Emergence of continual directed flow in Hamiltonian systems

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We propose a minimal model for the emergence of a directed flow in autonomous Hamiltonian systems. It is shown that internal breaking of the spatio-temporal symmetries, via localised initial conditions, that are unbiased with respect to the transporting degree of freedom, and transient chaos conspire to form the physical mechanism for the occurrence of a current. Most importantly, after passage through the transient chaos, trajectories perform solely regular transporting motion so that the resulting current is of continual ballistic nature. This has to be distinguished from the features of transport reported previously for driven Hamiltonian systems with mixed phase space where transport is determined by intermittent behaviour exhibiting power-law decay statistics of the duration of regular ballistic periods.

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Nonlinear transport processes of particles evolving in a spatially periodic potential have attracted considerable interest [1] (for a recent review see [2]). In most of the studies, the emergence of particle current is triggered by an external time-dependent field with zero mean (which can be of stochastic nature) or is provided by a deterministic periodic force. Recently, the Hamiltonian case has become the subject of intensive studies due to its relevance to the motion of cold atoms in an optical potential [3].

The necessary conditions for rectification of the current, based on symmetry investigations of the external field and the underlying static potential, have been presented in [4] and [5]. To be precise, all symmetries that, to each trajectory, generate a counterpart moving in the opposite direction, need to be broken. This is achievable by imposing a time-dependent external force that is periodic but not symmetric under time reversal [4]–[8]. Furthermore, the phase space has to possess a mixing property, with coexisting regular and chaotic dynamics [6]. In extended chaotic systems a nonzero current can be obtained as the time-averaged velocity of an ensemble of trajectories in the chaotic component of phase space, and the chaotic transport proceeds ballistically and directedly [6], [8].

Extensions to studies of autonomous Hamiltonian systems of one-dimensional billiard chains have followed [9], [10]. The necessity of creating chaos requires at least two degrees of freedom. As an example of such a system, a classical magnetic billiard for particles carrying an electric charge has been studied in [9]. In order to break the time-reversal invariance, an external static magnetic field, penetrating the plane of motion perpendicularly, has been applied. In addition, achieving directed transport demands breaking the remaining spatial symmetry, which can be achieved, e.g., by properly placed asymmetric obstacles inside the billiard [9], [10]. Uni-directional motion in a serpent billiard chain has been reported in [11].

The aim of the current work is to demonstrate that chaotic directed transport as achieved in systems with a mixed phase space is not the only option to obtain directed transport in Hamiltonian systems. In driven Hamiltonian systems the directed transport necessitates a mixed phase space for which chaotic trajectories can stick to the boundaries of regular regions inducing long periods of nearly regular motion so that finite asymptotic currents can be observed [4]–[8]. On the other hand, sustained transport seems to be impeded by the intermittent behaviour exhibiting power-law decay statistics of the duration of periods of regular motion. Therefore, for the sake of stability and reliability of a transporting regime it is desirable to find ways to accomplish continual regular transport. As we show, when dealing with nonintegrable systems, it is advantageous when chaos is only of transient nature and serves to guide trajectories onto *regular transporting* motion. Furthermore, we also show that, in contrast to the studies quoted above, a directed flow can arise in autonomous Hamiltonian systems even without the application of a time reversibility symmetry breaking external field: specifically, we show that while the system as a whole is time-reversible, physically-relevant sets of localised initial conditions lead to current despite being *unbiased* with respect to the transporting degree of freedom.

We consider a Hamiltonian system with n degrees of freedom of the form

$$H(p,q) = \frac{1}{2}p^2 + U(q) \tag{1}$$

with $(p,q) \in \mathbb{R}^{2n}$ and $U(q)$ as a potential function. It is assumed that the system possesses an open component, by which we mean that constant energy surfaces may be unbounded in the coordinate(s), allowing for transient chaos.

In the following, we discuss the spatio-temporal symmetry properties of the Hamiltonian equations $\dot{p} = -\partial H/\partial q$ and $\dot{q} = \partial H/\partial p$. Firstly, the system of equations exhibits time reversible invariance. Solutions are of the form $X(t) = (p(t), q(t))$. Applying the time-reversal operator yields $\widehat{\tau}(p(t), q(t)) = (-p(-t), q(-t))$ and, hence, if X is a solution, then so is $\widehat{\tau}X$. As for the implication of time-reversibility symmetry with regard to the net flow, let a solution, starting from some initial condition $X(0)$, be evolved in time up to a finite observation time T at which the *forward* trajectory arrives at the point $X(T)$ on the constant energy surface. Subsequently, under application of the time-reversal operation, the signs of the momenta at this point, $X(T)$, are reversed, and letting the solution evolve once again, with $\widehat{\tau}X(T)$ as the initial condition, the corresponding *backward* trajectory traces back the path of the forward trajectory in coordinate space. For a microcanonical ensemble, the initial conditions $X(0)$ and $\widehat{\tau}X(T)$ are equally selected points from the constant energy surface. Thus, for systems with time-reversibility symmetry and uniformly distributed initial conditions populating the whole energy surface there is no preferred direction of the flow preventing the emergence of a current.

However, energy surfaces that are unbounded (along the coordinates of the open component) may not, in practise, be completely populated with a finite set of initial conditions. In this sense, any finite set of initial conditions can be regarded as being *localised* in the coordinates of the spatially-open system. Localised initial coordinates are frequently used in applications such as for the problem of a particle flow emerging when the particles are initially trapped in a well of a spatially infinitely extended, multiple-well potential (see below). Therefore, for practical purposes, it is supposed that the initial conditions for coordinates are localised in the domain $-q_{j,l} \le q_j(0) \le q_{j,r}$ and $1 \le j \le n$. We define a trajectory of the spatially-open system as *transporting* (with respect to a specified observation time, T) if: (i) at least one of its coordinates, $q_j(t)$, escapes from the domain of the localised initial conditions at some instant of time $0 < t_{escape} \ll T$ and, (ii) performs subsequently directed net motion, that is $\langle p_j(t)\rangle \neq 0$ for $T \ge t \ge t_{escape}$, where $\langle\,\cdot\,\rangle$ denotes the time average. This leads the trajectory away from the domain of localised initial conditions so that at the end of the observation time, at least one of the terminal coordinates of a transporting forward trajectory lies outside the domain of the localised initial values; $q_j(T) < -q_{j,l}$ or $q_j(T) > q_{j,r}$. Consequently, the initial condition of the corresponding backward trajectory (which would compensate the contribution of the forward trajectory to the net current) is not contained in the set of localised initial conditions. It needs to be stressed that this alone does not imply the emergence of a current in the system for such sets of initial conditions.

Furthermore, although the equations of motion are time reversal symmetric, not all of their solutions necessarily obey this symmetry. Regarding the selection of an initial condition, $X(0)$, it holds that unless a trajectory is self-reversed, i.e., for the selected initial condition and its time-reversed counterpart, $\widehat{\tau}X(0)$, the corresponding motion coincides in phase space, i.e., $X = \widehat{\tau}X$, time reversibility is broken. In fact, selecting an initial condition, $X(0)$, may break the time-reversibility symmetry, which is the case when the trajectory X and its time reversal counterpart $\widehat{\tau}X$ are distinct. Examples of time-reversibility breaking are provided by the (unbounded) rotating trajectories of a pendulum whereas the (bounded) librating trajectories are self-reversed and thus time reversible symmetric. For potential systems of the form of Eq. (1), time-reversibility is manifested in coordinate space in the spatial symmetry features induced by reflections on the time-reversal symmetry hypersurfaces on which the time evolution of the trajectories satisfies $p(-t) = -p(t)$ and $q(-t) = q(t)$. They are represented by $(n-1)$-dimensional manifolds in the coordinate space and are determined by the condition $\dot{p}_k = 0$. This gives

$$S_k\;:\qquad \frac{\partial U}{\partial q_k} = F_k(q) = 0\,,\qquad 1 \le k \le n\,. \qquad (2)$$

(For the sake of illustration, we consider the case where each condition in (2) has a single solution and the resulting symmetry manifolds intersect in a single point representing an equilibrium of the system.) Let us consider reflections of a trajectory, projected onto coordinate space, on the symmetry manifolds S_k induced by the corresponding operators $\widehat{R}_k$. Self-reversed trajectories are left invariant upon reflections on the symmetry manifolds, $\prod_{k=1}^{n} \widehat{R}_k(q) = q$, since they are mapped pointwise onto themselves on equipotentials $U(q) = U(\widehat{R}_k(q))$ such that the sign on the r.h.s. in the equations of motion for $\dot{p}_k$ is reversed, $F_k(q) \to -F_k(\widehat{R}_k(q))$, $1 \le k \le n$. Observe that upon reflecting on all symmetry manifolds the relation $U(q) = U(\prod_{k=1}^{m} \widehat{R}_k(q))$, $1 \le m \le n$ is left invariant under permutations of the reflection operators. In fact, time-reversing symmetry is in coordinate space tantamount to invariance with respect to reflections on the symmetry manifolds. In more detail, any self-reversed trajectory, projected onto coordinate space, repeatedly crosses every symmetry manifold S_k each time with an opposite sign of the corresponding force $-F_k(q) < \infty$. Moreover, successive crossings of a single symmetry manifold, S_k, occur in alternating directions. Thus there must be turning points for the trajectory implying bounded motion and no directed flow can arise. Notice that no assumptions with regard to the spatial symmetries of the trajectory are needed. In contrast, as transporting (unbounded) trajectories are not invariant with respect to reflections on the symmetry manifolds, preservation of time-reversing symmetry is not possible for such a single trajectory. A transporting trajectory may escape without having crossed a symmetry manifold at all. However, if it does cross then after all such crossings of a symmetry manifold, the escaping trajectory promotes directed transport. Nevertheless, reflections on the symmetry manifolds, mapping a transporting trajectory onto another transporting one, can induce spatial symmetries such that these two trajectories mutually compensate each others contribution to the net flow. Let the point q_O in coordinate space be an initial condition associated with a transporting trajectory.

Reflecting in coordinate space on the symmetry manifolds S_k transforms an original point, q_O, into its image point, q_I, according to $\widehat{R}_k q_O = q_{I,k}$. While the value of the potential energy is maintained, $U(q_O) = U(q_{I,k})$, the sign on the r.h.s. in the equations of motion for $\dot{p}_k$ is reversed; $F_k(\widehat{R}_k q_O) \to -F_k(q_{I,k})$. However, the magnitude of the gradients, $F_k = \partial U/\partial q_k$, is not necessarily maintained. Reflection on all of the symmetry manifolds yields $\prod_{k=1}^n \widehat{R}_k q_O = q_I$, reversing the sign in all of the r.h.s. of the equations of motion for the evolution of the momenta $F_k(q_I) \to -F_k(q_O), 1 \leq k \leq n$. With the time evolution of a coordinate expressed as $q(t) = q(0) + \int_0^t dt' \{p(0) + \int_0^{t'} dt''[-F(q(t''))]\}$, we conclude that, for the pair of trajectories emanating from q_O and q_I, symmetry (zero net flow) results if $(p_I, q_I) = (-p_O, -q_O)$ so that $F_k(q_I) = -F_k(q_O), 1 \leq k \leq n$. This is the case when the potential is even in the coordinates, that is $U(q) = U(-q)$. Then there exist pairs of current-annihilating *counterpropagating* trajectories, $X(t)$, starting from $X(0)$, and $-X(t)$, starting from $-X(0)$, respectively. In other words, *reversion symmetry* under reflections on the symmetry manifolds is needed for zero net flow which, together with invariance with respect to changes of the sign of the momenta, amounts to parity-symmetry of the system $H(p, q) = H(-p, -q)$. Conversely, violation of reversion symmetry with respect to at least one of the coordinates q_k establishes a prerequisite for the occurrence of directed flow.

For an illustration, we consider the conservative and deterministic dynamics of a particle whose coordinate q evolves in a one-dimensional periodic, spatially-symmetric washboard potential of unit period which is given by

$$U(q) = U(q + 1) = \frac{1 - \cos(2\pi q)}{2\pi}\,. \tag{3}$$

The particle is assumed to interact with a local oscillator of amplitude Q evolving in a harmonic potential being subjected to a tilt force of strength F

$$V(Q) = \frac{1}{2}\omega^2 Q^2 + F\,Q + \frac{1}{2}\left(\frac{F}{\omega}\right)^2\,. \tag{4}$$

Coupling between the particle and the harmonic oscillator arises from the interaction potential

$$W(q, Q) = D\left[1 - \frac{1}{\cosh(q - Q)}\right]\,, \tag{5}$$

that depends on the relative distance $|q - Q|$. The parameter D regulates the strength of the coupling between the particle and the harmonic oscillator degree of freedom. Most importantly, the interaction is of local character, having negligible influence on the motion outside the so called *interaction region*, in which when the relative distance $|q - Q|$ is sufficiently small so that a significantly large gradient of the interaction potential results. The system of coupled equations of motion is given by

$$\ddot{q} \;=\; -\sin(2\pi q) - D\frac{\tanh(q - Q)}{\cosh(q - Q)} \tag{6}$$

$$\ddot{Q} \;=\; -Q - F + D\frac{\tanh(q - Q)}{\cosh(q - Q)}\,. \tag{7}$$

For $D = 0$, the system decouples into two integrable subsystems and the dynamics is characterized by individual regular motions of the particle in the washboard potential, and bounded oscillations of the harmonic degree of freedom, respectively. For $D \neq 0$, the subsystems interact, exchanging energy. While the Q-oscillator performs solely bounded motion, there is the possibility that, for an escaping particle, the corresponding coordinate, $|q|$ (representing the spatially-open component), attains large values and thus the related interaction forces, $\partial W/\partial q$ and $\partial W/\partial Q$, vanish asymptotically, allowing transient chaos [12]-[19]. That is, for large distance $|q - Q| \gg 1$, the interaction vanishes asymptotically, with the result that the two degrees of freedom effectively decouple, rendering the dynamics regular. (Although this decoupling is a feature of the specific example taken here, it is not essential for current emergence in general.)

We focus interest on the situation when the particle is initially at rest at the bottom of one well of the washboard potential with initial condition $p(0) = q(0) = 0$ (representing localised initial conditions in the spatially-open component, situated in the centre of the interaction region). The total energy is initially deposited in the harmonic degree of freedom. The particle can then escape from the well only if it gains sufficient energy from its interaction with the excited harmonic degree of freedom. The total energy is fixed at $E = 1.625$, which exceeds several times the barrier height, $E_b = 1/\pi \simeq 0.318$, of the washboard potential. A typical trajectory is shown in Fig. 1. Strikingly, the particle gains sufficient energy from the harmonic degree of freedom that it manages to escape from the potential well and subsequently, avoids becoming trapped in wells of the washboard potential indefinitely. Thus, the particle moves directedly to the right (continually increasing the coordinate, q) while the amplitude, Q, of the harmonic degree of freedom, performs small oscillations around the bottom of the potential $V(Q)$. In other words, after a chaotic transient the particle departs from the interaction region and the dynamics of the two degrees of freedom settles on individual regular motion. (That the particle does, indeed, settle on regular motion, and that the observed behaviour is not merely a long-lived transient of the system, is proved in what follows.)

We emphasise that the coordinate of the washboard particle tends to infinity and the interaction term in the equations of motion vanishes exponentially, excluding the return of the trajectory to the region of the initial conditions, as we prove below. (Remarkably, already at $t = 50$ the magnitude of the interaction term $D\tanh(q - Q)/\cosh(q - Q)$ has fallen below 10^{-7}.) In other words, the Poincaré recurrence time is infinity.

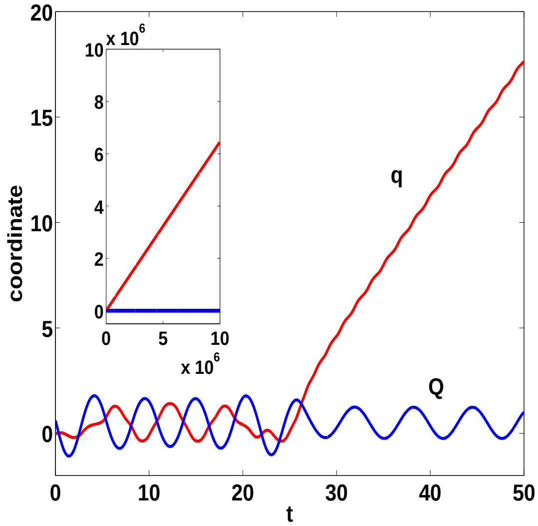

FIG. 1: Typical time evolutions of the coordinates as indicated in the plot for $D = 0.94$, $F = -0.5$, and $\omega = 1$. Note the time scale on the inset figure; $0 \leq t \leq 10^7$.

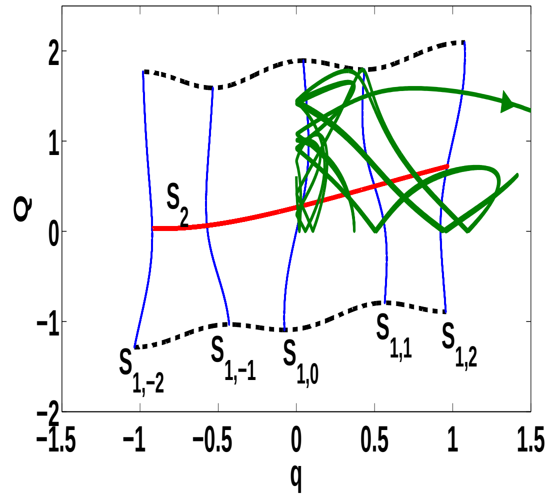

FIG. 3: Typical time evolutions of the coordinates as indicated in the plot for $D = 0.94$, $F = -0.5$, and $\omega = 1$. The total energy is $E = 1.625$ and the boundaries of the energetically allowed region are indicated by the horizontal dashed lines. Superimposed is the trajectory shown in Fig. 1.

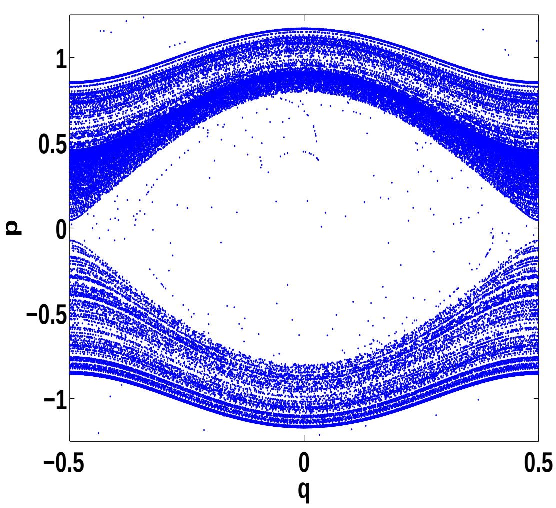

FIG. 2: Poincaré surfaces of section represented in the (p, q)-plane for $D = 0.94$, $F = -0.5$, and $\omega = 1$.

In order to illustrate the dynamics of an ensemble, evolving in the four-dimensional phase space on the three-dimensional energy hypersurface, we invoke a Poincaré surface of section taken as $\Sigma = \{\, q, p \,|Q = 0, P > 0\,\}$. Fig. 2 shows the Poincaré surface of section for $D = 0.94$ and $F = -0.5$. For the computation an ensemble of 10^5 initial momenta and coordinates $(P(0), Q(0))$ of the harmonic degree of freedom, is taken. These initial values are uniformly distributed on the level curve $E = 0.5P^2 + V(Q) + W(0, Q)$ in the (P, Q)-plane. Note that the initial conditions are *unbiased*, obeying the symmetry $P \leftrightarrow -P$, and possess spatial symmetry with regard to the washboard potential. The simulation time interval is $T = 10^5$ being equivalent to almost 4×10^4 times the period of harmonic oscillations near the bottom of a well of the washboard potential as well as of the potential of the harmonic oscillator. Likewise, the simulation time exceeds by far the time it takes for the particle to escape from the potential well. The Poincaré surface of section, shown in Fig. 2, is characterized by a few scattered points related with the dynamics of chaotic transients (followed by the trajectories during the particle's escape process) and densely covered curves associated with the regular, rotational motion to which the dynamics eventually adjusts. Crucially, there are in fact far more trajectories evolving in the range of positive momenta than in the negative range. Hence, net motion to the right arises.

It is illustrative to consider the symmetry properties underlying the equations of motion, Eqs. (6) and (7). Their structure is determined by the potential functions given in Eqs. (3)–(5), which together give rise to an effective potential $U_{\text{eff}}(q, Q) = U(q) + V(Q) + W(q, Q)$. Regarding only the washboard potential, it is periodic in q with period 1 and invariant under reflection in q, i.e., $U(q) = U(-q)$. This amounts to symmetry with respect to every $q_n = n/2$ with integer n. The interaction potential is invariant with respect to changes of the sign of its argument, viz., $(q - Q) \leftrightarrow -(q - Q)$. In contrast, the tilted harmonic oscillator potential $V(Q)$ is without reflection symmetry and, consequently, the system of Eqs. (6) and (7) is not invariant under inversions $Q \leftrightarrow -Q$. The symmetry lines are given by

$$S_1: \qquad \sin(2\pi q) + D\frac{\tanh(q-Q)}{\cosh(q-Q)} = 0 \qquad (8)$$

$$S_2: \qquad Q + F - D\frac{\tanh(q-Q)}{\cosh(q-Q)} = 0\,. \qquad (9)$$

The symmetry line S_1 exhibits the following symmetry:

$$Q \to -Q\,, \qquad \frac{n}{2} + q \to -\frac{n}{2} - q\,: \qquad S_{1,n} \to -S_{1,-n} \qquad (10)$$

with integer n labelling the several branches of the symmetry line as $S_{1,n}$. A single branch, S_2, arises from the second equation with no apparent symmetry as shown in coordinate space in Fig. 3. The branches of the symmetry lines $S_{1,n}$ related to the starting potential well and its neighbour to either side are displayed in the figure, i.e., those with $n = -2, -1, 0, 1, 2$. Crucially, reflections in the (q, Q)-coordinate plane on the symmetry line S_2 do not respect inversion symmetry, and thus, no pairs of counterpropagating trajectories are produced by the localised, but otherwise unbiased, set of initial conditions. Symmetry breaking of the trajectory displayed in Fig. 1 becomes apparent by superimposing the latter on the diagram in coordinate space in Fig. 3. In particular after an ultimate crossing of the symmetry line $S_{1,2}$ the trajectory proceeds exclusively to the right as indicated by the arrow. This spatial symmetry breaking is a prerequisite for the occurrence of directed net motion. In conclusion, even though the initial conditions are unbiased for the ensuing dynamics, the time-reversibility symmetry is broken.

We now prove that there exist trajectories which, once they have entered a certain region in phase space, provide continual transport of washboard particles. The Hamiltonian associated with the system in Eqs. (6),(7) reads as

$$\begin{aligned} H &= \frac{1}{2}p^2 + U(q) + \frac{1}{2}P^2 + V(Q) + W(q, Q) \\ &\equiv H_1(p, q) + H_2(P, Q) + W(q, Q)\,, \end{aligned} \qquad (11)$$

with U, V, W given in (3),(4),(5).

Theorem: Consider the Cauchy problem for the Hamiltonian (11) with initial data $(p(0), q(0), P(0), Q(0))$.

Let E denote the total energy of the system and let $\widehat{Q}$ denote the positive root of the equation $V(\widehat{Q}) = E$, i.e., the maximal coordinate that the oscillator could attain if all of the energy were in the oscillator degree of freedom.

Assume that the energies of the washboard particle and of the harmonic oscillator satisfy the constraints

$$E_1(0) \geq E_a + D\frac{1}{\cosh(q(0) - \widehat{Q})}\,, \qquad (12)$$

and

$$E_2(0) < E - D - E_a\,, \qquad (13)$$

respectively, where

$$E_a > \frac{1}{\pi}\,, \qquad (14)$$

the latter being the energy barrier height of the washboard degree of freedom.

If, in addition, the intial values of the coordinates of the washboard and oscillator satisfy the constraint

$$q(0) - Q(0) \geq q(0) - \widehat{Q} > \sinh^{-1}(1)\,, \qquad (15)$$

then it holds that, for some $\widehat{p}$,

$$p(t) \geq \widehat{p} > 0\,, \qquad (16)$$

for all times $t > 0$.

Proof: The proof utilises that existence of solutions with $p > 0$ is guaranteed by the asymptotic behaviour of the washboard coordinate as $q \to \infty$. The momentum p is represented as follows:

$$\frac{1}{2}p^2 = E - U(q) - E_2 - W(q, Q)\,. \qquad (17)$$

With the help of (13) we obtain the lower bound for the initial value

$$\begin{aligned} \frac{1}{2}p^2(0) &= E_a - \max[U(q(0))] + D\frac{1}{\cosh(q(0) - Q(0))} \\ &\geq E_a - \frac{1}{\pi} + D\frac{1}{\cosh(q(0) - Q(0))}\,. \end{aligned} \qquad (18)$$

From the lower inequality (14) follows that the momentum $p(0)$ is positive, regardless of the value of the positive term $D/\cosh(q(0) - Q(0))$.

The change in energy of the two subsystems under the dynamics is determined by

$$\frac{dE_1}{dt} = \{E_1, H\} = -D\frac{\tanh(q-Q)}{\cosh(q-Q)}p\,, \qquad (19)$$

$$\frac{dE_2}{dt} = \{E_2, H\} = D\frac{\tanh(q-Q)}{\cosh(q-Q)}P\,. \qquad (20)$$

Suppose (in order to reach a contradiction) that the trajectory reaches a point at which $p = 0$, i.e., that the trajectory reaches a stationary point in the washboard coordinate, q, after which it might turn back. Let $q_1 > q_0$ denote the first point at which that happens. Using that $p = dq/dt$ and $P = dQ/dt$ one obtains

$$E_1(1) - E_1(0) = -D\int_{q_0}^{q_1} \frac{\tanh(q-Q)}{\cosh(q-Q)}dq\,, \qquad (21)$$

$$E_2(1) - E_2(0) = D\int_{Q_0}^{Q_1} \frac{\tanh(q-Q)}{\cosh(q-Q)}dQ\,. \qquad (22)$$

The maxima of the integrand $\tanh(q-Q)/\cosh(q-Q)$ are located at $|q - Q| = \sinh^{-1}(1)$ and for

$$|q - Q| > \sinh^{-1}(1) \qquad (23)$$

the integrand remains non-negative. Hence, provided the coordinates stay in the range given by (15) the washboard degree of freedom loses energy while the harmonic oscillator gains energy. We derive an upper bound on the magnitude of the energy loss of the washboard particle and demonstrate that, provided the above assumptions on the initial data hold, this loss cannot in fact be sufficient to halt or turn back the particle. To this end, the integration in (21) is performed yielding

$$\begin{aligned} E_1(1) - E_1(0) &= -D \int_{q_0}^{q_1} \frac{\tanh(q-Q)}{\cosh(q-Q)} dq \qquad (24) \\ &\geq -D \int_{q_0}^{q_1} \frac{\tanh(q-\widehat{Q})}{\cosh(q-\widehat{Q})} dq \\ &= -D \left[-\frac{1}{\cosh(q_1-\widehat{Q})} + \frac{1}{\cosh(q_0-\widehat{Q})} \right] \\ &\geq -D \left[\frac{1}{\cosh(q_0-\widehat{Q})} \right]. \qquad (25) \end{aligned}$$

Suppose that initial data for the coordinate $q(0)$ satisfies the inequality (15) and, in addition, the washboard degree of freedom satisfies our energy constraints (12) and (14), then

$$\begin{aligned} E_1(1) &\geq E_1(0) - D \left[\frac{1}{\cosh(q_0-\widehat{Q})} \right] \qquad (26) \\ &\geq E_a > \frac{1}{\pi}. \qquad (27) \end{aligned}$$

Thus, the energy of the particle in the washboard degree of freedom remains above the separatrix level, contradicting the supposition that the particle halts (attaining $p = 0$ at some position q_1), and we therefore have that $p(t) > 0$ for all times $t \geq 0$. Indeed, since the energy E_1 of the washboard particle is bounded away from the barrier level, $1/\pi$, it follows that $p(t) \geq \widehat{p} > 0$ for some $\widehat{p}$, which implies that $q \to \infty$ as $t \to \infty$. □

That regions of phase space satisfying the assumptions of the theorem do, indeed, exist, will be shown below. Firstly, we consider the asymptotic behaviour of such a transporting trajectory. Given that $q \to \infty$, the asymptotic energy of the washboard particle is bounded by

$$E_1(\infty) \geq E_1(0) - \frac{D}{\cosh(q_0-\widehat{Q})}. \qquad (28)$$

With the assumption in Eqs. (12) and (14) we deduce that

$$E_1(\infty) \geq E_a > \frac{1}{\pi}. \qquad (29)$$

The last property means that the washboard particle loses only a finite amount of energy and the energy of the washboard particle approaches the level $E = E_a$ from above. At the same time the energy of the harmonic oscillator asymptotically approaches $E_2 = E - E_a - D$ from below. Hence, the coordinate of the harmonic oscillator performs bounded oscillations around $Q = 0$ which, together with growing q, assures that the distance $q(t) - Q(t)$ effectively grows as time progresses. Hence,

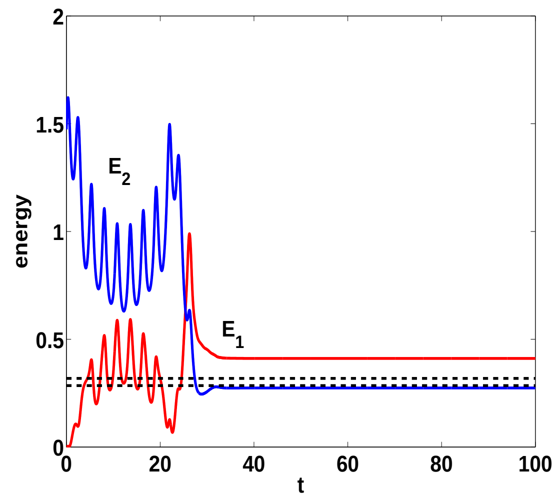

FIG. 4: Time evolution of the energy of the washboard particle E_1 and the harmonic oscillator E_2. The two dashed horizontal lines correspond to the energy levels $1/\pi$ and $E-D-E_a$ bounding the energies $E_1 = E_a$ and $E_2 = E - D - E_a$ in the asymptotic regime.

for initial coordinates fulfilling the constraints of the theorem, the integrand in (24) vanishes in an exponential fashion.

In conclusion, as soon as the inequalities (12),(13) and (15), imposing conditions in phase space, are satisfied, the process of redistributing the finite (albeit small) amount of energy $D/\cosh(q-Q)$, that the washboard degree of freedom possesses in excess of the asymptotic level E_a, into the harmonic oscillator is *irreversible.* This asymptotically decouples the two subsystems and, most importantly, the washboard particle, being equipped with energy $E_1 \geq E_a > 1/\pi$, moves in an unhindered and unidirectional manner, which establishes continual transport that cannot be terminated. In Fig. 4, we illustrate a trajectory that shows an initial energy redistribution from the harmonic oscillator into the washboard particle, after which the phase space region conforming to the assumptions of the theorem is reached. Notice the pronounced growth of the energy E_1 at the expense of E_2 for $t \gtrsim 25$. Crucially the energy E_2 falls below the level $E - D - E_a$ at $t \simeq 27$, that is at the *moment of no return.* Strikingly, this happens at the moment when the difference $q - Q$ starts to increase beyond the value $\sinh^{-1}(1)$. Afterwards the above mentioned (small) energy loss (gain) in E_1 (E_2) takes place as a result of which the energy E_1 settles onto E_a. The latter value lies above the lower bound $1/\pi$ as given in (14) (represented by the lower dashed line in Fig. 4) while the energy E_2 attains an asymptotic level below $E - D - E_a$ (represented by the upper dashed line in Fig. 4).

In summary, we have demonstrated that it is possible to obtain directed motion in autonomous Hamiltonian systems under the minimal conditions that (i) transient chaos is supported and (ii) some of the degrees of freedom serve to break the time reversibility with regard to a set of unbiased localised initial conditions. (The model system is minimal since these two conditions constitute the only indispensable prerequisites. Notice in particular that no additional modulation field is required in order to break the necessary spatio-temporal symmetries.) That chaos is needed only in an initial stage of the dynamics in order to guide the trajectory into the range of regular motion has a drastic implication, namely the directed net motion is provided by regular motion and thus is of *continual* nature. This is at variance with previous studies of transport in non-autonomous [4]–[8] and autonomous Hamiltonian systems [9],[10], where a mixing phase space is needed to support directed chaotic transport. Despite the fact that finite asymptotic currents were observed, sustaining transport in general in nonintegrable systems seems to be hampered because of the intermittent character of the dynamics inducing power-law decay statistics of the duration of periods of regular motion. On the other hand, provided the minimal prerequisites mentioned above are given, application of our approach to accomplish directed net motion in other Hamiltonian systems, such as infinite lattice systems, is straightforward. For example, in the context of the Holstein model, the charge being initially trapped in a confined region of the molecular chain constitutes localised initial conditions. Charge motion along a molecular chain can be directed by taking into account non-reversion symmetric oscillators representing the intra-molecular vibrational degrees of freedom.

[1] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, New York, 1989).
[2] P. Hänggi and F. Marchesoni, Rev. Mod. Phys. **81**, 387 (2009).
[3] T.S. Monteiro *et al*, Phys. Rev. Lett. **89**, 194102 (2002); M. Schiavoni, L. Sanchez-Palencia, F. Renzoni, and G. Grynberg, Phys. Rev. Lett. **90**, 094101 (2003); R. Gommers, S. Denisov, and F. Renzoni, Phys. Rev. Lett. **96**, 240604 (2006).
[4] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, Phys. Rev. Lett. **84**, 2358 (2000).
[5] S. Denisov, S. Flach, A.A. Ovchinnikov, O. Yevtushenko, and Y. Zolotaryuk, Phys. Rev. E **66**, 041104 (2002).
[6] H. Schanz, M.-F. Otto, R. Ketzmerick, and T. Dittrich, Phys. Rev. Lett. **87**, 070601 (2001).
[7] S. Denisov and S. Flach, Phys. Rev. E **64**, 056236 (2001); S. Denisov, J. Klafter, M. Urbakh, and S. Flach, Physica D **170**, 131 (2002); S. Denisov, J. Klafter, and M. Urbakh, Phys. Rev. E **66**, 046217 (2002).
[8] H. Schanz, T. Dittrich, and R. Ketzmerick, Phys. Rev. E **71**, 026228 (2005).
[9] W. Acevedo and T. Dittrich, Prog. Theor. Phys. Suppl. **150**, 313 (2003).
[10] H. Schanz and M. Prusty, Phys. A: Math. Gen. **38**, 10085 (2005).
[11] M. Horvat and T. Prosen, J. Phys. A: Math. Gen. **37**, 3133 (2004).
[12] S. Bleher, C. Grebogi, and E. Ott, Physica D **46**, 87 (1990).
[13] A. Pikovsky, J. Phys. A **25**, L477 (1992).
[14] G. Contopolous, H.E. Kandrup, and D. Kaufman, Physica D **64**, 310 (1993).
[15] T. Tel, J. Vollmer, and W. Breymann, Europhys. Lett. **35**, 659 (1996).
[16] E. Ott *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 2002).
[17] T. Tel, in *Directions in Chaos* (Ed.: Bai-lin Hao, World Scientific, Singapore, Vol. 3, 1990).
[18] T. Tel and M. Gruiz *Chaotic Dynamics* Cambridge University Press, Cambridge, 2006).
[19] M. Zaslavsky, *Chaos in Dynamical Systems* (Harwood, New York, 1985), *Physics of Chaos in Hamiltonian Systems* (Imperial College Press, London, 1998).