})\rho_{de} + aQ_{de}\,, \qquad Q_{de} = -Q_c\,, \tag{2}$$

where $\mathcal{H} = a'/a$, $w_{de} = p_{de}/\rho_{de}$ is the dark energy equation of state parameter, a prime indicates derivative with respect to conformal time τ, and Q_c is the rate of transfer of dark matter density due to the interaction.

Various forms for Q_c have been investigated (see, e.g. Wetterich (1995); Amendola (1999); Billyard & Coley (2000); Zimdahl & Pavon (2001); Farrar & Peebles (2004); Chimento et al. (2003); Olivares et al. (2005); Koivisto (2005); Guo et al. (2007); Sadjadi & Alimohammadi (2006); Boehmer et al. (2008); He & Wang (2008); Quartin et al. (2008); Pereira & Jesus (2009); Quercellini et al. (2008); Valiviita et al. (2008); He et al. (2009); Bean et al. (2008); Chongchitnan (2009); Corasaniti (2008); Caldera-Cabral et al. (2009); Gavela et al. (2009); Jackson et al. (2009)). *We consider* models where the interaction has the form of a decay of one species into another - as in simple models of reheating and of curvaton decay (Malik et al. 2003; Sasaki et al. 2006; Assadullahi et al. 2007). Such a model was introduced by Boehmer et al. (2008); Valiviita et al. (2008). It is not derived from a Lagrangian [in contrast with e.g. Wetterich (1995); Amendola (1999)], but it is motivated physically as a simple phenomenological model for decay of dark matter particles into dark energy. In this sense, it improves on most other phenomenological models, which are typically designed for mathematical simplicity, rather than as models of interaction. The methods that we use here and in the companion paper (Valiviita et al. 2009) may readily be extended to other interactions, including those based on a Lagrangian. We assume that in the background the interation takes the form (Boehmer et al. 2008;

Valiviita et al. 2008)

$$Q_c = -\Gamma \rho_c \,, \tag{3}$$

where Γ is the constant rate of transfer of dark matter density. Positive Γ corresponds to the decay of dark matter into dark energy, while negative Γ indicates a transfer of energy from dark energy to dark matter.

In Valiviita et al. (2008) we considered the case of fluid dark energy with a constant equation of state parameter $-1 < w_{de} \leqslant -4/5$, and found a serious large-scale non-adiabatic instability in the early radiation era. This instability grows stronger as w_{de} approaches -1. Phantom models, $w_{de} < -1$, do not suffer from this instability, but we consider them to be unphysical.

The instability is determined by the early-time value of w_{de}. We will show that the models are not affected by the large-scale non-adiabatic instability during early radiation domination if at early times $w_{de} > -4/5$. If we allow w_{de} to vary, such a large early w_{de} can be still consistent with Supernovae Type Ia (SN) and baryon acoustic oscillation (BAO) observations, provided that at late times $w_{de} \sim -1$. In this paper, we represent w_{de} via the parametrization $w_{de} = w_0 + w_a(1-a)$ (Chevallier & Polarski 2001; Linder 2003), which we rewrite as

$$w_{de} = w_0 a + w_e(1-a)\,, \tag{4}$$

where $w_e = w_0 + w_a$ is the early-time value of w_{de}, while w_0 is the late-time value.

There are two critical features of the analysis of interacting models, which are not always properly accounted for in the literature:

- The background energy transfer rate Q_c does not in itself determine the interaction in the perturbed universe. One should also specify the momentum transfer rate, preferably via a physical assumption. We make the physical assumption that the momentum transfer vanishes in the dark matter rest-frame; this requires that the energy-momentum transfer rate is given covariantly by (Kodama & Sasaki 1984; Valiviita et al. 2008)

$$Q_c^\mu = Q_c u_c^\mu = -Q_{de}^\mu\,, \qquad Q_c = -\Gamma \rho_c (1+\delta_c), \tag{5}$$

where u_c^μ is the dark matter 4-velocity, and $\delta_c = \delta\rho_c/\rho_c$ is the cold dark matter (CDM) density contrast.

- Adiabatic initial conditions in the presence of a dark sector interaction require a very careful analysis of the early-radiation solution, both in the background and in the perturbations. We derive these initial conditions by generalizing the methods of Doran et al. (2003) to the interacting case, thereby extending our previous results (Valiviita et al. 2008).

We give here the first systematic analysis of the initial conditions for perturbations in the interacting model given by Eq. (5) – and our methods can be adjusted to deal with other forms of interaction. In the companion paper Valiviita et al. (2009) we report the results of our full Monte Carlo Markov Chain likelihood scans for this model. Cosmological perturbations of other interacting models have been investigated, e.g., in Amendola et al. (2003); Koivisto (2005); Olivares et al. (2006); Mainini & Bonometto (2007); Bean et al. (2008); Vergani et al. (2009); Pettorino & Baccigalupi (2008); La Vacca & Colombo (2008); Schäfer (2008); Schaefer et al. (2008); He et al. (2009); Bean et al. (2008); Corasaniti (2008); Chongchitnan (2009); Jackson et al. (2009); Gavela et al. (2009); La Vacca et al. (2009); He et al. (2009); Caldera-Cabral et al. (2009); He et al. (2009); Koyama et al. (2009); Kristiansen et al. (2009).

2 PERTURBATION EQUATIONS

The scalar perturbations of the spatially flat Friedmann-Robertson-Walker metric are given by

$$ds^2 = a^2\Big\{-(1+2\phi)d\tau^2 + 2\partial_i B\, d\tau dx^i + \Big[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E\Big]dx^i dx^j\Big\}\,. \tag{6}$$

In the perturbed universe, the dark sector interaction involves a transfer of momentum as well as energy. The covariant form of energy-momentum transfer for a fluid component A is $\nabla_\nu T_A^{\mu\nu} = Q_A^\mu$, where $Q_c^\mu = a^{-1}(Q_c, \vec{0}) = -Q_x^\mu$ in the background. The perturbed energy-momentum transfer 4-vector can be split as (Valiviita et al. 2008)

$$Q_0^A = -a\left[Q_A(1+\phi) + \delta Q_A\right], \qquad Q_i^A = a\partial_i\left(f_A - Q_A\frac{\theta}{k^2}\right), \tag{7}$$

where k is the comoving wavenumber, f_A is the intrinsic momentum transfer potential and $\theta = (\rho+p)^{-1}\sum(\rho_A + p_A)\theta_A$ is the total velocity perturbation ($\theta = -k^2 v$). The evolution equations for density perturbations and velocity perturbations for a generic fluid are (Valiviita et al. 2008; Kodama & Sasaki 1984)

$$\begin{aligned}&\delta_A' + 3\mathcal{H}(c_{sA}^2 - w_A)\delta_A + (1+w_A)\theta_A + 3\mathcal{H}\big[3\mathcal{H}(1+w_A)(c_{sA}^2 - w_A) + w_A'\big]\frac{\theta_A}{k^2}\\ &\quad - 3(1+w_A)\psi' + (1+w_A)k^2(B - E') = \frac{aQ_A}{\rho_A}\left[\phi - \delta_A + 3\mathcal{H}(c_{sA}^2 - w_A)\frac{\theta_A}{k^2}\right] + \frac{a}{\rho_A}\,\delta Q_A\,,\end{aligned} \tag{8}$$

$$\theta_A' + \mathcal{H}(1-3c_{sA}^2)\theta_A - \frac{c_{sA}^2}{(1+w_A)}\,k^2\delta_A + \frac{2w_A}{3(1+w_A)}\,k^2\pi_A - k^2\phi = \frac{aQ_A}{(1+w_A)\rho_A}\big[\theta - (1+c_{sA}^2)\theta_A\big] - \frac{a}{(1+w_A)\rho_A}\,k^2 f_A\,. \tag{9}$$

Here c_{sA}^2 is the sound speed, and π_A is the anisotropic stress. For our model $\pi_{de} = 0$, and we set $c_{s\,de}^2 = 1$, as in standard non-interacting quintessence models, in order to avoid adiabatic instabilities (see discussion in Valiviita et al. (2008)).

For the interaction defined by Eq. (5), we find from Eq. (7) that

$$f_c = \Gamma \frac{\rho_c}{k^2} (\theta_c - \theta) = -f_{de} \,. \tag{10}$$

Then we can write the dark energy and cold dark matter perturbation equations for our model:

$$\delta'_{de} + 3\mathcal{H}(1-w_{de})\delta_{de} + (1+w_{de})\left[\theta_{de} + k^2(B-E')\right] + 9\mathcal{H}^2(1-w_{de}^2)\frac{\theta_{de}}{k^2} - 3a\mathcal{H}^2 w_a \frac{\theta_{de}}{k^2} - 3(1+w_{de})\psi' =$$
$$a\Gamma \frac{\rho_c}{\rho_{de}}\left[\delta_c - \delta_{de} + 3\mathcal{H}(1-w_{de})\frac{\theta_{de}}{k^2} + \phi\right], \tag{11}$$
$$\theta'_{de} - 2\mathcal{H}\theta_{de} - \frac{k^2}{(1+w_{de})}\delta_{de} - k^2\phi = \frac{a\Gamma}{(1+w_{de})}\frac{\rho_c}{\rho_{de}}(\theta_c - 2\theta_{de}) \,, \tag{12}$$
$$\delta'_c + \theta_c + k^2(B-E') - 3\psi' = -a\Gamma\phi \,, \tag{13}$$
$$\theta'_c + \mathcal{H}\theta_c - k^2\phi = 0 \,. \tag{14}$$

3 BACKGROUND SOLUTION IN EARLY RADIATION ERA

The background solution in the early radiation era ($\rho_{\rm tot} \simeq \rho_r$) is important for finding the initial conditions for the integration of cosmological perturbations. In what follows we use occasionally the Hubble parameter $H = a^{-1}\mathcal{H}$ instead of the conformal Hubble parameter $\mathcal{H}$. In the radiation era we have

$$\mathcal{H} = \tau^{-1} \qquad \text{and} \qquad a = \mathcal{H}_0\sqrt{\Omega_{r0}}\,\tau \,, \tag{15}$$

where $\mathcal{H}_0$ is the conformal Hubble parameter today, and $\Omega_{r0} = \rho_{r0}/\rho_{\rm crit0} \approx 2.47 \times 10^{-5}h^{-2}$ is the radiation energy density parameter today. Here h is defined by $H_0 = h \times 100\,{\rm km\,s^{-1}Mpc^{-1}}$, and as $a_0 = 1$, we have $H_0 = \mathcal{H}_0$. Furthermore, we have $H = (2t)^{-1}$ and $a = (2H_0\sqrt{\Omega_{r0}}\,t)^{1/2}$ where t is the cosmic time. By Eq. (15) we find

$$t = (H_0\sqrt{\Omega_{r0}}\,/2)\tau^2 \,. \tag{16}$$

We define the ratio of dark energy to cold dark matter density $r = \rho_{de}/\rho_c$. Then, employing Eqs. (1) and (2),

$$\dot{r} = -\left\{\frac{3w_e}{2t} - \left[\Gamma + 3w_a\left(\frac{H_0\sqrt{\Omega_{r0}}}{2t}\right)^{1/2}\right]\right\} r + \Gamma \,, \tag{17}$$

where the dot indicates derivative with respect to cosmic time. At early enough times, $|\Gamma + 3w_a(H_0\sqrt{\Omega_{r0}}/2t)^{1/2}| \ll 3|w_e|/(2t)$, and we can neglect the term in square brackets, so that the solution is

$$r = r_{\rm ref}\left(\frac{t}{t_{\rm ref}}\right)^{-3w_e/2} + \frac{2\Gamma}{3w_e+2}t \,, \tag{18}$$

where $r_{\rm ref}$ is an integration constant corresponding to ρ_{de}/ρ_c at some (early) reference time $t_{\rm ref}$ in the case where $\Gamma = 0$. From Eq. (18) we find that we have two regimes, depending on the value of the early-time equation of state parameter w_e. If $w_e \leqslant -2/3$, then the second term dominates over the first as $t \to 0$, and we recover the solution of Valiviita et al. (2008):

$$\frac{\rho_{de}}{\rho_c} = \frac{a\Gamma}{3w_e+2}\mathcal{H}^{-1} = \tilde{C}(k\tau)^2 \,, \qquad \tilde{C} = \frac{H_0\Gamma}{k^2}\frac{\sqrt{\Omega_{r0}}}{3w_e+2} \,. \tag{19}$$

If $w_e > -2/3$, then the first term in Eq. (18) dominates:

$$\frac{\rho_{de}}{\rho_c} \simeq r_{\rm ref}\left(\frac{t}{t_{\rm ref}}\right)^{-3w_e/2} = C(k\tau)^{-3w_e} \,, \qquad C = r_{\rm ref}\left(\frac{H_0\sqrt{\Omega_{r0}}}{2t_{\rm ref}k^2}\right)^{-3w_e/2} \,. \tag{20}$$

For the background evolution of ρ_c in the radiation dominated era,

$$\dot{\rho}_c = -\left(\frac{3}{2t} + \Gamma\right)\rho_c \,, \tag{21}$$

the second term in brackets is negligible relative to the first at times $t \ll 3/(2\Gamma)$, or $\tau \ll \tau_{\rm switch} = (\sqrt{\Omega_{r0}}|\Gamma/H_0|/3)^{-1/2}H_0^{-1}$. For these times, $\rho_c \propto a^{-3}$. In typical models that provide a good fit to Cosmic Microwave Background (CMB) data, $H_0^{-1} = \mathcal{O}(10)\,$Gpc, and the conformal time at matter-radiation equality is $\tau_{eq} = \mathcal{O}(100)\,$Mpc. If we demand that the evolution of ρ_c is effectively standard during the whole radiation dominated era, i.e. $\tau_{\rm switch} > \tau_{eq}$, we require

$$\left|\frac{\Gamma}{H_0}\right| \lesssim \frac{30\,000}{\sqrt{\Omega_{r0}}} \approx 10^6\,h \,, \tag{22}$$

where $h \sim 0.7$. As we study in this paper coupling strengths $|\Gamma/H_0| \lesssim 1$, we can safely assume that the cold dark matter evolution during radiation domination is completely the standard non-interacting one $\rho_c = \rho_c^{eq}(a/a_{eq})^{-3}$, where ρ_c^{eq} and a_{eq}

are the dark matter energy density and the scale factor at matter-radiation equality, respectively. Noticing that the radiation energy density can be written as $\rho_r = \rho_r^{eq}(a/a_{eq})^{-4}$, and that $\rho_c^{eq} = \rho_r^{eq}$ by definition, we find that in the radiation dominated era

$$\frac{\rho_c}{\rho_r} = \frac{a}{a_{eq}} = \omega_2\, k\tau\,, \qquad \omega_2 = \frac{H_0}{k}\frac{\sqrt{\Omega_{r0}}}{a_{eq}}, \tag{23}$$

where we used Eq. (15). In the non-interacting case we could continue by setting $a_{eq} = \Omega_{r0}/\Omega_{c0}$, but in the interacting case the dark matter evolution from $\tau_{\rm switch}$ up to today (τ_0) differs from $\propto a^{-3}$: by Eq. (21), it follows that at recent times, for a positive Γ, the dark matter density decreases faster, and with a negative Γ it decreases slower than a^{-3}. Therefore we cannot do the "a^{-3} scaling" all the way up to today, but instead have to stop at some early enough reference time. Here we choose the time of matter-radiation equality.

An upper limit to the early dark energy (DE) equation of state w_e could be set by requiring dark matter domination over DE at early times. Then Eq. (20) would set the constraint $w_e < 0$. However, if the DE equation of state is close to 0 at early times, it could well mimic the behaviour of cold dark matter. On the other hand, if w_e is close to 1/3, the "DE" component would behave like radiation at early times. So, for $0 \leqslant w_e < 1/3$, we conclude that the fluid which at late times behaves like dark energy, behaves at early times like a combination of matter and radiation. As this case cannot be ruled out, we set a conservative upper bound on w_e by demanding that in the early universe DE does not dominate over radiation, i.e., for $\tau \to 0$, we have $\rho_{de}/\rho_r \to 0$. Using Eqs. (20) and (23), we find

$$\frac{\rho_{de}}{\rho_r} = \frac{\rho_{de}}{\rho_c}\frac{\rho_c}{\rho_r} = C\omega_2 (k\tau)^{1-3w_e}\,, \tag{24}$$

which implies, as expected, $w_e < 1/3$.

4 SUPER-HORIZON INITIAL CONDITIONS FOR PERTURBATIONS

In order to solve numerically the perturbation equations we need to specify initial conditions in the early radiation era. The wavelength of the relevant fluctuations is far outside the horizon during this period: $k\tau \ll 1$. To compute the initial conditions we start by writing the perturbation equations of each species and the perturbed Einstein equations in terms of the gauge-invariant variables developed by Bardeen (1980):

$$\Phi = -\psi + \mathcal{H}(B-E')\,, \qquad \Psi = \phi + \mathcal{H}\left(B-E'\right) + \left(B-E'\right)'\,,$$
$$\Delta_A = \delta_A + \mathcal{H}^{-1}\frac{\rho_A'}{\rho_A}\psi\,, \qquad V_A = k^{-1}\theta_A + k(B-E')\,, \qquad \Pi_A = \pi_A. \tag{25}$$

The general evolution equations for the density, velocity and anisotropic stress perturbations Δ_A, V_A and Π_A and the Einstein equations for the metric perturbations Φ, Ψ are given in Kodama & Sasaki (1984).

We use and generalize the systematic method of Doran et al. (2003) in order to analyze the initial conditions in the interacting DE model with time-varying w_e. The results are derived below, but let us summarize the key points before going to the details. The conclusion is that we can use adiabatic initial conditions for

$$-\frac{4}{5} \leqslant w_e \leqslant -\frac{2}{3} \qquad \text{or} \qquad -\frac{2}{3} < w_e < \frac{1}{3}\,. \tag{26}$$

For both of these intervals, the initial conditions for all non-dark energy quantities are the same as in the non-interacting case. For the second w_e interval, the initial dark energy density perturbation is the same as the standard one, $\Delta_{de} = 3(1+w_e)\Delta_\gamma/4$, whereas for the first interval, we find a non-standard initial condition $\Delta_{de} = \Delta_\gamma/4$. The difference arises because of the different background evolution in the two cases (as given in the previous section). Note that for $w_e < -1$ it is also possible to have adiabatic initial conditions, but we consider this case to be unphysical. For $-1 \leqslant w_e \leqslant -4/5$, we recover the non-adiabatic blow-up case of Valiviita et al. (2008).

Similar considerations could be extended to the early matter dominated era. The key difference there is that the background behaves differently for the interval $-1/2 < w_e < 1/3$ than for $w_e \leqslant -1/2$, where the interaction modifies the DE evolution. In the matter era, a non-adiabatic blow-up may thus happen if

$$-1 < w_e \leqslant -1/2\,. \tag{27}$$

A detailed analysis shows however that in the interval

$$-2/3 < w_e < -1/2\,, \tag{28}$$

the "blow-up" mode is in fact a decaying mode, and hence (non-standard) adiabatic evolution on super-Hubble scales is possible. In the interval $-1 < w_e < -2/3$ the non-adiabatic "blow-up" mode is rapidly increasing, and will dominate unless $|\Gamma|$ is suitably small. Therefore, a blow-up in the matter era will make large interaction models with $-1 < w_e < -2/3$ non-viable, while the blow-up in the radiation era ruins all interacting models (no matter how weak) with $-1 < w_e < -4/5$. We summarize these results in Table 1.

Table 1. The evolution of perturbations on super-Hubble scales with various values of the early dark energy equation of state parameter in the radiation and matter dominated eras (RD and MD respectively). "Adiabatic" means that it is possible to specify adiabatic initial conditions so that the total gauge invariant curvature perturbation ζ stays constant on super-Hubble scales. "Adiabatic (standard)" means that the behaviour of perturbations at early times on super-Hubble scales is the same as in the non-interacting model.

w_{de} in the RD or MD era	Radiation dominated era (RD)	Matter dominated era (MD)	Viable?
$w_{de} < -1$	adiabatic	adiabatic	viable, but phantom
$-1 < w_{de} < -4/5$	"blow-up" isocurvature growth	"blow-up" isocurvature growth	non-viable
$-4/5 \leqslant w_{de} < -2/3$	adiabatic	isocurvature growth	viable, if Γ small enough
$-2/3 \leqslant w_{de} < -1/2$	adiabatic (standard)	adiabatic	viable
$-1/2 \leqslant w_{de} < +1/3$	adiabatic (standard)	adiabatic (standard)	viable

Assuming tight coupling between photons and baryons, so that $V_b = V_\gamma$, passing from conformal time τ to the time variable $x = k\tau$, using a rescaled velocity $\tilde{V}_A = V_A/x$ and a rescaled anisotropic stress $\tilde{\Pi}_A = \Pi_A/x^2$, as in Doran et al. (2003), we obtain the following evolution equations:

$$\frac{d\Delta_c}{d\ln x} = -x^2\tilde{V}_c - \frac{\Gamma}{\mathcal{H}_0}\,\alpha\, x^2\left(\frac{3}{2}(1+w)\tilde{V} + 2\Omega_\nu\tilde{\Pi}_\nu + 2\Psi\right), \tag{29}$$

$$\frac{d\tilde{V}_c}{d\ln x} = -2\tilde{V}_c + \Psi\,, \tag{30}$$

$$\frac{d\Delta_\gamma}{d\ln x} = -\frac{4}{3}x^2\tilde{V}_\gamma\,, \tag{31}$$

$$\frac{d\tilde{V}_\gamma}{d\ln x} = \frac{1}{4}\Delta_\gamma - \tilde{V}_\gamma + \Omega_\nu\tilde{\Pi}_\nu + 2\Psi\,, \tag{32}$$

$$\frac{d\Delta_b}{d\ln x} = -x^2\tilde{V}_\gamma\,, \tag{33}$$

$$\frac{d\Delta_\nu}{d\ln x} = -\frac{4}{3}x^2\tilde{V}_\nu\,, \tag{34}$$

$$\frac{d\tilde{V}_\nu}{d\ln x} = \frac{1}{4}\Delta_\nu - \tilde{V}_\nu - \frac{1}{6}x^2\tilde{\Pi}_\nu + \Omega_\nu\tilde{\Pi}_\nu + 2\Psi\,, \tag{35}$$

$$\frac{d\tilde{\Pi}_\nu}{d\ln x} = \frac{8}{5}\tilde{V}_\nu - 2\tilde{\Pi}_\nu\,, \tag{36}$$

$$\begin{aligned}\frac{d\Delta_{de}}{d\ln x} &= 3(w_e-1)\left\{\Delta_{de} + 3(1+w_e)\left[-\Psi - \Omega_\nu\tilde{\Pi}_\nu\right] + (1+w_e)\left[3 - \frac{x^2}{3(w_e-1)}\right]\tilde{V}_{de}\right\} \\ &\quad + \frac{\Gamma}{\mathcal{H}_0}\alpha x^2\frac{\rho_c}{\rho_{de}}\left[\Delta_c - \Delta_x + 3(1-w_e)\tilde{V}_{de} - (3w_e-5)\left[\Psi + \Omega_\nu\tilde{\Pi}_\nu\right] + \frac{3}{2}(1+w)\tilde{V}\right]\end{aligned} \tag{37}$$

$$\frac{d\tilde{V}_{de}}{d\ln x} = \frac{\Delta_{de}}{1+w_e} + \tilde{V}_{de} + 3\Omega_\nu\tilde{\Pi}_\nu + 4\Psi + \frac{\Gamma}{\mathcal{H}_0}\alpha x^2\frac{\rho_c}{\rho_{de}}\frac{\tilde{V}_c - 2\tilde{V}_{de} - \Omega_\nu\tilde{\Pi}_\nu - \Psi}{1+w_e}\,. \tag{38}$$

Here $\alpha = (\mathcal{H}_0/k)^2\sqrt{\Omega_{r0}}$, and we used the Einstein equations

$$\Phi = \frac{3}{2}x^{-2}\left\{\Delta + 3(1+w)\left[\tilde{V} - \Phi\right]\right\}, \tag{39}$$

$$\frac{d\Phi}{d\ln x} = \Psi - \frac{3}{2}(1+w)\tilde{V}\,, \tag{40}$$

$$\Phi = -\Psi - 3w\tilde{\Pi}\,, \tag{41}$$

to eliminate Φ in favour of Ψ.

Using Eqs. (39) and (41), we have

$$\Psi = -\frac{\sum_{A=c,b,\gamma,\nu,de}\Omega_A\left[\Delta_A + 3\,(1+w_A)\tilde{V}_A\right]}{\sum_{A=c,b,\gamma,\nu,de}3\,(1+w_A)\Omega_A + \frac{2}{3}x^2} - \Omega_\nu\tilde{\Pi}_\nu\,. \tag{42}$$

The total velocity appearing in Eqs. (29) and (37) is

$$\tilde{V}(1+w) = \sum_{A=c,b,\gamma,\nu,de}\Omega_A(1+w_A)\tilde{V}_A\,. \tag{43}$$

Recalling that $\rho = \sum\rho_A$, $(1+w) = \sum\Omega_A(1+w_A)$, $\Delta = \sum\Omega_A\Delta_A$, and $\tilde{\Pi} = \sum\Omega_A\tilde{\Pi}_A = \Omega_\nu\tilde{\Pi}_\nu$, we then see that Eqs. (29–38) form a set of 10 linear differential equations for 10 perturbation variables Δ_A, $\tilde{V}_A$ and $\tilde{\Pi}_\nu$. (Note that $\Pi_A = 0$ for $A \neq \nu$.)

Since we are interested in the early radiation era, we make the approximation $\Omega_A = \rho_A/\rho \simeq \rho_A/\rho_r$. Using Eqs. (15), (19), (23) and (24), and the (standard non-interacting) background evolution of photons, baryons and neutrinos, we obtain

$$\Omega_b = \frac{\rho_b}{\rho_r} = \frac{\Omega_{b0}}{\Omega_{r0}}\, a = \frac{\Omega_{b0}}{\sqrt{\Omega_{r0}}}\frac{\mathcal{H}_0}{k}\, x = \omega_1\, x\,, \qquad \Omega_c = \omega_2\, x\,,$$
$$\Omega_{de} = \tilde{C}\omega_2 x^3 \;\text{ for } w_e \leqslant -2/3, \qquad \text{and} \qquad \Omega_{de} = C\omega_2 x^{1-3w_e} \;\text{ for } -2/3 < w_e < 1/3\,,$$
$$\Omega_\nu = \rho_\nu/\rho_r = R_\nu\,, \qquad \Omega_\gamma = 1 - \Omega_b - \Omega_c - \Omega_{de} - \Omega_\nu\,. \tag{44}$$

The next step of the method proposed in Doran et al. (2003) consists in writing the system of differential equations (29–38) in a matrix form:

$$\frac{d\boldsymbol{U}}{d\ln x} = \mathbf{A}(x)\boldsymbol{U}(x)\,, \tag{45}$$

where

$$\boldsymbol{U}^T = \left\{\Delta_c,\, \tilde{V}_c,\, \Delta_\gamma,\, \tilde{V}_\gamma,\, \Delta_b,\, \Delta_\nu,\, \tilde{V}_\nu,\, \tilde{\Pi}_\nu,\, \Delta_{de},\, \tilde{V}_{de}\right\}\,, \tag{46}$$

and the matrix $\mathbf{A}(x)$ can be read from Eqs. (29–38) after substituting Eqs. (42–44) and the background evolution of ρ_c/ρ_{de} from (19) or (20), depending on the value of w_e.

The initial conditions are specified for modes well outside the horizon, i.e. for $x \ll 1$. There will be several independent solution vectors to Eq. 45, that we write as $x^{\lambda_i}\boldsymbol{U}^{(i)}$. If no term of $\mathbf{A}(x)$ diverges for $x \to 0$, then we can approximate $\mathbf{A}$ by a constant matrix $\mathbf{A}_0 = \lim_{x\to 0}\mathbf{A}(x)$. If we require more accuracy we can expand $\mathbf{A}(x)$ up to a desired order in x. For example, up to order x^3 the matrix $\mathbf{A}(x)$ contains the constant term $\mathbf{A}_0$ as well as terms proportional to x, x^2, and x^3 in the case where $w_e \leqslant -2/3$. However, in the case $-2/3 < w_e \leqslant 1/3$, $\mathbf{A}(x)$ contains in addition to integer powers of x also non-integer powers $1-3w_e$, $2-3w_e$, $3-3w_e$, etc., and $2+3w_e$, $3+3w_e$, $4+3w_e$, etc. The listed ones and possibly their multiples can fall in the range $(0,3)$. For a given w_e, however, one should drop those which turn out to be higher order than x^3.

Thus going beyond zeroth order, up to order x^3, we can expand $\mathbf{A}$ and each solution $x^{\lambda_i}\boldsymbol{U}^{(i)}$ as

$$\mathbf{A}(x) \simeq \mathbf{A}_0 + \mathbf{A}_1\, x + \mathbf{A}_2\, x^2 + \mathbf{A}_3\, x^3 + \sum_{j=0}^{3}\left[\sum_{n=1}^{N}\left(\mathbf{B}_{nj}\, x^{n(1-3w_e)+j}\right) + \mathbf{C}_j\, x^{2+3w_e+j}\right]\,, \tag{47}$$

$$x^{\lambda_i}\boldsymbol{U}^{(i)}(x) \simeq x^{\lambda_i}\left\{\boldsymbol{U}_0^{(i)} + \boldsymbol{U}_1^{(i)}\, x + \boldsymbol{U}_2^{(i)}\, x^2 + \boldsymbol{U}_3^{(i)}\, x^3 + \sum_{j=0}^{3}\left[\sum_{n=1}^{N}\left(\boldsymbol{U}_{Bnj}^{(i)} x^{n(1-3w_e)+j}\right) + \boldsymbol{U}_{Cj}^{(i)} x^{2+3w_e+j}\right]\right\}\,, \tag{48}$$

where $\mathbf{A}_j$, $\mathbf{B}_{nj}$, and $\mathbf{C}_j$ are constant (not depending on the time variable x) matrices, and $\boldsymbol{U}_j^{(i)}$ are constant vectors. Note that for $w_e \leqslant -2/3$ all $\mathbf{B}_{nj}$, $\mathbf{C}_j$, $\boldsymbol{U}_{Bnj}^{(i)}$, and $\boldsymbol{U}_{Cj}^{(i)}$ terms vanish. For simplicity, we demonstrate below this case, which leads to only integer powers. Substituting Eqs. (47) and (48) into the evolution equation (45) and equating order by order, we obtain

$$\mathbf{A}_0\boldsymbol{U}_0^{(i)} = \lambda_i\boldsymbol{U}_0^{(i)}\,,\ \text{i.e. } \lambda_i \text{ is an eigenvalue of } \mathbf{A}_0, \text{ and } \boldsymbol{U}_0^{(i)} \text{ is an eigenvector of } \mathbf{A}_0, \tag{49}$$
$$\boldsymbol{U}_1^{(i)} = -\left[\mathbf{A}_0 - (\lambda_i+1)\mathbb{1}\right]^{-1}\left[\mathbf{A}_1\boldsymbol{U}_0^{(i)} + \mathbf{A}_0\boldsymbol{U}_1^{(i)}\right]\,, \tag{50}$$
$$\boldsymbol{U}_2^{(i)} = -\left[\mathbf{A}_0 - (\lambda_i+2)\mathbb{1}\right]^{-1}\left[\mathbf{A}_2\boldsymbol{U}_0^{(i)} + \mathbf{A}_1\boldsymbol{U}_1^{(i)} + \mathbf{A}_0\boldsymbol{U}_2^{(i)}\right]\,, \tag{51}$$
$$\boldsymbol{U}_3^{(i)} = -\left[\mathbf{A}_0 - (\lambda_i+3)\mathbb{1}\right]^{-1}\left[\mathbf{A}_3\boldsymbol{U}_0^{(i)} + \mathbf{A}_2\boldsymbol{U}_1^{(i)} + \mathbf{A}_1\boldsymbol{U}_2^{(i)} + \mathbf{A}_0\boldsymbol{U}_3^{(i)}\right]\,. \tag{52}$$

Now the general solution to the differential equation (45) is a linear combination of solutions $x^{\lambda_i}\boldsymbol{U}^{(i)}$:

$$\boldsymbol{U}(x) = \sum_i c_i\, x^{\lambda_i}\, \boldsymbol{U}^{(i)}(x)\,, \tag{53}$$

where c_i are dimensionless constants. If we define an initial reference time $t_{\rm init}$, then the constants $\tilde{c}_i = c_i x_{\rm init}^{\lambda_i}$ represent the initial contribution of the vector $\boldsymbol{U}^{(i)}$ to the total perturbation vector $\boldsymbol{U}(x_{\rm init})$. The imaginary part of λ_i represents oscillations of $x^{\lambda_i}\boldsymbol{U}_0^{(i)}(x)$, while the real part gives its power-law behaviour: $x^{\lambda_i}\boldsymbol{U}_0^{(i)}(x) = x^{{\rm Re}(\lambda_i)}\cos[{\rm Im}(\lambda_i)\ln x]$. The contribution corresponding to the eigenvalue(s) with largest real part, ${\rm Re}(\lambda_i)$, will dominate as time goes by, while initial contributions from eigenvectors corresponding to λ_i with smaller real part will become negligible compared to the dominant mode. Hence, to set initial conditions deep in the radiation era but well after inflation, it is sufficient to specify the contribution coming from mode(s) with largest ${\rm Re}(\lambda_i)$.

From now on we divide the treatment into two cases $-2/3 < w_e < 1/3$ and $w_e \leqslant -2/3$. Before proceeding, we should point out that the matrix method presented in Doran et al. (2003) and applied to non-interacting constant-w_{de} dark energy, represents a systematic and efficient approach for finding initial conditions. Once the matrix $\mathbf{A}(x)$ has been read from the set of first order differential equations (29–38), one can feed it into a symbolic mathematical program such as Maple or Mathematica and easily extract the constant part $\mathbf{A}_0$ as well as the other parts (such as $\mathbf{A}_1$, $\mathbf{A}_2$, ..., $\mathbf{B}_{nj}$, $\mathbf{C}_j$) up to any desired order. Then it is simple linear algebra to find the eigenvalues λ_i and eigenvectors $\boldsymbol{U}_0^{(i)}$ of $\mathbf{A}_0$ and, if higher order solutions in $k\tau$ are needed, to substitute these step by step into Eqs. (50)–(52) etc. in order to find the solutions $x^{\lambda_i}\boldsymbol{U}^{(i)}(x)$.

4.1 Case $-2/3 < w_e < 1/3$

We substitute Ψ from Eq. (42), $\tilde{V}$ from Eq. (43), and the energy density parameters from Eq. (44) into Eqs. (29–38). Then, using Eq. (20) for ρ_c/ρ_{de} in the last two of them, and taking the limit $x \to 0$, we find the $\mathbf{A}_0$ matrix:

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & \frac{\mathcal{N}}{4} & \mathcal{N} & 0 & -\frac{R_\nu}{4} & -R_\nu & -R_\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2R_\nu-1}{4} & 2R_\nu - 3 & 0 & -\frac{R_\nu}{2} & -2R_\nu & -R_\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathcal{N}}{2} & 2\mathcal{N} & 0 & \frac{1-2R_\nu}{4} & -1-2R_\nu & -R_\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{5} & -2 & 0 & 0 \\ 0 & 0 & \frac{9\mathcal{N}({w_e}^2-1)}{4} & 9\mathcal{N}\left({w_e}^2-1\right) & 0 & \frac{9R_\nu(1-{w_e}^2)}{4} & 9R_\nu\left(1-{w_e}^2\right) & 0 & 3\left(w_e-1\right) & 9\left({w_e}^2-1\right) \\ 0 & 0 & \mathcal{N} & 4\mathcal{N} & 0 & -R_\nu & -4R_\nu & -R_\nu & \frac{1}{1+w_e} & 1 \end{pmatrix} \tag{54}$$

where $\mathcal{N} = R_\nu - 1$.

The eigenvalues of $\mathbf{A}_0$ are

$$\lambda_i = \left\{-2, -1, 0, 0, 0, 0, -\frac{5}{2} - \frac{\sqrt{1-32\,R_\nu/5}}{2}, -\frac{5}{2} + \frac{\sqrt{1-32\,R_\nu/5}}{2}, \lambda_{\rm d}^-, \lambda_{\rm d}^+\right\}, \tag{55}$$

where

$$\lambda_{\rm d}^\pm = \frac{-2+3\,w_e}{2} \pm \frac{\sqrt{-20+12\,w_e+9\,{w_e}^2}}{2}. \tag{56}$$

For the range $0 < R_\nu < 0.405$ and $-2/3 < w_e < 1/3$, all eigenvalues have a non-positive real part. In (56) the term inside the square root, $-20+12\,w_e+9\,{w_e}^2$, falls between -24 and -15, and hence $\mathrm{Re}(\lambda_{\rm d}^\pm) = -1+3\,w_e/2$, falls between -2 and $-1/2$.

As explained in Doran et al. (2003), since the eigenvalue with largest real part, $\lambda = 0$, is fourfold degenerate, it is possible to choose a basis from the subspace spanned by the eigenvectors with eigenvalue $\lambda = 0$, so that physically meaningful choices can be made for the initial condition vector. One can form 4 independent linear combinations from the four vectors with $\lambda = 0$. The physical choices are an adiabatic mode and 3 isocurvature modes. Here we choose adiabatic initial conditions, specified by the condition that the gauge-invariant entropy perturbations S_{AB} of every A, $B = \gamma$, ν, c, b vanish, where (Malik et al. 2003)

$$S_{AB} = -3\mathcal{H}\frac{\rho_A}{\rho'_A}\Delta_A + 3\mathcal{H}\frac{\rho_B}{\rho'_B}\Delta_B\,. \tag{57}$$

We will show later the interesting new result that demanding adiabaticity between the standard constituents automatically guarantees adiabaticity with respect to DE.

We should remind the reader that for the interacting constituents the coupling appears in the continuity equation, and we should not use blindly the standard result

$$S_{AB} = \frac{\Delta_A}{(1+w_A)} - \frac{\Delta_B}{(1+w_B)}, \tag{58}$$

where the $1+w_A$ factors result from applying the continuity equation to ρ'_A/ρ_A. Indeed, for cold dark matter in the early radiation era we find, using Eqs.(1), (3), and (15),

$$-3\mathcal{H}\frac{\rho_c}{\rho'_c}\Delta_c = \frac{\Delta_c}{(1+w_c) + (H_0\sqrt{\Omega_{r0}}/3k^2)(k\tau)^2}, \tag{59}$$

where $w_c = 0$. For DE, we find using Eqs.(2), (3), (15), and (20)

$$-3\mathcal{H}\frac{\rho_{de}}{\rho'_{de}}\Delta_{de} = \frac{\Delta_{de}}{(1+w_{de}) - (H_0\sqrt{\Omega_{r0}}/3k^2C)(k\tau)^{2+w_e}}. \tag{60}$$

At zeroth order in $x = k\tau$ we incidentally regain the standard non-interacting result (58). From $S_{AB} = 0$ it then follows that

$$\Delta_c = \Delta_b = \frac{3}{4}\Delta_\gamma = \frac{3}{4}\Delta_\nu\,. \tag{61}$$

Imposing this condition on a linear combination of the four eigenvectors with eigenvalue $\lambda = 0$ we obtain

$$\boldsymbol{U}_0^{(\mathrm{adi})} = \begin{pmatrix} \Delta_c \\ \tilde{V}_c \\ \Delta_\gamma \\ \tilde{V}_\gamma \\ \Delta_b \\ \Delta_\nu \\ \tilde{V}_\nu \\ \tilde{\Pi}_\nu \\ \Delta_{de} \\ \tilde{V}_{de} \end{pmatrix}_{\mathrm{adiabatic}} = C_1 \begin{pmatrix} 3/4 \\ -(5/4)\mathcal{P} \\ 1 \\ -(5/4)\mathcal{P} \\ 3/4 \\ 1 \\ -(5/4)\mathcal{P} \\ -\mathcal{P} \\ (3/4)\,(1+w_e) \\ -(5/4)\mathcal{P} \end{pmatrix}, \tag{62}$$

where $\mathcal{P} = (15 + 4\,R_\nu)^{-1}$, and C_1 is a dimensionless normalization constant corresponding to, e.g., Δ_γ and Δ_ν at the initial time.

The vector (62) is identical to the standard adiabatic initial condition vector (see Doran et al. (2003)). In particular, it should be noticed that although we did not require adiabaticity of DE, (62) automatically satisfies the condition $S_{de,A} = 0$ for all $A = \gamma$, ν, c, b. In Doran et al. (2003) this result was found for non-interacting dark energy. Here we have now shown that also interacting dark energy is automatically adiabatic, once cold dark matter, baryons, photons and neutrinos are set to be adiabatic.

Finally, since all components of the vector (62) are different from zero, it is not necessary to compute terms to higher order in x, and we use Eq. (62) as our adiabatic initial condition for the computation of the CMB power spectrum for models with $-2/3 < w_e < 1/3$.

Lee et al. (2006) have reported that the quintessence isocurvature mode decays away (in an interacting quintessence model which is quite similar to our set-up). After our systematic derivation of initial conditions, this decay can be tracked down to the fact that $\mathrm{Re}(\lambda_{\mathrm{d}}^{\pm})$ in Eq. (55) are negative. The reason for this is that in quintessence models the early-time equation of state parameter is typically larger than $-2/3$, indeed positive [but in Lee et al. (2006) less than $+2/3$]. So $\mathrm{Re}(\lambda_{\mathrm{d}}^{\pm})$ is negative, and hence the isocurvature mode decays. In the next subsection we will see that also in the range $-4/5 \leqslant w_e \leqslant -2/3$ (or $w_e < -1$) the DE isocurvature mode decays (although in this case the interaction affects the evolution of DE perturbations), whereas in the range $-1 < w_e < -4/5$ the DE isocurvature mode is a rapidly growing mode as recently realised by Valiviita et al. (2008).

4.2 Case $w_e \leqslant -2/3$

Since at early times the equation of state can be approximated as a constant, $w_e = w_0 + w_a$, this case has already been studied in Valiviita et al. (2008), where a constant $-1 < w_{de} \leqslant -2/3$ was analysed. A serious non-adiabatic large-scale instability that excludes these models was found whenever $-1 < w_{de} < -4/5$, no matter how weak the interaction was. However, we notice that there is a limited region of parameter space, $-4/5 \leqslant w_{de} \leqslant -2/3$, where the instability can possibly be avoided. In the case of a constant DE equation of state parameter this range would be observationally disfavoured, since for example supernova data require that w_{de} is closer to -1 at recent times. In the case of time varying $w_{de}(a)$ we do not have this problem as w_0 can be close to -1 while $-4/5 < w_e \leqslant -2/3$. In the following we repeat the analysis of initial conditions done in Valiviita et al. (2008), but using the matrix method of Doran et al. (2003), extended to include the interaction, and give the conditions for a viable cosmology.

Substituting Ψ from Eq. (42), $\tilde{V}$ from Eq. (43), and the energy density parameters from Eq. (44) into Eqs. (29–38), as well as ρ_c/ρ_{de} from Eq. (19) into the last two of them, and taking the limit $x \to 0$, we find the $\mathbf{A}_0$ matrix, which is very similar to our previous result, Eq. (54). This happens because everything remains unchanged, except that we must replace in Eqs. (37) and (38) the evolution of ρ_c/ρ_{de} with Eq. (19), and whenever Ω_{de} appears we must now substitute the $\propto x^3$ behaviour from (44), instead of the $\propto x^{1-3w_e}$ behaviour. Therefore only the last two rows in (54) are modified, and will now read

$$\begin{pmatrix} 2+3w_e & 0 & \frac{\mathcal{N}(1+9w_e)}{4} & 3\mathcal{N}\,(w_e-1) & 0 & -\frac{R_\nu(1+9w_e)}{4} & -3R_\nu\,(w_e-1) & 0 & -5 & 3\,(w_e-1) \\ 0 & \frac{2+3w_e}{1+w_e} & \frac{\mathcal{N}(2+w_e)}{4(1+w_e)} & \frac{\mathcal{N}(2+w_e)}{1+w_e} & 0 & -\frac{R_\nu(2+w_e)}{4(1+w_e)} & -\frac{R_\nu(2+w_e)}{1+w_e} & -R_\nu & \frac{1}{1+w_e} & -\frac{3+5w_e}{1+w_e} \end{pmatrix}. \tag{63}$$

The eigenvalues of $\mathbf{A}_0$ are

$$\lambda_i = \left\{-2, -1, 0, 0, 0, 0, -\frac{5}{2} - \frac{\sqrt{1-32R_\nu/5}}{2}, -\frac{5}{2} + \frac{\sqrt{1-32R_\nu/5}}{2}, \lambda_{\mathrm{g}}^-, \lambda_{\mathrm{g}}^+\right\}, \tag{64}$$

where

$$\lambda_{\rm g}^{\pm} \equiv \frac{-5w_e - 4 \pm \sqrt{3w_e^2 - 2}}{1 + w_e} \,. \tag{65}$$

The first eight eigenvalues coincide with the previous case, Eq. (55). Of those, four have a negative real part, corresponding thus to modes that will decay away quickly, and that we can neglect. The last two eigenvalues, $\lambda_{\rm g}^{\pm}$, are instead very different from the previous case, and depend on the value of w_e. The eigenvalue with a largest real part, $\lambda_{\rm g}^{+}$, is real and positive for $-1 < w_e \leqslant -\sqrt{2/3}$. In addition to this, $\mathrm{Re}(\lambda_{\rm g}^{+})$ is positive also in the small range $-\sqrt{2/3} < w_e < -4/5$. Therefore $\mathrm{Re}(\lambda_{\rm g}^{+}) > 0$ for $-1 < w_e < -4/5$. This corresponds to the blow-up solution found in Valiviita et al. (2008); $\lambda_{\rm g}^{+}$ is larger, the closer w_e is to -1. There is no blow-up of perturbations for $-4/5 \leqslant w_e \leqslant -2/3$, because then the largest $\mathrm{Re}(\lambda_i)$ are zero.

4.2.1 Case $-1 < w_e < -4/5$; non-adiabatic blow-up

We now calculate the initial condition vector $\boldsymbol{U}^{(\rm g)}(x)$ corresponding to the fastest growing mode, $\lambda_{\rm g}^{+}$. At zeroth order in x, it is given by

$$\boldsymbol{U}_0^{(\rm g)T}(x) = \left\{0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, -1 + \sqrt{3w_e^2 - 2},\, 1\right\} \,. \tag{66}$$

In this case, since only the last two components of the vector are different from zero, we need to compute higher order corrections. It turns out that an expansion up to x^3 is necessary, and as explained both before and after Eqs. (47) and (48), the expansion contains only integer powers of x, when $w_e \leqslant -2/3$. Therefore we have

$$\mathbf{A}(x) \simeq \mathbf{A}_0 + \mathbf{A}_1\, x + \mathbf{A}_2\, x^2 + \mathbf{A}_3\, x^3 \,, \tag{67}$$

$$\boldsymbol{U}^{(g)}(x) \simeq \boldsymbol{U}_0^{(g)} + \boldsymbol{U}_1^{(g)}\, x + \boldsymbol{U}_2^{(g)}\, x^2 + \boldsymbol{U}_3^{(g)}\, x^3 \,. \tag{68}$$

By substituting $\mathbf{A}(x)$ and $x^{\lambda_{\rm g}^{+}} \boldsymbol{U}^{(g)}(x)$ into the evolution equation (45) and equating order by order, we obtain

$$\boldsymbol{U}_1^{(g)} = -\left[\mathbf{A}_0 - (\lambda_{\rm g}^{+} + 1)\mathbb{1}\right]^{-1} \mathbf{A}_1 \boldsymbol{U}_0 \,, \tag{69}$$

$$\boldsymbol{U}_2^{(g)} = -\left[\mathbf{A}_0 - (\lambda_{\rm g}^{+} + 2)\mathbb{1}\right]^{-1} (\mathbf{A}_2 \boldsymbol{U}_0 + \mathbf{A}_1 \boldsymbol{U}_1) \,, \tag{70}$$

$$\boldsymbol{U}_3^{(g)} = -\left[\mathbf{A}_0 - (\lambda_{\rm g}^{+} + 3)\mathbb{1}\right]^{-1} (\mathbf{A}_3 \boldsymbol{U}_0 + \mathbf{A}_2 \boldsymbol{U}_1 + \mathbf{A}_1 \boldsymbol{U}_2) \,. \tag{71}$$

Using these formulas we find corrections to Eq. (66). Keeping for each perturbation variable only the leading order (in x) terms, we obtain the following initial condition vector:

$$\boldsymbol{U}^{(\rm g)}(x) = \begin{pmatrix} \Delta_c \\ \tilde{V}_c \\ \Delta_\gamma \\ \tilde{V}_\gamma \\ \Delta_b \\ \Delta_\nu \\ \tilde{V}_\nu \\ \tilde{\Pi}_\nu \\ \Delta_{de} \\ \tilde{V}_{de} \end{pmatrix}_{\rm g} = C_2 \begin{pmatrix} 0 \\ \frac{\frac{\Gamma}{\mathcal{H}_0}\omega_3 (1+w_e)^3 [8R_\nu (1+w_e)\mathcal{Q} - 15(w_e-1)(1-w_e+\mathcal{Q})] x^3}{4(w_e-1)(2+3w_e)\mathcal{M}} \\ 0 \\ \frac{-15\frac{\Gamma}{\mathcal{H}_0}\omega_3 (1+w_e)^3 (2+\mathcal{Q}) x^3}{2(2+3w_e)\mathcal{M}} \\ 0 \\ 0 \\ \frac{-15\frac{\Gamma}{\mathcal{H}_0}\omega_3 (1+w_e)^3 (2+\mathcal{Q}) x^3}{2(2+3w_e)\mathcal{M}} \\ \frac{4\frac{\Gamma}{\mathcal{H}_0}\omega_3 (1+w_e)^3 \left[3\left(7+8w_e+{w_e}^2\right)+(5+3w_e)\mathcal{Q}\right] x^3}{(2+3w_e)(\mathcal{Q}-2w_e)\mathcal{B}} \\ \mathcal{Q} \\ 1 \end{pmatrix} , \tag{72}$$

where $\omega_3 = \omega_2 (H_0/k)^2 \sqrt{\Omega_{r0}} = (H_0/k)^3 \Omega_{r0}/a_{eq}$, $\mathcal{Q}$, $\mathcal{M}$ and $\mathcal{B}$ are $\mathcal{Q} = \sqrt{3{w_e}^2 - 2} - 1$, $\mathcal{M} = 5\left[6 + 7w_e + 3{w_e}^3 + (3 + 5w_e)\mathcal{Q}\right] - 4R_\nu (1 + w_e)^2 (\mathcal{Q} - 1 - 3w_e)$ and $\mathcal{B} = 8R_\nu\, (1 + w_e)^2\, (5 + 3w_e + 2\mathcal{Q}) + 5\,\{9\mathcal{Q} - 3 + w_e\, [13 + 14\mathcal{Q} + 3w_e\, (13 + 5w_e + 3\mathcal{Q})]\}$. This solution coincides with equations (63)–(70) of Valiviita et al. (2008), after substituting $n_\psi = \lambda_{\rm g}^{+} + 3$, $J = 1 - 16R_\nu [\,5(n_\psi + 2)(n_\psi + 1) + 8R_\nu\,]^{-1}$, converting into Newtonian gauge (by using Eqs. (25) with $B = E = 0$) and conveniently renormalizing the vector. Equation (72) is the initial condition vector for the case $-1 < w_e \leqslant -4/5$, when the dominant eigenvector is that corresponding to $\lambda_{\rm g}^{+}$.

The initial condition (72) is trivially adiabatic with respect to γ, ν, c and b, but not with respect to DE. Indeed, for DE we find using Eqs.(2), (3), and (19)

$$-\,3\mathcal{H}\frac{\rho_{de}}{\rho'_{de}}\Delta_{de} = \frac{\Delta_{de}}{(1 + w_{de}) - (3w_{de} + 2)/3} = 3\Delta_{de} \,. \tag{73}$$

Therefore

$$S_{de\,A} = 3\Delta_{de} = C_2 \mathcal{Q} x^{\lambda_{\rm g}^{+}} \,, \tag{74}$$

for any $A = \gamma$, ν, c or b. Even if we were able to set the initial conditions at $\tau = 0$ and demanded adiabaticity there, after a short time the solution would not be adiabatic. Thus Eq. (72) represents the non-adiabatic "blow-up" solution (Valiviita et al. 2008) for the case $-1 < w_e < -4/5$.

4.2.2 Cases $w_e < -1$ or $-4/5 \leqslant w_e \leqslant -2/3$; adiabatic initial conditions

In the range $-4/5 < w_e \leqslant -2/3$, as well as for $w_e < -1$, we have $\mathrm{Re}(\lambda_{\rm g}^{\pm}) < 0$, so that the largest eigenvalue is the fourfold degenerate $\lambda = 0$. [If $w_e = -4/5$, then $\lambda = 0$ is fourfold degenerate, and there are also two oscillating solutions with $\mathrm{Re}(\lambda_{\rm g}^{\pm}) = 0$.] We look for a linear combination of the four eigenvectors (corresponding to $\lambda = 0$) that satisfies adiabaticity (see Eq. (61)) of photons, neutrinos, baryons and cold dark matter. The resulting eigenvector is

$$\boldsymbol{U}_0^{(\rm adi)} = \begin{pmatrix} \Delta_c \\ \tilde{V}_c \\ \Delta_\gamma \\ \tilde{V}_\gamma \\ \Delta_b \\ \Delta_\nu \\ \tilde{V}_\nu \\ \tilde{\Pi}_\nu \\ \Delta_{de} \\ \tilde{V}_{de} \end{pmatrix}_{\rm adiabatic} = \mathbf{C}_3 \begin{pmatrix} 3/4 \\ -(5/4)\mathcal{P} \\ 1 \\ -(5/4)\mathcal{P} \\ 3/4 \\ 1 \\ -(5/4)\mathcal{P} \\ -\mathcal{P} \\ 1/4 \\ -(5/4)\mathcal{P} \end{pmatrix}, \tag{75}$$

where $\mathcal{P} = (15 + 4\,R_\nu)^{-1}$. This corresponds to equations (59–61) of Valiviita et al. (2008). All components except Δ_{de} are equal to the initial conditions for $-2/3 < w_e < 1/3$, Eq. (62). However, as pointed out in Valiviita et al. (2008), $\Delta_{de} = \Delta_\gamma/4$ corresponds exactly to the adiabaticity condition for DE: $S_{de\,A} = 0$. Namely, substituting the result (73) into definition (57), we find

$$S_{de\,\gamma} = 3\Delta_{de} - \frac{3}{4}\Delta_\gamma\,. \tag{76}$$

Thus Eq. (75) is an adiabatic initial condition vector for the cases $-4/5 \leqslant w_e \leqslant -2/3$ or $w_e < -1$.

5 CONCLUSION

We have presented, for the first time, a *systematic* derivation of initial conditions for perturbations in a model of interacting dark matter - dark energy fluids, in the early radiation era. These initial conditions are essential for studying the further evolution of perturbations up to today's observables. They are the initial values for perturbations in any Boltzmann integrator which solves the multipole hierarchy and produces the theoretical predictions for the CMB temperature and polarization angular power spectrum, as well as the matter power spectrum. We have focused on the interaction $Q_c^\mu = -\Gamma\rho_c(1+\delta_c)u_c^\mu$, where Γ is a constant rate of energy density transfer [see Eqs. (1) and (2)]. Generalising a previous result for non-interacting dark energy in Doran et al. (2003), we find that, in our interacting model, requiring adiabaticity between all the other constituents (photons, neutrinos, baryons, and cold dark matter) leads automatically also to dark energy adiabaticity, if its early-time equation of state parameter is $w_e < -1$ or $-4/5 \leqslant w_e \leqslant 1/3$. In our previous work (Valiviita et al. 2008), we showed that if the equation of state parameter for dark energy is $-1 < w_{de} < -4/5$ in the radiation or matter eras, the model suffers from a serious non-adiabatic instability on large scales. In this paper, the systematic derivation of initial conditions confirms that result. However, in this paper we have shown that the instability can easily be avoided, if we allow for suitably time-varying dark energy equation of state. The main results are verbally summarised in Table 1 on page 5.

In the companion paper (Valiviita et al. 2009) we modified the CAMB Boltzmann integrator[1] (Lewis et al. 2000), using the adiabatic initial conditions derived here for the interacting model, and performed full Monte Carlo Markov Chain likelihood scans for this model as well as for the non-interacting (Γ=0) model for a reference, with various combinations of publicly available data sets: WMAP (Komatsu et al. 2009), WMAP & ACBAR (Reichardt et al. 2009), SN (Kowalski et al. 2008), BAO (Percival et al. 2007), WMAP&SN, WMAP&BAO, WMAP&SN&BAO.

With the parametrization $w_{de} = w_0 a + w_e(1-a)$, viable interacting cosmologies result for w_0 close to -1 and $w_e < -1$ or $-4/5 < w_e \leqslant 1/3$, as long as $w_0 + 1$ and $w_e + 1$ have the same sign (Valiviita et al. 2009). These particular conclusions apply exclusively to the interaction model we considered in this paper.

However, the method can be easily adapted for studying different interactions: one only needs to modify the background evolution and interaction terms in Eqs. (29), (30), (37), and (38), before reading a new matrix $\mathbf{A}(x)$ from them. Based on

[1] http://camb.info

section IV of Valiviita et al. (2008), the other interacting fluid models [$aQ_c = -\alpha\mathcal{H}\rho_c$ or $aQ_c = -\beta\mathcal{H}(\rho_c + \rho_{de})$, where α, $\beta \lesssim 1$ are dimensionless constants], that are common in the literature, behave in a very similar way to the model studied here, i.e., for $-1 < w_e < w_{\rm crit}$ the models are not viable due to the early-time large-scale blow-up of perturbations, for $w_{\rm crit} < w_e < w_{\rm adiab}$ the models can be viable and *non-standard* adiabatic initial conditions maybe found, and for $w_e > w_{\rm adiab}$ (or $w_e < -1$) the models are viable and standard (non-interacting) adiabatic initial conditions can be found. The critical value $w_{\rm crit}$ is determined by demanding that the 'blow-up' mode is actually a decaying mode and the fastest 'growing' curvature perturbation mode is a constant, i.e., that the largest real part of the eigenvalues λ_i is zero, which with the notation of Valiviita et al. (2008) is guaranteed whenever $\mathrm{Re}(n_+) \leqslant 3$. In our model the critical value, $w_{\rm crit} = -4/5$, is independent of the strength of interaction, but in the above-mentioned models it depends on α or β, as indicated by equations (85) and (98) in Valiviita et al. (2008). In general, our results show that the (early-time) dark energy equation of state plays, together with the interaction model, an important role in the (in)stability of perturbations.

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