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Non-gaussianity from the trispectrum in general single field inflation

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We compute the fourth order action in perturbation theory for scalar and second order tensor perturbations for a minimally coupled single field inflationary model, where the inflaton's lagrangian is a general function of the field's value and its kinetic energy. We obtain the fourth order action in two gauges, the comoving gauge and the uniform curvature gauge. Using the comoving gauge action we calculate the trispectrum at leading order in slow roll, finding agreement with a previously known result in the literature. We point out that in general to obtain the correct leading order trispectrum one cannot ignore second order tensor perturbations as previously done by others. The next-to-leading order corrections may become detectable depending on the shape and we provide the necessary formalism to calculate them.

I. INTRODUCTION

The theory of slow-roll inflation generically predicts that the observed cosmic microwave background radiation (CMBR) anisotropies are nearly scale invariant and very gaussian. Indeed, the latest observations of CMBR by WMAP3 [1] confirm these expectations. This constitutes one of the biggest achievements of modern cosmology.

Despite its successes the theory of inflation still has many open questions. For example, we do not know the origin of the scalar field whose energy drives inflation, not to mention that we have never detected directly in the laboratory these kind of particles. The energy scale at which inflation happened is unknown by many orders of magnitude. There are many models of inflation that give similar predictions for the power spectrum of primordial perturbations, so which one (if any) is the correct one?

For us to move a step forward in our understanding of the very early universe we have to work in two fronts. First the observational side. In the next few years, with improved experiments like the Planck satellite, we will measure the CMBR anisotropies to an incredible resolution. For example the observational bounds on the bispectrum (the three point correlation function of the primordial curvature perturbation ζ) will shrink from the present WMAP3 value $-50 < f_{NL} < 114$ [1] to $|f_{NL}| \lesssim 5$ [2], where the parameter f_{NL} parameterizes the size of the bispectrum. It's because this parameter is constrained to be small that we say that the CMBR anisotropies are very gaussian. The observational bounds on the trispectrum (four point function) will also tighten significantly from the rather weak present constraint of $|\tau_{NL}| < 10^8$ [3, 4] to the future constraint of $|\tau_{NL}| \sim 560$ [5], where τ_{NL} denotes the size of the trispectrum. These previous observational bounds on the non-linearity parameters are for non-gaussianity of the local type. These bounds change depending on the shape of the wave vectors' configuration [6]. This is one of the reasons why it is important to calculate the shape dependence of the non-gaussianity.

In face of these expected observational advances, it is then imperative to push forward our theoretical knowledge of our theories and calculate more observational consequences of the different inflationary models to make a comparison with observations possible. One possible direction to be followed by us and many others is to calculate higher order statistics (like the trispectrum) of the primordial curvature perturbation. These higher order statistics contain much more information about the inflationary dynamics and if we observe them they will strongly constrain our models. Because these higher order statistics have a non-trivial momentum dependence (shape) they will help to discriminate between models that have a similar power spectrum (two point function).

Calculations of the bispectrum for a single field inflationary model were done by Maldacena [7]. He showed that the primordial bispectrum is too small (of the order of the slow-roll parameters) to be observed even with Planck. Subsequent work generalized Maldacena's result to include more fields and more complicated kinetic terms [8, 9, 10]. In [10], Chen *et al.* have calculated the bispectrum for a quite general model of single field inflation. They showed that for some models even the next-to-leading order corrections in the slow-roll expansion may be observed.

In this paper, we will focus our attention on the calculation of the trispectrum. In [11], Seery *et al.* have calculated the trispectrum for slow-roll multi-field models (with standard kinetic terms) and they showed that at horizon crossing it is too small to be observed. But there are models of single field inflation, well motivated from more fundamental theories, that can produce a significant amount of non-gaussianity, such as Dirac-Born-Infeld (DBI) inflation [12, 13].

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In [13] the authors have computed the trispectrum for a model where the inflaton's lagrangian is a general function of the field's kinetic energy and the field's value, their result was obtained using a simple method [14], that only gives the correct leading order answer. In this paper we will provide the equations necessary to calculate the next-to-leading order corrections to the trispectrum. We argue that for some models these corrections might become equally observable in the future. In fact, we will calculate the fourth order action in the uniform curvature gauge that is exact in the slow-roll expansion and therefore in principle one could calculate all slow-roll corrections to the trispectrum of the field perturbations.

We will also compute the exact fourth order action for the curvature perturbation ζ in the comoving gauge. For a simpler inflation model (with the standard kinetic term) this was recently done in [15]. However [15] did not consider second order tensor perturbations. We will argue that this is an oversimplification that leads to erroneous results. The reason simply being that at second order in perturbation theory, scalar degrees of freedom will source second order tensor perturbations and this will give a non-zero contribution for the fourth order action and hence the trispectrum.

There are other reasons why we will perform the calculation in the comoving gauge. First of all, in doing so we work all the time with the gauge invariant variable ζ that is directly related with the observational quantities. The comoving gauge action can also be used other practical purposes. For example, it can be used to calculate loop effects that can possibly have important observational consequences. It can also be used to calculate the trispectrum of models where the potential has a "feature" (see [16, 17] for an example of such calculation for the bispectrum). In the vicinity of the sudden potential "jump" the slow-roll approximation temporarily fails and one might get an enhancement of the trispectrum. There are well motivated models of brane inflation where the throat's warp factor suddenly jumps [18].

This paper is organized as follows. In the next section, we introduce the model under consideration. In section III we shall study non-linear perturbations. First, we compute the fourth order action in the comoving gauge including both scalar and second order tensor degrees of freedom. After that we compute the fourth order action in the uniform curvature gauge. In section IV, we present the formalism needed to calculate the trispectrum. In section V, we calculate the leading order trispectrum using the comoving gauge action. We comment on previous works and on the observability of next-to-leading order corrections. Section VI is devoted to conclusions. Finally in Appendix, we present the second order gauge transformation between the two previous gauges and a way to extract the transverse and traceless part of a tensor.

II. THE MODEL

In this work, we will consider a fairly general class of models described by the following action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[M_{Pl}^2 R + 2P(X,\phi)\right], \tag{1}$$

where ϕ is the inflaton field, M_{Pl} is the Planck mass that we will set to unity hereafter, R is the Ricci scalar and $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the inflaton's kinetic energy. $g_{\mu\nu}$ is the metric tensor. We label the inflaton's lagrangian by P and we assume that it is a well behaved function of two variables, the inflaton field and its kinetic energy.

This general field lagrangian includes as particular cases the common slow-roll inflation model, DBI-inflation [19] [12] and K-inflation [20].

We are interested in flat, homogeneous and isotropic Friedmann-Robertson-Walker universes described by the line element

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{2}$$

where $a(t)$ is the scale factor. The Friedmann equation and the continuity equation read

$$3H^2 = E, \tag{3}$$

$$\dot{E} = -3H\left(E+P\right), \tag{4}$$

where the Hubble rate is $H = \dot{a}/a$, E is the energy of the inflaton and it is given by

$$E = 2XP_{,X} - P, \tag{5}$$

where $P_{,X}$ denotes the derivative of P with respect to X.

It was shown in [21] that for this model the speed of propagation of scalar perturbations ("speed of sound") is c_s given by

$$c_s^2 = \frac{P_{,X}}{E_{,X}} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}. \tag{6}$$

We define the slow variation parameters, analogues of the slow-roll parameters, as:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{XP_{,X}}{H^2}, \tag{7}$$

$$\eta = \frac{\dot{\epsilon}}{\epsilon H}, \tag{8}$$

$$s = \frac{\dot{c}_s}{c_s H}. \tag{9}$$

We should note that these slow variation parameters are more general than the usual slow-roll parameters and that the smallness of these parameters does not imply that the field in rolling slowly. We assume that the rate of change of the speed of sound is small (as described by s) but c_s is otherwise free to change between zero and one.

It is convenient to introduce the following parameters that describe the non-linear dependence of the lagrangian on the kinetic energy:

$$\Sigma = XP_{,X} + 2X^2 P_{,XX} = \frac{H^2\epsilon}{c_s^2}, \tag{10}$$

$$\lambda = X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX}, \tag{11}$$

$$\Pi = X^3 P_{,XXX} + \frac{2}{5} X^4 P_{,XXXX}. \tag{12}$$

These parameters are related to the size of the bispectrum and trispectrum. The power spectrum of the primordial quantum fluctuation was first derived in [21] and reads

$$P_k^\zeta = \frac{1}{36\pi^2} \frac{E^2}{E+P} = \frac{1}{8\pi^2} \frac{H^2}{c_s \epsilon}, \tag{13}$$

where it should be evaluated at the time of horizon crossing $c_{s*} k = a_* H_*$. The spectral index is

$$n_s - 1 = \frac{d \ln P_k^\zeta}{d \ln k} = -2\epsilon - \eta - s. \tag{14}$$

WMAP observations of the perturbations in the CMBR tell us that the previous power spectrum is almost scale invariant therefore implying that the three slow variation parameters should be small at horizon crossing, roughly of order 10^{-2}.

III. NON-LINEAR PERTURBATIONS

In this section, we will consider perturbations of the background (2) beyond linear order. There is a vast literature on second order perturbations that are important when one is interested in calculating three point correlation functions, see for example [7, 8, 10, 22]. In the present paper, we are interested in non-gaussianities that come from the trispectrum and so we need to use third order perturbation theory. For that we need to compute the fourth order in the perturbation action. In this section we will obtain the fourth order action in two different gauges. As a check on our calculations we will compute the leading order (in slow roll) trispectrum in both gauges. We will follow the pioneering approach developed by Maldacena [7] and used in several subsequent papers [8, 9, 11, 13].

For reasons that will become clear later it is convenient to use the ADM metric formalism [23]. The ADM line element reads

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{15}$$

where N is the lapse function, N^i is the shift vector and h_{ij} is the 3D metric.

The action (1) becomes

$$S = \frac{1}{2}\int dt d^3x \sqrt{h} N \left({}^{(3)}R + 2P\right) + \frac{1}{2}\int dt d^3x \sqrt{h} N^{-1}\left(E_{ij}E^{ij} - E^2\right). \tag{16}$$

The tensor E_{ij} is defined as

$$E_{ij} = \frac{1}{2}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \tag{17}$$

and it is related to the extrinsic curvature by $K_{ij} = N^{-1}E_{ij}$. ∇_i is the covariant derivative with respect to h_{ij} and all contra-variant indices in this section are raised with h_{ij} unless stated otherwise.

The hamiltonian and momentum constraints are respectively

$$\begin{aligned} {}^{(3)}R + 2P - 2\pi^2 N^{-2} P_{,X} - N^{-2}\left(E_{ij}E^{ij} - E^2\right) &= 0, \\ \nabla_j\left(N^{-1}E^j_i\right) - \nabla_i\left(N^{-1}E\right) &= \pi N^{-1}\nabla_i\phi P_{,X}, \end{aligned} \tag{18}$$

where π is defined as

$$\pi \equiv \dot{\phi} - N^j \nabla_j \phi. \tag{19}$$

We decompose the shift vector N^i into scalar and intrinsic vector parts as

$$N_i = \tilde{N}_i + \partial_i \psi, \tag{20}$$

where $\partial_i \tilde{N}^i = 0$, here indices are raised with δ_{ij}.

Before we consider perturbations around our background let us count the number of degrees of freedom (dof) that we have. There are five scalar functions, the field ϕ, N, ψ, $\det h$ and $h_{ij} \sim \partial_i\partial_j H$, where H is a scalar function and $\det h$ denotes the determinant of the 3D metric. Also, there are two vector modes $\tilde{N}^i$ and $h_{ij} \sim \partial_i \chi_j$, where χ^j is an arbitrary vector. Both $\tilde{N}^i$ and χ^j satisfy a divergenceless condition and so carry four dof. Furthermore, we also have a transverse and traceless tensor mode γ_{ij} that contains two additional dof. Because our theory is invariant under change of coordinates we can eliminate some of these dof. For instance, a spatial reparametrization like $x^i = \tilde{x}^i + \partial^i \tilde{\epsilon}(\tilde{x}, \tilde{t}) + \epsilon^i_{(t)}(\tilde{x}, \tilde{t})$, where $\tilde{\epsilon}$ and $\epsilon^i_{(t)}$ are arbitrary and $\partial_i \epsilon^i_{(t)} = 0$, can be chosen so that it removes one scalar dof and one vector mode. A time reparametrization would eliminate another scalar dof. Constraints in the action will eliminate further two scalar dof and a vector mode. In the end we are left with one scalar, zero vector and one tensor modes that correspond to three physical propagating dof.

In the next subsection we shall use two different gauges that correctly parameterize these dof. Because physical observables are gauge invariant we know that both gauges have to give the same result for the trispectrum for instance. It seems then unnecessary to perform the calculation twice in different gauges. In practice, we will see that both gauges have advantages and disadvantages and one is more suitable for some applications than the other. Furthermore, it provides a good consistency check on the calculation.

A. Non-linear perturbations in the comoving gauge

In this subsection, we will compute the fourth order action for the general model (1) in the comoving gauge. In this gauge the scalar degree of freedom is the so-called curvature perturbation ζ that is also gauge invariant. There are a few works on this subject using this gauge, see e.g. [15], where the authors have calculated the fourth order action for a standard kinetic term inflation but they neglected second order tensor perturbations. We will show that this is an oversimplification that may lead to an erroneous result for the four point correlation function.

In the comoving gauge, the inflaton fluctuations vanish and the 3D metric is perturbed as

$$\begin{aligned} &\delta\phi = 0, \\ &h_{ij} = a^2 e^{2\zeta}\hat{h}_{ij}, \quad \hat{h}_{ij} = \delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma^k_j + \cdots \end{aligned} \tag{21}$$

where $\det\hat{h}=1$, γ_{ij} is a tensor perturbation that we assume to be a second order quantity, i. e. $\gamma_{ij}=\mathcal{O}(\zeta^2)$. It obeys the traceless and transverse conditions $\gamma^i_i=\partial^i\gamma_{ij}=0$ (indices are raised with δ_{ij}). ζ is the gauge invariant scalar perturbation. In (21), we have ignored the first order tensor perturbations ${}^{(1)}\gamma_{ij\,GW}$. This is because any correlation function involving this tensor mode will be smaller than a correlation function involving only scalars, see results of [7]. In the literature the second order tensor perturbations are often neglected, however they should be taken into account. The reason for this is because at second order the scalars will source the tensor perturbations equation. Later in this section, we will elaborate further on this point. Higher order tensor perturbations, like ${}^{(3)}\gamma_{ij}$, do not contribute to the fourth order action.

We expand N and N^i in power of the perturbation ζ

$$N = 1+\alpha_1+\alpha_2+\cdots\,, \tag{22}$$

$$\tilde{N}_i = \tilde{N}_i^{(1)}+\tilde{N}_i^{(2)}+\cdots\,, \tag{23}$$

$$\psi = \psi_1+\psi_2+\cdots\,, \tag{24}$$

where α_n, $\tilde{N}_i^{(n)}$ and ψ_n are of order ζ^n.

Some useful expressions for the quantities appearing in (18), valid to all orders in perturbations but for $\gamma_{ij}=0$:

$${}^{(3)}R = -2a^{-2}e^{-2\zeta}\left(\partial_i\zeta\partial^i\zeta+2\partial_i\partial^i\zeta\right), \tag{25}$$

$$\begin{aligned}
E_{ij}E^{ij}-E^2 \;=\;& -6\left(H+\dot{\zeta}\right)^2+\frac{4H}{a^2}\left(1+\frac{\dot{\zeta}}{H}\right)e^{-2\zeta}\left(\partial^2\psi+\partial^k\zeta\partial_k\psi+\partial^k\zeta\tilde{N}_k\right)\\
&+a^{-4}e^{-4\zeta}\Big[\frac{1}{2}\partial_i\tilde{N}_j\left(\partial^i\tilde{N}^j+\partial^j\tilde{N}^i\right)+2\partial^i\partial^j\psi\partial_i\tilde{N}_j+\partial_i\partial_j\psi\partial^i\partial^j\psi-\partial^2\psi\partial^2\psi\\
&-2\partial_i\tilde{N}_j\left(\partial^j\zeta\tilde{N}^i+\partial^i\zeta\tilde{N}^j\right)-4\tilde{N}_k\partial_i\zeta\partial^i\partial^k\psi-2\partial_i\tilde{N}_j\left(\partial^j\zeta\partial^i\psi+\partial^i\zeta\partial^j\psi\right)\\
&-4\partial_i\partial_j\psi\partial^j\zeta\partial^i\psi+2\partial^j\zeta\left(\partial_j\zeta\tilde{N}_i\tilde{N}^i+2\partial_j\zeta\tilde{N}_i\partial^i\psi+\partial_j\zeta\partial_i\psi\partial^i\psi\right)\Big],
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\nabla_j\left(N^{-1}E^j_i\right)-\nabla_i\left(N^{-1}E\right)=\\
&-N^{-2}\Big[-2\left(H+\dot{\zeta}\right)\partial_iN+\frac{a^{-2}e^{-2\zeta}}{2}\left(-\partial^jN\left(\partial_iN_j+\partial_jN_i\right)+2\partial_jN\left(\partial^j\zeta N_i+\partial_i\zeta N^j\right)+2\partial_iN\partial^2\psi\right)\Big]\\
&+N^{-1}\Big[-2\partial_i\dot{\zeta}+a^{-2}e^{-2\zeta}\left(\frac{1}{2}\left(\partial^j\zeta\partial_j\tilde{N}_i-\partial_j\zeta\partial_i\tilde{N}^j\right)+\partial_j\zeta\partial^j\zeta N_i-\partial_i\zeta\partial^j\zeta N_j+\partial_i\partial^j\zeta N_j+\partial^2\zeta N_i-\frac{1}{2}\partial^2\tilde{N}_i\right)\Big].
\end{aligned} \tag{27}$$

In the previous equations, indices in the right-hand side are raised with δ_{ij} while indices in the left-hand side are raised with h_{ij}. In the rest of this section indices will be raised with δ_{ij}.

Now, the strategy is to solve the constraint equations for the lapse function and shift vector in terms of ζ and then plug in the solutions in the expanded action up to fourth order.

At first order in ζ, a particular solution for equations (18) is [7, 8]:

$$\alpha_1=\frac{\dot{\zeta}}{H},\quad \tilde{N}_i^{(1)}=0,\quad \psi_1=-\frac{\zeta}{H}+\chi,\quad \partial^2\chi=a^2\frac{\epsilon}{c_s^2}\dot{\zeta}. \tag{28}$$

At second order, the constraint equation for the lapse function gives

$$\begin{aligned}
\frac{4H}{a^2}\partial^2\psi_2 \;=\;& -2a^{-2}\partial_i\zeta\left(\partial^i\zeta+2H\partial^i\psi_1\right)-4\alpha_1\left(a^{-2}\partial_i\partial^i\zeta-2\Sigma\zeta\right)-2\alpha_1^2\left(\Sigma+6\lambda\right)\\
&-a^{-4}\left(\partial^i\partial_k\psi_1\partial_i\partial^k\psi_1-\partial^2\psi_1\partial^2\psi_1\right)+4\alpha_2\left(\Sigma-3H^2\right),
\end{aligned} \tag{29}$$

and the equation for the shift vector gives

$$2H\partial_i\alpha_2-\frac{1}{2}a^{-2}\partial^2\tilde{N}_i^{(2)}=-a^{-2}\left(\partial_k\alpha_1\partial^k\partial_i\psi_1-\partial_i\alpha_1\partial^2\psi_1+\partial^2\zeta\partial_i\psi_1+\partial_i\partial^k\zeta\partial_k\psi_1\right). \tag{30}$$

Due to the fact that $\tilde{N}^i$ is divergenceless and that any vector can be separated into a incompressible and irrotational part one can separate the contributions from α_2 and $\tilde{N}_i^{(2)}$ in the previous equation. The irrotational part of Eq. (30) gives

$$2H\alpha_2 = \partial^{-2}\partial^i F_i, \tag{31}$$

and the incompressible part gives

$$\frac{1}{2a^2}\tilde{N}_i^{(2)} = -\partial^{-2}F_i + \partial^{-4}\partial_i\partial^k F_k, \tag{32}$$

where F_i is define as the right-hand side of equation (30). The operator ∂^{-2} is defined by $\partial^{-2}(\partial^2\varphi) = \varphi$ and in Fourier space it just bring in a factor of $-1/k^2$.

It was shown in [10] that to compute the effective action of order ζ^n, within the ADM formalism, one only needs to use the solution for the Lagrange multipliers N and N^i up to order ζ^{n-2}. Therefore in order to calculate the fourth order effective action the knowledge of the Lagrange multipliers up to second order is required. It is given in equations (28), (31) and (32).

The second order action is

$$S_2 = \int dtd^3x \left[a^3\frac{\epsilon}{c_s^2}\dot{\zeta}^2 - a\epsilon\left(\partial\zeta\right)^2\right]. \tag{33}$$

The third order action is [7, 8, 10]

$$\begin{aligned} S_3 \;=\; \int dtd^3x \Bigg[& -\epsilon a\zeta\left(\partial\zeta\right)^2 - a^3\left(\Sigma + 2\lambda\right)\frac{\dot{\zeta}^3}{H^3} + \frac{3a^3\epsilon}{c_s^2}\zeta\dot{\zeta}^2 \\ & + \frac{1}{2a}\left(3\zeta - \frac{\dot{\zeta}}{H}\right)\left(\partial_i\partial_j\psi_1\partial_i\partial_j\psi_1 - \partial^2\psi_1\partial^2\psi_1\right) - \frac{2}{a}\partial_i\psi_1\partial_i\zeta\partial^2\psi_1\Bigg]. \end{aligned} \tag{34}$$

The scalar fourth order action is

$$\begin{aligned} S_4 \;=\; \frac{1}{2}\int dtd^3x a^3 \Bigg[& -a^{-2}\epsilon\zeta^2\left(\partial\zeta\right)^2 + \alpha_1^4\left(2\Sigma + 9\lambda + \frac{10}{3}\Pi\right) - 6\zeta\alpha_1^3\left(\Sigma + 2\lambda\right) + 9\zeta^2\alpha_1^2\Sigma \\ & -2\alpha_2^2\left(\Sigma - 3H^2\right) + a^{-4}\left(\frac{\zeta^2}{2} + \zeta\alpha_1 + \alpha_1^2\right)\left(\partial^k N_j^{(1)}\partial_k N^{j\,(1)} - \partial^j N_j^{(1)}\partial^k N_k^{(1)}\right) \\ & -2a^{-4}\left(\zeta + \alpha_1\right)\partial^k N^{j\,(1)}\left(\partial_k N_j^{(2)} - \delta_{kj}\partial^n N_n^{(2)} - 2\partial_j\zeta N_k^{(1)}\right) \\ & +a^{-4}\left(-4\partial_k N_j^{(1)}\partial^j\zeta N^{k\,(2)} + 2N_k^{(1)}\partial_j\zeta N^{k\,(1)}\partial^j\zeta\right) \\ & +\frac{a^{-4}}{2}\partial_k\tilde{N}_j^{(2)}\partial^k\tilde{N}^{j(2)} - 2a^{-4}\partial_k N_j^{(2)}\left(\partial^j\zeta N^{k\,(1)} + \partial^k\zeta N^{j\,(1)}\right)\Bigg]. \end{aligned} \tag{35}$$

Here, no slow-roll approximation has been made. The previous action has to be supplemented with the action containing terms with one and two tensors

$$S_{\gamma^2} = \frac{1}{8}\int dtd^3x\left[a^3\dot{\gamma}_{ij}\dot{\gamma}^{ij} - a\partial_k\gamma_{ij}\partial^k\gamma^{ij}\right], \tag{36}$$

$$S_{\gamma\zeta^2} = \int dtd^3x\left[-2\frac{a}{H}\gamma_{ij}\partial^i\dot{\zeta}\partial^j\zeta - a\gamma_{ij}\partial^i\zeta\partial^j\zeta - \frac{1}{2}a\left(3\zeta - \frac{\dot{\zeta}}{H}\right)\dot{\gamma}_{ij}\partial^i\partial^j\psi_1 + \frac{1}{2}a^{-1}\partial_k\gamma_{ij}\partial^i\partial^j\psi_1\partial^k\psi_1\right]. \tag{37}$$

1. Canonical variable for quantization ζ_n

In order to calculate the quantum four point correlation function we follow the standard procedure in quantum field theory. However there is an important subtlety here. The gauge invariant quantity ζ is not the correct variable to be quantized, because it is not a canonical field . The canonical field to be quantized is the field perturbation $\delta\phi$, or a convenient parameterization ζ_n defined by

$$\zeta_n = -\frac{H}{\dot{\phi}_0}\delta\phi, \tag{38}$$

where ϕ_0 is the background value of the field. We will see that ζ is related to ζ_n by a non-linear transformation so for the power spectrum calculation both procedures of quantizing ζ or ζ_n give the same answer because the difference between these two variables is a second order quantity. However, for the calculation of higher order correlation functions (like the bispectrum or trispectrum) ζ_n is the correct variable to be quantized as it is linear in $\delta\phi$. For the trispectrum, quantizing ζ gives different results from quantizing ζ_n. We will find the relation between ζ and ζ_n through the gauge transformation equations from the uniform curvature gauge (discussed in detail in the next subsection) to the comoving gauge. In the uniform curvature gauge the ansatz is

$$\begin{aligned}&\phi(x,t) = \phi_0 + \delta\phi(x,t),\\ &h_{ij} = a^2\hat{h}_{ij}, \quad \hat{h}_{ij} = \delta_{ij} + \tilde{\gamma}_{ij} + \frac{1}{2}\tilde{\gamma}_{ik}\tilde{\gamma}^k_j + \cdots\end{aligned} \tag{39}$$

where $\det\hat{h} = 1$ and $\tilde{\gamma}_{ij}$ is a tensor perturbation that we assume to be a second order quantity, i.e., $\tilde{\gamma}_{ij} = \mathcal{O}(\delta\phi^2)$. The gauge transformations are

$$\zeta(\mathbf{x}) \ = \ \zeta_n(\mathbf{x}) + F_2(\zeta_n(\mathbf{x})) + F_3(\zeta_n(\mathbf{x})), \tag{40}$$

$$\gamma_{ij} \ = \ \tilde{\gamma}_{ij}(t) + \mu_{ij}, \tag{41}$$

where $F_2(\zeta_n), \mu_{ij} = \mathcal{O}(\zeta_n^2)$, $F_3(\zeta_n) = \mathcal{O}(\zeta_n^3)$ are the terms coming from the second and third order gauge transformations respectively, they can be found explicitly in Appendix A or in [7, 15].

We now need to find the fourth order action for the variable ζ_n. Schematically, we write the different contributions as

$$S_{\zeta_n} = S_4(\zeta_n) + S_{\gamma^2}(\zeta_n) + S_{\gamma\zeta^2}(\zeta_n) + S_3(F_2(\zeta_n)) + S_2(F_2(\zeta_n)). \tag{42}$$

The first three terms come from Eqs. (35-37) when we substitute ζ with ζ_n and γ_{ij} with Eq. (41). Due to the non-linear relation between ζ and ζ_n, the third order action for ζ, Eq. (34), will after the change of variables give a contribution to the fourth order action like $S_3(F_2(\zeta_n))$. Similarly, the second order action, Eq. (33), will also contribute with $S_2(F_2(\zeta_n))$. In principle, one would also need to compute the third order gauge transformation as the second order action gives origin to fourth order terms like $\dot{\zeta}_n\dot{F}_3(\zeta_n)$ (where $F_3(\zeta_n)$ is the third order piece of the gauge transformation). Fortunately, terms involving F_3 can be shown to be proportional to the first order equations of motion for ζ_n, therefore when computing the trispectrum these terms will vanish and we don't need to calculate the third order gauge transformation explicitly at this point. It can be easily seen that equations (34), (35), (36), (37) or their counterparts in terms of ζ_n, Eq. (42), have terms that are not slow roll suppressed. However, because in pure de Sitter space ζ is a gauge mode we expect the action (42) to be slow-roll suppressed (of order ϵ). One can perform many integrations by parts to show that the unsuppressed terms of (42) can be reduced to total derivative terms and slow-roll suppressed terms given by

$$\begin{aligned}S_{parts} = \int dt\, d^3x \Big\{ &-\frac{3\epsilon}{aH^2}\Big[\frac{1}{4}\left(\partial_j\zeta_n\partial^j\zeta_n\right)^2 + \frac{1}{8}\zeta_n^2\partial^2\left(\partial_j\zeta_n\partial^j\zeta_n\right) + \partial_j\zeta_n\partial^j\zeta_n\partial^{-2}\partial^l\partial^k\left(\partial_l\zeta_n\partial_k\zeta_n\right)\\ &\qquad +2\partial_i\left(\partial^k\zeta_n\partial^i\zeta_n\right)\partial^{-2}\partial_l\left(\partial_k\zeta_n\partial^l\zeta_n\right) + \frac{1}{2}\left(\partial^{-2}\partial_j\partial_k\left(\partial^j\zeta_n\partial^k\zeta_n\right)\right)^2\Big]\\ &-\frac{5\epsilon}{8a^3H^4}\Big[\partial_l\zeta_n\partial^l\zeta_n\partial^k\partial^j\left(\partial_k\zeta_n\partial_j\zeta_n\right) - \frac{1}{2}\partial_j\zeta_n\partial^j\zeta_n\partial^2\left(\partial_k\zeta_n\partial^k\zeta_n\right)\\ &\qquad -\frac{1}{2}\partial^k\partial^j\left(\partial_k\zeta_n\partial_j\zeta_n\right)\partial^{-2}\partial^m\partial^l\left(\partial_m\zeta_n\partial_l\zeta_n\right)\Big]\Big\}.\end{aligned} \tag{43}$$

For us to be able to obtain the previous result it is crucial to include the contributions from the tensor actions (36) and (37), otherwise the trispectrum calculated using the $\mathcal{O}(\epsilon^0)$ of (35) does not vanish, giving the wrong leading order result. Neglecting tensor perturbations (sourced by the scalars) for the calculation of the trispectrum is not consistent and leads to wrong results. This is one of the results of our work. The contribution from $\tilde{\gamma}_{ij}$ that comes through γ_{ij} in (36) and (37) will result in terms that are already slow-roll suppressed and no further integrations by parts are need on these terms (see Eq. (59) of next subsection). The final action for ζ_n is then

$$S_{4\zeta_n} = S^{(\epsilon)}_{\zeta_n} + S_{parts}, \tag{44}$$

where $S^{(\epsilon)}_{\zeta_n}$ denotes the terms of (42) that are suppressed by at least one slow-roll parameter. This final action is slow-roll suppressed as expected and no slow-roll approximation was made so it is also exact.

B. Non-linear perturbations in the uniform curvature gauge

In order to calculate the intrinsic four point correlation function of the field perturbation we need to compute the action of fourth order in the perturbations. In this subsection we will obtain the fourth order action in the uniform curvature gauge. In this gauge, the scalar degree of freedom is the inflaton field perturbation $\delta\phi(x^\mu)$. There are several works in the literature where the authors also calculate the trispectrum. In [11], Seery *et al.* calculate the trispectrum of a multi-field inflation model, however the result is only valid for fields with standard kinetic energies, i.e. $P(X_1,\ldots,X_n,\phi_1,\ldots,\phi_n) = X_1+\cdots+X_n-V$, where X_n is the kinetic energy of ϕ_n and V is the potential. In this paper we will generalize their result for an arbitrary function $P(X,\phi)$ but for single field only. Recently, Huang and Shiu have obtained the fourth order action for the model under consideration (1). However the result was obtained by only perturbing the field lagrangian. This procedure gives the right result as long as we are interested in the leading order contribution in the small speed of sound limit and in the slow-roll approximation, which was their case of interest. In the present section we will compute the fourth order action that is valid to all orders in slow roll and in the sound of speed expansion. To do that we have to perturb the full action (1) up to fourth order in the field perturbations. The procedure to obtain the fourth order action in this gauge is similar to the one used in subsection III A.

In this gauge, the inflaton perturbation does not vanish and the 3D metric takes the form

$$\begin{aligned}
&\phi(x,t) = \phi_0 + \delta\phi(x,t), \\
&h_{ij} = a^2\hat{h}_{ij}, \quad \hat{h}_{ij} = \delta_{ij} + \tilde{\gamma}_{ij} + \frac{1}{2}\tilde{\gamma}_{ik}\tilde{\gamma}^k_j + \cdots
\end{aligned} \tag{45}$$

where $\det\hat{h} = 1$ and $\tilde{\gamma}_{ij}$ is a tensor perturbation that we assume to be a second order quantity, i.e., $\tilde{\gamma}_{ij} = \mathcal{O}(\delta\phi^2)$. It obeys the traceless and transverse conditions $\tilde{\gamma}^i_i = \partial^i\tilde{\gamma}_{ij} = 0$ (indices are raised with δ_{ij}). In the literature the second order tensor perturbations are often neglected, however based on our results it should be taken into account. We can always use the gauge freedom at second order to eliminate the trace and the vector perturbations of h_{ij}. The presence of $\tilde{\gamma}_{ij}$ makes the three dimensional hypersurfaces non-flat so using the name uniform curvature gauge might be misleading. We will continue to use that name because in the literature that is the name given to the gauge where $\delta\phi(x,t) = 0$.

We expand N and N^i in powers of the perturbation $\delta\phi(x,t)$

$$N = 1 + \alpha_1 + \alpha_2 + \cdots, \tag{46}$$

$$\tilde{N}_i = \tilde{N}_i^{(1)} + \tilde{N}_i^{(2)} + \cdots, \tag{47}$$

$$\psi = \psi_1 + \psi_2 + \cdots, \tag{48}$$

where α_n, $\tilde{N}_i^{(n)}$ and ψ_n are of order $\delta\phi^n$ and $\phi_0(t)$ is the background value of the field. At first order in $\delta\phi$, a particular solution for equations (18) is [7, 11]:

$$\alpha_1 = \frac{1}{2H}\dot{\phi}_0\delta\phi P_{,X}, \quad \tilde{N}_i^{(1)} = 0, \quad \partial^2\psi_1 = \frac{a^2\epsilon}{c_s^2}\frac{d}{dt}\left(-\frac{H}{\dot{\phi}}\delta\phi\right). \tag{49}$$

At second order, the constraint equation for the lapse function gives

$$\begin{aligned}
\frac{4H}{a^2}\partial^2\psi_2 \;=\; & \frac{1}{a^4}\left(\partial^2\psi_1\partial^2\psi_1 - \partial_i\partial_j\psi_1\partial^i\partial^j\psi_1\right) + 6H^2\left(3\alpha_1^2 - 2\alpha_2\right) \\
& + \frac{8H\alpha_1}{a^2}\partial^2\psi_1 - \left(3\dot{\phi_0}^2\alpha_1^2 - 2\dot{\phi_0}^2\alpha_2 + \dot{\delta\phi}^2 - 4\dot{\phi}_0\dot{\delta\phi}\alpha_1\right)P_{,X} - \frac{1}{a^2}\left(\partial_i\delta\phi\partial^i\delta\phi - 2\dot{\phi}_0\partial_i\delta\phi\partial^i\psi_1\right)P_{,X} \\
& + \delta\phi^2 P_{,\phi\phi} + 2\dot{\phi}_0\delta\phi\left(\dot{\phi}_0\alpha_1 - \dot{\delta\phi}\right)P_{,X\phi} \\
& - \frac{\dot{\phi_0}^2}{a^2}\left(-10a^2\dot{\phi}_0\dot{\delta\phi}\alpha_1 - 2\dot{\phi}_0\partial_i\delta\phi\partial^i\psi_1 + 6a^2\dot{\phi_0}^2\alpha_1^2 - 2\dot{\phi_0}^2a^2\alpha_2 + 4a^2\dot{\delta\phi}^2 - \partial_i\delta\phi\partial^i\delta\phi\right)P_{,XX} \\
& - \dot{\phi_0}^4\left(\dot{\phi_0}^2\alpha_1^2 + \dot{\delta\phi}^2 - 2\dot{\phi}_0\dot{\delta\phi}\alpha_1\right)P_{,XXX} + 2\dot{\phi_0}^3\delta\phi\left(\dot{\phi}_0\alpha_1 - \dot{\delta\phi}\right)P_{,XX\phi} - \dot{\phi_0}^2\delta\phi^2 P_{,X\phi\phi},
\end{aligned} \tag{50}$$

and the equation for the shift vector gives

$$
\begin{aligned}
2H\partial_i\alpha_2 - \frac{1}{2}a^{-2}\partial^2\tilde{N}_i^{(2)} \;=\;& 4\alpha_1 H\partial_i\alpha_1 + a^{-2}\left(\partial_i\alpha_1\partial^2\psi_1 - \partial_k\alpha_1\partial_i\partial^k\psi_1\right) + \partial_i\delta\phi\left(\dot{\delta\phi} - \dot{\phi}_0\alpha_1\right)P_{,X} \\
&+\partial_i\delta\phi\dot{\phi_0}^2\left(\dot{\delta\phi} - \dot{\phi}_0\alpha_1\right)P_{,XX} + \partial_i\delta\phi\dot{\phi}_0\delta\phi P_{,X\phi},
\end{aligned}
\tag{51}
$$

where $P_{,\phi}$ means derivative of P with respect to ϕ.

Due to the fact that $\tilde{N}^i$ is divergenceless and that any vector can be separated into a incompressible and irrotational part one can separate the contributions from α_2 and $\tilde{N}_i^{(2)}$ in the previous equation. The irrotational part of Eq. (51) gives

$$
2H\alpha_2 = \partial^{-2}\partial^i F_i, \tag{52}
$$

and the incompressible part gives

$$
\frac{1}{2a^2}\tilde{N}_i^{(2)} = -\partial^{-2}F_i + \partial^{-4}\partial_i\partial^k F_k, \tag{53}
$$

where F_i is defined as the right-hand side of equation (51).

The scalar fourth order action, where no slow-roll approximation has been made, is

$$
S_4 = S_A + S_B, \tag{54}
$$

where

$$
\begin{aligned}
S_A \;=\;& \int dtd^3x\Big[-\frac{a\delta\phi}{2}\big[a^2\alpha_1\dot{\delta\phi}^2 + 2\partial_i\delta\phi\partial^i\psi_1\left(\dot{\delta\phi} - \alpha_1\dot{\phi}_0\right) + 2\dot{\phi}_0\partial^i\delta\phi\left(\tilde{N}_i^{(2)} + \partial_i\psi_2\right) + a^2\dot{\phi}_0^2\alpha_1^3 + \alpha_1(\partial\delta\phi)^2\big]P_{,X\phi} \\
&-\frac{1}{8a}\big[4\dot{\phi}_0^4a^4\alpha_2\left(\alpha_2 - 2\alpha_1^2\right) + 3\dot{\phi}_0^4a^4\alpha_1^4 - 18\dot{\phi}_0^2a^4\alpha_1^2\dot{\delta\phi}^2 + 8\dot{\phi}_0^2a^2\partial^i\delta\phi\left(\tilde{N}_i^{(2)} + \partial_i\psi_2\right)\left(\dot{\delta\phi} - \dot{\phi}_0\alpha_1\right) \\
&-4\dot{\phi}_0\partial_i\delta\phi\partial^i\psi_1\left((\partial\delta\phi)^2 - 3\dot{\phi}_0^2a^2\alpha_1^2 + 8\dot{\phi}_0a^2\alpha_1\dot{\delta\phi} - 3a^2\dot{\delta\phi}^2\right) + 12\dot{\phi}_0a^4\alpha_1\dot{\delta\phi}^3 + 4\dot{\phi}_0^3a^4\alpha_1^3\dot{\delta\phi} \\
&-4\,\dot{\phi}_0^2\partial_i\delta\phi\partial^i\psi_1\partial_k\delta\phi\partial^k\psi_1 - 2\dot{\phi}_0^2a^2\alpha_1^2(\partial\delta\phi)^2 - 4\alpha_1\dot{\phi}_0\dot{\delta\phi}a^2(\partial\delta\phi)^2 - a^4\left(\dot{\delta\phi}^2 - a^{-2}(\partial\delta\phi)^2\right)^2\big]P_{,XX} \\
&+\frac{\dot{\phi}_0^2a}{12}\big[6\dot{\phi}_0\partial_i\delta\phi\partial^i\psi_1\left(2\dot{\phi}_0\alpha_1\dot{\delta\phi} - \dot{\delta\phi}^2 - \dot{\phi}_0^2\alpha_1^2\right) + 3(\partial\delta\phi)^2\left(2\dot{\phi}_0\alpha_1\dot{\delta\phi} - \dot{\phi}_0^2\alpha_1^2\right) - 16\dot{\phi}_0a^2\alpha_1\dot{\delta\phi}^3 \\
&+24\dot{\phi}_0^2a^2\alpha_1^2\dot{\delta\phi}^2 - 12\dot{\phi}_0^3a^2\alpha_1^3\dot{\delta\phi} + \dot{\phi}_0^4a^2\alpha_1^4 + 3a^2\dot{\delta\phi}^2\left(\dot{\delta\phi}^2 - a^{-2}(\partial\delta\phi)^2\right)\big]P_{,XXX} \\
&+\frac{\dot{\phi}_0a\delta\phi}{2}\big[a^2\dot{\delta\phi}^3 + \left((\partial\delta\phi)^2 + 2\dot{\phi}_0\partial_i\delta\phi\partial^i\psi_1\right)\left(\dot{\phi}_0\alpha_1 - \dot{\delta\phi}\right) + 3\dot{\phi}_0^2a^2\alpha_1^2\dot{\delta\phi} - 4\dot{\phi}_0a^2\alpha_1\dot{\delta\phi}^2\big]P_{,XX\phi} \\
&-\frac{a\delta\phi^2}{8}\big[2(\partial\delta\phi)^2 - 2a^2\dot{\delta\phi}^2 + 4a^2\alpha_1\dot{\phi}_0\dot{\delta\phi} + 2a^2\dot{\phi}_0^2\alpha_1^2 + 4\dot{\phi}_0\partial_i\delta\phi\partial^i\psi_1\big]P_{,X\phi\phi} \\
&+\frac{1}{24}\dot{\phi}_0^4a^3\big[6\dot{\phi}_0^2\alpha_1^2\dot{\delta\phi}^2 - 4\dot{\phi}_0\alpha_1\dot{\delta\phi}^3 - 4\dot{\phi}_0^3\alpha_1^3\dot{\delta\phi} + \dot{\phi}_0^4\alpha_1^4 + \dot{\delta\phi}^4\big]P_{,XXXX} \\
&-\frac{1}{6}\dot{\phi}_0^3\delta\phi a^3\left(-\dot{\delta\phi}^3 + 3\dot{\phi}_0\alpha_1\dot{\delta\phi}^2 + \dot{\phi}_0^3\alpha_1^3 - 3\dot{\phi}_0^2\alpha_1^2\dot{\delta\phi}\right)P_{,XXX\phi} \\
&+\frac{1}{4}\dot{\phi}_0^2\delta\phi^2a^3\left(-2\alpha_1\dot{\phi}_0\dot{\delta\phi} + \dot{\delta\phi}^2 + \dot{\phi}_0^2\alpha_1^2\right)P_{,XX\phi\phi} - \frac{1}{6}\dot{\phi}_0\delta\phi^3a^3\left(\alpha_1\dot{\phi}_0 - \dot{\delta\phi}\right)P_{,X\phi\phi\phi}\Big],
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
S_B \;=\;& \int dtd^3x\Big[\alpha_1^3a^3\delta\phi P_{,\phi} + \frac{1}{2}a^3\alpha_1^2\delta\phi^2P_{,\phi\phi} + \frac{1}{6}a^3\alpha_1\delta\phi^3P_{,\phi\phi\phi} + \frac{1}{24}a^3\delta\phi^4P_{,4\phi} \\
&-\frac{1}{2a}\big[-(\partial_i\delta\phi\partial^i\psi_1)^2 - 2\alpha_1\dot{\delta\phi}a^2\partial_i\delta\phi\partial^i\psi_1 + 2a^2\partial_i\delta\phi\left(\tilde{N}_i^{(2)} + \partial_i\psi_2\right)\left(\dot{\delta\phi} - \alpha_1\dot{\phi}_0\right) \\
&+\alpha_1^2a^2(\partial\delta\phi)^2 + \dot{\phi}_0^2a^4\alpha_2\left(\alpha_2 - 2\alpha_1^2\right)\big]P_{,X} \\
&+\frac{1}{4a}\big[2\partial^i\tilde{N^j}^{(2)}\partial_{(i}\tilde{N_{j)}}^{(2)} - 4\alpha_1\partial_i\partial_k\psi_1\left(\partial^i\partial^k\psi_2 + \partial^i\tilde{N^k}^{(2)}\right) + 12a^4H^2\alpha_2\left(\alpha_2 - 2\alpha_1^2\right) + 4\alpha_1\partial^2\psi_1\partial^2\psi_2\big]\Big].
\end{aligned}
\tag{56}
$$

The previous actions should be supplemented with the pure tensor terms and the tensor-scalar coupling terms:

$$
S = \frac{1}{8}\int dtd^3x\left[a^3\dot{\tilde{\gamma}}_{ij}\dot{\tilde{\gamma}}^{ij} - a\partial_k\tilde{\gamma}_{ij}\partial^k\tilde{\gamma}^{ij}\right], \tag{57}
$$

$$S=\int dtd^3x\left[aP_{,X}\tilde{\gamma}^{ij}\partial_j\delta\phi\left(\frac{1}{2}\partial_i\delta\phi+\dot{\phi}_0\partial_i\psi_1\right)\right]. \tag{58}$$

This constitutes the main result of this subsection. It is a good check for our calculation to see that the previous action (54) reduces in some particular cases to previously know results present in the literature.

For example, if we restrict our model to the standard inflation case, i.e., $P(X,\phi)=X-V(\phi)$, where $V(\phi)$ is the inflaton potential, then all the terms in the scalar action (55) vanish and the only contribution to the fourth order action comes from (56). These terms exactly reproduce the result of Seery *et al.* [11], their equation (36), restricted to single field. However, in the total fourth order action there are also the tensor contributions (57) and (58). In general, to proceed one has to calculate the equation of motion for the second order tensor perturbations $\tilde{\gamma}_{ij}$ from eqs. (57), (58) to get

$$\tilde{\gamma}_{ij}''+2\frac{a'}{a}\tilde{\gamma}_{ij}'-\partial^2\tilde{\gamma}_{ij}=\Big(2P_{,X}\partial_j\delta\phi\partial_i\delta\phi+4P_{,X}\dot{\phi}_0\partial_j\delta\phi\partial_i\psi_1\Big)^{TT}, \tag{59}$$

where TT means the transverse and traceless parts of the expression inside the parenthesis (see Appendix B for details of how to extract the TT parts of a tensor) and then solve this equation to obtain $\tilde{\gamma}_{ij}$ as a function of $\delta\phi$. One can immediately see that at second order the scalars will source the tensor perturbation equation as it was previously shown by others [24, 25]. At this order in perturbation theory, equation (59) should also have a source term quadratic in the first order tensor perturbations, ${}^{(1)}\tilde{\gamma}_{ij\,GW}$. We neglect these terms because we expect that any correlation function where ${}^{(2)}\tilde{\gamma}_{ij\,GW}$ enters, which is sourced by the first order tensor modes squared, must be smaller than a correlation function with only scalars, see Ref. [7] for an example. In Fourier space, the source term of (59) is suppressed by k^2, where k is the wave number. Once we have the solution of $\tilde{\gamma}_{ij}$ in terms of $\delta\phi$ we can plug back the result in (57) and (58) to get the total fourth order scalar action.

IV. THE GENERAL FORMALISM TO CALCULATE THE TRISPECTRUM

A. The trispectrum of ζ_n

Now we shall give the basic equations needed to calculate the trispectrum [7, 26]. First we need to solve the second order equation of motion for ζ_n (obtained from (33)). Defining new variables

$$v_k=zu_k,\quad z=\frac{a\sqrt{2\epsilon}}{c_s}, \tag{60}$$

where the Fourier mode function u_k is given by

$$u_k=\int d^3x\zeta_n(t,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{61}$$

the equation of motion for ζ_n is

$$v_k''+c_s^2k^2v_k-\frac{z''}{z}v_k=0, \tag{62}$$

where prime denotes derivative with respect to conformal time τ. This is also known as the Mukhanov equation. The previous equation can be solved, at leading order in slow roll and if the rate of change of the sound speed is small [10], to give

$$u_k\equiv u(\tau,\mathbf{k})=\frac{iH}{\sqrt{4\epsilon c_s k^3}}\left(1+ikc_s\tau\right)e^{-ikc_s\tau}. \tag{63}$$

We do not need to impose any constraints in the sound speed and it can be arbitrary. Only its rate of change is assumed to be small. The next-to-leading order corrections to the previous solutions are also known and can be found in [10]. In the general case, we would have to solve Eq. (62) without assuming slow roll. This can be done numerically.

In order to calculate the ζ_n correlators we follow the standard procedure in quantum field theory. The curvature perturbation is promoted to an operator that can be expanded in terms of creation and annihilation operator as

$$\zeta_n(\tau,\mathbf{k})=u(\tau,\mathbf{k})a(\mathbf{k})+u^*(\tau,-\mathbf{k})a^\dagger(-\mathbf{k}). \tag{64}$$

The standard commutation relation applies

$$\left[a(\mathbf{k_1}), a^\dagger(\mathbf{k_2})\right] = (2\pi)^3\delta^{(3)}(\mathbf{k_1} - \mathbf{k_2}). \tag{65}$$

The vacuum expectation value of the four point operator in the interaction picture (at first order) is [7, 26]

$$\langle\Omega|\zeta_n(t,\mathbf{k_1})\zeta_n(t,\mathbf{k_2})\zeta_n(t,\mathbf{k_3})\zeta_n(t,\mathbf{k_4})|\Omega\rangle = -i\int_{t_0}^{t} d\tilde{t}\langle 0|\left[\zeta_n(t,\mathbf{k_1})\zeta_n(t,\mathbf{k_2})\zeta_n(t,\mathbf{k_3})\zeta_n(t,\mathbf{k_4}), H_I(\tilde{t})\right]|0\rangle, \tag{66}$$

where t_0 is some early time during inflation when the inflaton vacuum fluctuation is deep inside the horizon, t is some time after horizon exit. $|\Omega\rangle$ is the interacting vacuum which is different from the free theory vacuum $|0\rangle$. If one uses conformal time, it's a good approximation to perform the integration from $-\infty$ to 0 because $\tau \approx -(aH)^{-1}$. H_I denotes the interaction hamiltonian and it is given by $H_I = \pi\dot{\zeta}_n - L$, where π is defined as $\pi = \frac{\partial L}{\partial\dot{\zeta}_n}$ and L is the lagrangian. In this work, we will only calculate the contribution for the four point function that comes from a part of the interaction Hamiltonian determined by the 4th order Lagrangian $H_I = -L_4$, where L_4 is the total lagrangian obtained from the action (44). We should point out that the other terms that we do not consider here in the fourth order interaction hamiltonian are indeed important to obtain the full leading order result (see equation (75)) as was recently shown by [13].

Of course in the end we are interested in the four point correlation function of ζ and not of ζ_n. At leading order in slow roll these two correlation functions are equal but they will differ at next-to-leading order.

B. The trispectrum of ζ

In this subsection we calculate the relation between the trispectrum of ζ, on large scales, and the trispectrum of ζ_n calculated using the formalism of the previous subsection. This relation also involves lower-order correlation functions of ζ_n present in the literature. The variables ζ and ζ_n are related up to third order by

$$\zeta(\mathbf{x}) = \zeta_n(\mathbf{x}) + F_2(\zeta_n(\mathbf{x})) + F_3(\zeta_n(\mathbf{x})), \tag{67}$$

where $F_2(\zeta_n) = \mathcal{O}(\zeta_n^2)$, $F_3(\zeta_n) = \mathcal{O}(\zeta_n^3)$ are the terms coming from the second and third order gauge transformations respectively. F_2 can be found in Appendix A, it is

$$F_2(\zeta_n) = \left(\frac{\epsilon}{2} + \frac{\ddot{\phi}_0}{2H\dot{\phi}_0}\right)\zeta_n^2 + \frac{1}{H}\zeta_n\dot{\zeta}_n + \beta, \tag{68}$$

where β is given in Eq. (A6). In the large scale limit (super-horizon scales), we can ignore β as it contains gradient terms. F_3 was calculated in [15] and reads

$$F_3(\zeta_n) = \left(\frac{\dddot{\phi}_0}{3H^2\dot{\phi}_0} + \frac{\epsilon\ddot{\phi}_0}{H\dot{\phi}_0} + \frac{\epsilon^2}{3} + \frac{\epsilon\eta}{3}\right)\zeta_n^3 + \left(\frac{3\ddot{\phi}_0}{2H\dot{\phi}_0} + 2\epsilon\right)\frac{\dot{\zeta}_n\zeta_n^2}{H} + \frac{\zeta_n\dot{\zeta}_n^2}{H^2} + \frac{\ddot{\zeta}_n\zeta_n^2}{2H^2} + f_a(\zeta_n) + f_b(\zeta_n, \tilde{\gamma}_{ij}), \tag{69}$$

where f_a denotes terms that contain gradients (it can be found in [15]). f_b is the part of the third order gauge transformations that contains $\tilde{\gamma}_{ij}$. The explicit form of f_b is to the best of our knowledge still unknown. To find out the explicit dependence of these terms on ζ_n one would have to solve the equations of motion for $\tilde{\gamma}_{ij}$, equation (59). We do not do this in this work. We believe that these terms will vanish in the large scale limit and therefore do not contribute to our calculation.

A field redefinition like $\zeta = \zeta_n + a_1\zeta_n^{(a)}\zeta_n^{(b)} + a_2\zeta_n^{(c)}\zeta_n^{(d)}$, where $\zeta_n^{(a,b,c,d)}$ denote one of $\zeta_n, \dot{\zeta}_n, \ddot{\zeta}_n$, gives after using Wick's theorem a relation between both trispectrum like

$$\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\zeta(\mathbf{x}_3)\zeta(\mathbf{x}_4)\rangle_c = \{T\} + \{PB\} + \{PPP\} + \mathcal{O}(P_k^\zeta)^4, \tag{70}$$

where

$$\{T\} = \langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\zeta_n(\mathbf{x}_4)\rangle, \tag{71}$$

$$\begin{aligned}
\{PB\} \;=\; & a_1\Big[\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle \\
& +\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle + \langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\zeta_n(\mathbf{x}_4)\rangle \\
& +\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle \\
& +3\;perm.\Big] \\
+\; & a_2\Big[(a\to c, b\to d) + 3\;perm.\Big], \qquad (72)
\end{aligned}$$

$$\begin{aligned}
\{PPP\} \;=\; & a_1^2\Big[\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n^{(a)}(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle\Big) \\
& +\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n^{(b)}(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle\Big) \\
& +\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n^{(a)}(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle\Big) \\
& +\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n^{(b)}(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle\Big) \\
& +5\;perm.\Big] \\
+\; & a_2^2\Big[(a\to c, b\to d) + 5\;perm.\Big] \\
+\; & 2a_1a_2\Big[\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n^{(c)}(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(d)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(d)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle\Big) \\
& +\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n^{(d)}(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(c)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(c)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle\Big) \\
& +\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n^{(c)}(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(d)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(d)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle\Big) \\
& +\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n^{(d)}(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(c)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n^{(c)}(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle\Big) \\
& +5\;perm.\Big], \qquad (73)
\end{aligned}$$

where "perm" means the other permutations of the preceding terms and $\mathcal{O}(P_k^\zeta)^4$ denotes terms that are suppressed by successive powers of the power spectrum. $(a\to c, b\to d)$ means terms equal to the immediately preceding terms with a, b replaced by c, d respectively.

If the field redefinition contains third order pieces like $\zeta = \zeta_n + b_1\zeta_n^{(a)}\zeta_n^{(b)}\zeta_n^{(c)}$, they contribute with additional terms as

$$\begin{aligned}
\langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\zeta(\mathbf{x}_3)\zeta(\mathbf{x}_4)\rangle_c \;=\; & \langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\zeta_n(\mathbf{x}_4)\rangle \\
+\; & b_1\Big[\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(c)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(c)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\Big) \\
& +\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(c)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(c)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\Big) \\
& +\langle\zeta_n^{(c)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\Big(\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n^{(b)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n^{(a)}(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\Big) \\
& +3\;perm.\Big] + \mathcal{O}(P_k^\zeta)^4. \qquad (74)
\end{aligned}$$

To the best of our knowledge the expectation values involving operators containing derivatives of ζ_n have not yet been calculated in the literature. However, once the mode function equation (62) is solved, one has all the ingredients needed to calculate these expectation values, including the interaction hamiltonian.

V. CALCULATION OF THE LEADING ORDER TRISPECTRUM

In this section, we will use the formalism of the previous section and the fourth order exact interaction hamiltonian of subsection III A to calculate the leading order trispectrum, under the assumption that the "slow-roll" parameters (7-9) are always small until the end of inflation.

A. The leading order trispectrum of ζ_n

To calculate the leading order trispectrum of ζ_n in slow roll, we need to evaluate Eq. (66) where H_I is read from the order ϵ terms of the action (44). The interaction hamiltonian we get contains terms with $\tilde{\gamma}_{ij}$. Fortunately it can be shown that to compute the leading order trispectrum we don't need to know the solution for $\tilde{\gamma}_{ij}$ and the knowledge of its equation of motion (Eq. (59)) is sufficient. At this order we use the solution for the mode functions Eq. (63). The integrals in Eq. (66) can then be performed analytically to give

$$\begin{aligned}\langle\Omega|\zeta_n(\mathbf{k}_1)\zeta_n(\mathbf{k}_2)\zeta_n(\mathbf{k}_3)\zeta_n(\mathbf{k}_4)|\Omega\rangle &= (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{H^6}{\epsilon^3 c_s^3}\frac{1}{\Pi_i k_i^3}\\ &\left[\frac{3}{4}\left(10\Pi+3\lambda\right)\frac{c_s^2}{H^2\epsilon}A_1-\frac{1}{2^6}\left(3\lambda-\frac{H^2\epsilon}{c_s^2}+H^2\epsilon\right)\frac{1}{H^2\epsilon}A_2-\frac{1}{2^8}\frac{c_s^2-1}{c_s^4}A_3\right],\end{aligned} \tag{75}$$

where the momentum dependent functions A_i are defined as

$$\begin{aligned}A_1 &= \frac{\Pi_i k_i^2}{K^5},\\ A_2 &= \frac{k_1^2k_2^2(\mathbf{k}_3\cdot\mathbf{k}_4)}{K^3}\left(1+\frac{3(k_3+k_4)}{K}+\frac{12k_3k_4}{K^2}\right)+\text{perm.},\\ A_3 &= \frac{(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_3\cdot\mathbf{k}_4)}{K}\left[1+\frac{\sum_{i<j}k_ik_j}{K^2}+\frac{3k_1k_2k_3k_4}{K^3}\left(\sum_i\frac{1}{k_i}\right)+12\frac{k_1k_2k_3k_4}{K^4}\right]+\text{perm.},\end{aligned} \tag{76}$$

and "perm." refers to the 24 permutations of the four momenta. Note that the quantities of Eq. (75) are evaluated at the moment τ_* at which the total wave number $K=\sum_{i=1}^4 k_i$ exits the horizon, i.e., when $Kc_{s*}=a_*H_*$. This leading order result that comes from the 4th order Lagrangian agrees with the result of Huang and Shiu [13] [1] that did their calculation in the uniform curvature gauge and using a simpler method that is only valid to calculate the leading order contribution for models with $c_s \ll 1$. In fact, we can compare our uniform curvature gauge result (54) with the result of Huang and Shiu [13]. We see that the last terms of the fourth, sixth and ninth lines of equation (55) are exactly the ones obtained by [13], their equation (15), using the method of just expanding the field lagrangian as in [14, 27]. For a model with a general field lagrangian these terms are the ones that give the leading order contribution for the trispectrum, in the small sound speed limit, equation (75). The contribution coming from the tensor part will be of next-to-leading order in this case.

For standard kinetic term inflation, $\Pi=\lambda=0$ and $c_s=1$ and Eq. (75) vanishes, the leading order is then given by the next order in slow roll.

B. The leading order trispectrum of ζ

It is well known that if the slow-roll conditions are satisfied until the end of inflation and we can ignore gradient terms then the gauge invariant curvature perturbation ζ remains constant on super-horizon scales to all order in perturbation theory. In this subsection, we will see that this fact greatly simplifies the relation between the trispectrum of ζ and ζ_n.

In the large scales limit, Eq. (67) simplifies to give

$$\zeta=\zeta_n+a\zeta_n^2+\frac{1}{H}\zeta_n\dot{\zeta}_n+\mathcal{O}(\zeta_n^3), \tag{77}$$

where a is defined as

$$a=\frac{\epsilon}{2}+\frac{\ddot{\phi}_0}{2H\dot{\phi}_0}. \tag{78}$$

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[1] The full leading order result for the four point function can be found in the revised version of [13] which takes into account all the contributions for the fourth order interaction hamiltonian.

Using the fact that $\dot{\zeta} = 0$ on super-horizon scales and the equation resulting from a time derivative of Eq. (77) one can show that

$$\dot{\zeta}_n = -\dot{a}\zeta_n^2 + \mathcal{O}(\zeta_n^3). \tag{79}$$

This equation has a simple interpretation. The variable ζ_n is not constant outside the horizon, only the gauge invariant quantity ζ is. This is the reason why the term $\frac{1}{H}\zeta_n\dot{\zeta}_n$ in the second order gauge transformation cannot be ignored when one is calculating the trispectrum of ζ. Substituting Eq. (79) in Eq. (67) and taking the large scale limit we get

$$\zeta = \zeta_n + a\zeta_n^2 + b\zeta_n^3 + \cdots \tag{80}$$

where $\cdots$ means cubic terms that contain at least one time derivative of ζ_n and that will only give a contribution to the five point function. The variable b is defined as

$$\begin{aligned} b &= \frac{\dddot{\phi}_0}{3H^2\dot{\phi}_0} + \frac{\epsilon\ddot{\phi}_0}{H\dot{\phi}_0} + \frac{\epsilon^2}{3} + \frac{\epsilon\eta}{3} - \frac{\dot{a}}{H} \\ &= -\frac{\dddot{\phi}_0}{6H^2\dot{\phi}_0} + \frac{\epsilon\ddot{\phi}_0^2}{2H\dot{\phi}_0} + \frac{\ddot{\phi}_0^2}{2H^2\dot{\phi}_0^2} + \frac{\epsilon^2}{3} - \frac{\eta\epsilon}{6}. \end{aligned} \tag{81}$$

We shall now compare (80) with the result given by the δN formalism [28, 29, 30, 31]. In the δN approach ζ is expanded in series in terms of the field perturbation as

$$\zeta = N'\delta\phi + \frac{1}{2}N''\delta\phi^2 + \frac{1}{6}N'''\delta\phi^3 + \mathcal{O}(\delta\phi)^4, \tag{82}$$

where N is the number of e-folds and a prime denotes derivative with respect to ϕ. Now comparing Eq. (80) with the previous equation and observing that $\zeta_n = -\frac{H}{\dot{\phi}_0}\delta\phi$ we expect

$$\frac{N''}{2} = \frac{H^2}{\dot{\phi}_0^2}a, \quad \frac{N'''}{6} = -\frac{H^3}{\dot{\phi}_0^3}b. \tag{83}$$

We verified that this is indeed the case.

Using Wick's theorem one can now relate the connected part of the four point correlation function of ζ with the four point correlation function of ζ_n calculated in the previous section [32, 33]. This relation also involves lower order correlation functions of ζ_n, like the bispectrum $\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle$ (the leading and next-to-leading order in slow roll bispectrum was previously calculated in [10]). The relation is

$$\begin{aligned} \langle\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\zeta(\mathbf{x}_3)\zeta(\mathbf{x}_4)\rangle_c &= \langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\zeta_n(\mathbf{x}_4)\rangle \\ &+ 2a\Big[\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\zeta_n(\mathbf{x}_4)\rangle + \langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle \\ &\quad +\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle + 3\,perm\Big] \\ &+ 4a^2\Big[\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_4)\rangle \\ &\quad +\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle\langle\zeta_n(\mathbf{x}_2)\zeta_n(\mathbf{x}_3)\rangle + 5\,perm\Big] \\ &+ 6b\Big[\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_2)\rangle\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_3)\rangle\langle\zeta_n(\mathbf{x}_1)\zeta_n(\mathbf{x}_4)\rangle + 3\,perm\Big] + \mathcal{O}(P_k^\zeta)^4, \end{aligned} \tag{84}$$

where "perm" means the other permutations of the preceding terms and $\mathcal{O}(P_k^\zeta)^4$ denotes terms that are suppressed by successive powers of the power spectrum. Now, one can easily see that at leading order in slow roll the trispectrum for ζ and ζ_n are equal, this is because the constants a and b are slow-roll suppressed. These terms will only contribute to the next-to-leading order corrections.

C. The next-to-leading order corrections for the trispectrum

In subsection V A, we showed that for standard kinetic term inflation the leading order result, equation (75), vanishes and in fact in this case the leading order of the trispectrum of ζ_n is of order ϵ^{-2} (the next-to-leading order is the leading order). To obtain these leading order contributions it is easier to perform the calculation using the uniform curvature gauge action Eq. (55)-(58). Eq. (55) vanishes exactly for standard kinetic term inflation. The action (56) is exact in the slow-roll approximation but it is instructive to determine the slow-roll order of the different terms. One can see that the leading order contribution comes from terms of order $\mathcal{O}(\epsilon^0)$, as pointed out in [11]. If we take $P(X,\phi) = X - V(\phi)$ then the leading order (in slow roll) source of Eq. (59) will be of order $\mathcal{O}(\epsilon^0)$. We therefore don't expect $\tilde{\gamma}_{ij}$ to be slow-roll suppressed and the actions (57), (58) will contain unsuppressed terms of the same order as the leading order term of the action (56). These tensor contributions were absent in the analysis of [11] and we have shown that they are of the same order as the fourth order action considered in [11], our Eq. (56). It is still an open question how these new contributions will change the trispectrum result of Seery *et al.*.

For the general lagrangian case, the leading order trispectrum was given in the previous subsection and in [13]. Contrary to the method of [13], our method of obtaining the fourth order action (54) does not rely on any approximation and therefore the action (54) is valid to all orders in slow roll and in the sound speed expansion and it can be used to study the next-to-leading order corrections. Depending on the momentum shape of these next-to-leading terms they might become big enough to be observed in the next generation of experiments. A similar argument applies for the next-to-leading order corrections for the bispectrum, as it was shown in [10]. For example, for DBI inflation, [13] showed that the leading order non-gaussianity parameters τ_{NL} scales like $\tau_{NL} \sim 0.1/c_s^4$ (for a specific momentum configuration) and $f_{NL} \sim 1/c_s^2$. They argue that if $c_s \sim 0.1$ then f_{NL} is still inside the value range allowed by observations but $\tau_{NL} \sim 10^3$ could be detected with the Planck satellite CMBR experiment. Therefore, assuming that the slow-roll parameter ϵ is of order $\epsilon \sim 0.01$ (at horizon crossing) these next-to-leading order corrections for the trispectrum could possible be observed with the Planck satellite. A more careful and systematic study of the momentum dependence of these new terms is required and it is left for future work.

VI. CONCLUSION

We have computed the fourth order action for scalar and second order tensor perturbations in the comoving gauge. Our result is exact in the slow-roll (SR) expansion but practically it is useful to study the SR suppression of the different terms. We were able to show that after many integrations by parts the unsuppressed terms contained in the previous action can be reduced to total derivatives terms plus corrections that are SR suppressed. The resulting action has the correct order in SR. It is suppressed by ϵ as it should be, because in pure de Sitter space the curvature perturbation is a pure gauge mode. An important lesson from our work is that in order to obtain the correct SR order for the action, the second order tensor perturbations cannot be ignored as assumed in previous works [11] and [15]. We found the explicit form of these tensor perturbations in the comoving gauge by using the gauge transformations from the uniform curvature gauge. Fortunately, for a general inflation model like (1), we showed that we do not need to solve the equations of motion for the tensor perturbations if we are interested in calculating the leading order trispectrum. However, to calculate the next-to-leading order corrections to that result, or to calculate the leading order trispectrum for standard kinetic term inflation, we do need to solve explicitly the equations of motion for the tensor perturbations. This will be left for future work.

Using the comoving gauge action we have calculated the leading order in SR trispectrum of ζ. We compared our result with the result of [13], obtained using the uniform curvature gauge, and we found an agreement.

For the uniform curvature gauge action, that is also exact in the SR expansion, we identified the terms that will contribute to the next-to-leading order corrections to the trispectrum. We pointed out that depending on the model and on the momentum configuration, some of these corrections might be observable with the Planck satellite. After taking particular limits, the previous action nicely reduces to previously know results [11], [13] with the caveat that the above mentioned works ignore tensor contributions.

Finally we have obtained the relations between the trispectrum of ζ and $\delta\phi$ (on large scales) using the third order gauge transformations and compared the result with the δN formalism.

To conclude, we have provided the necessary equations (fourth order action and the relation between ζ and $\delta\phi$) to calculate the trispectrum for a fairly general model of inflation that are also valid for models where SR is temporarily interrupted, i.e., around a "step" in the inflaton's lagrangian [18]. In this case, it is impossible to apply the δN approach and it is required to evaluate the n-point functions numerically [16] (see [34] for a different approach). We leave this more practical application of our results for future work.

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APPENDIX A: GAUGE TRANSFORMATIONS UP TO SECOND ORDER

In this Appendix we will find the change of variables that one needs to perform to go from the uniform curvature gauge (45) to the comoving gauge (21). A similar result can be found in [7]. In order to go from the gauge (45) where the field fluctuation is not zero to the gauge (21) where $\delta\phi = 0$ we need a change of variables that satisfy $\phi(t + T(t)) + \delta\phi(t + T(t)) = \phi(t)$.

At first order in perturbation theory we only need to do a time reparametrization. Let t and $\tilde{t}$ be the time coordinates in the gauges (21) and (45) respectively. The time reparametrization is $\tilde{t} = t + T$. At first order

$$T = -\frac{\delta\phi}{\dot{\phi}_0} = \frac{\zeta}{H}, \quad \zeta = -\frac{H}{\dot{\phi}_0}\delta\phi. \tag{A1}$$

At second order the time reparametrization is

$$T = -\frac{\delta\phi}{\dot{\phi}_0} - \frac{\ddot{\phi}_0\delta\phi^2}{2\dot{\phi}_0{}^3} + \frac{\delta\dot{\phi}\delta\phi}{\dot{\phi}^2}. \tag{A2}$$

At this order we also need to perform a spatial reparametrization given by $\tilde{x}^i = x^i + \epsilon^i(x,t)$, where ϵ^i is of second order in the perturbations. The metric in the gauge (21) becomes

$$h_{ij} = -\frac{\partial T}{\partial x^i}\frac{\partial T}{\partial x^j} + N_j^{(1)}\frac{\partial T}{\partial x^i} + N_i^{(1)}\frac{\partial T}{\partial x^j} + a^2 T\dot{\tilde{\gamma}}_{ij} + a^2\left(\frac{\partial \epsilon_j}{\partial x^i} + \frac{\partial \epsilon_i}{\partial x^j}\right) + a^2 e^{2HT + \dot{H}T^2}\left(\delta_{ij} + \tilde{\gamma}_{ij}(t) + \frac{1}{2}\tilde{\gamma}_{ik}\tilde{\gamma}_j^k\right), \tag{A3}$$

where $N_i^{(1)}$ is the first order shift vector in the gauge (45). If the vector ϵ^i obeys the equation

$$a^{-2}\delta h_{ij} + \frac{\partial \epsilon_j}{\partial x^i} + \frac{\partial \epsilon_i}{\partial x^j} = 2\beta\delta_{ij} + \mu_{ij}, \tag{A4}$$

with μ_{ij} being a transverse and traceless tensor and δh_{ij} being defined as the first four terms of Eq. (A3), then the gauge transformation equations are given by

$$\begin{aligned} \zeta &= HT + \frac{\dot{H}T^2}{2} + \beta, \\ \gamma_{ij} &= \tilde{\gamma}_{ij}(t) + \mu_{ij}. \end{aligned} \tag{A5}$$

To obtain the quantities β and μ_{ij} it proves to be useful to decompose ϵ^i in $\epsilon^i = \partial^i\tilde{\epsilon} + \epsilon_t^i$ with $\partial_i\epsilon_t^i = 0$. After a few mathematical manipulations of equation (A4) one can obtain

$$\beta = \frac{a^{-2}}{4}\left(\delta h_i^i - \partial^{-2}\partial^i\partial^j\delta h_{ij}\right), \tag{A6}$$

$$\mu_{ij} = a^{-2}\left(\delta h_{ij} - \frac{1}{2}\delta_{ij}\delta h_k^k - \partial^{-2}\partial_i\partial^k\delta h_{kj} - \partial^{-2}\partial_j\partial^k\delta h_{ki} + \frac{1}{2}\delta_{ij}\partial^{-2}\partial^l\partial^k\delta h_{lk} + \frac{1}{2}\partial^{-2}\partial_i\partial_j\delta h_k^k + \frac{1}{2}\partial^{-4}\partial_i\partial_j\partial^l\partial^k\delta h_{lk}\right), \tag{A7}$$

where δh_{ij} can be written explicitly as

$$\delta h_{ij} = -\frac{1}{H^2}\partial_i\zeta_n\partial_j\zeta_n + \frac{1}{H}\left(\partial_i\zeta_n\partial_j\psi_1 + \partial_j\zeta_n\partial_i\psi_1\right) + \frac{a^2}{H}\zeta_n\dot{\tilde{\gamma}}_{ij}, \tag{A8}$$

where ψ_1 is from the uniform curvature gauge and we have used the variable ζ_n introduced before, Eq. (38). As $\tilde{\gamma}_{ij}$ is of second order now, the term $\zeta_n\dot{\tilde{\gamma}}_{ij}$ is of third order. We kept it in Eq. (A3) for the sake of comparison with the result of [7]. For ζ we have

$$\zeta = \zeta_n + \frac{\epsilon}{2}\zeta_n^2 + \frac{\ddot{\phi}_0}{2\dot{\phi}_0 H}\zeta_n^2 + \frac{1}{H}\zeta_n\dot{\zeta}_n + \beta. \tag{A9}$$

APPENDIX B: EXTRACTION OF TT PART OF A TENSOR

Let T_{ij} to be a given 3D symmetric tensor, as it is the case for the source of equation (59). Then it can be decomposed into a trace part and a traceless part as

$$T_{ij} = \frac{T}{3}\delta_{ij} + \tilde{T}_{ij}. \tag{B1}$$

The traceless part (5 degrees of freedom) can be written like

$$\tilde{T}_{ij} = D_{ij}\chi + \partial_i\chi_j + \partial_j\chi_i + \chi_{ij}, \tag{B2}$$

with $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2$, $\partial^i\chi_i = 0$ and $\partial^i\chi_{ij} = 0 = \chi^i_i$, where indices are raised by δ_{ij}. The equation $\partial^i T_{ij} = \frac{1}{3}\partial_j T + \frac{2}{3}\partial_j\partial^2\chi + \partial^2\chi_j$ can be solved using a similar method as the one we used to solve the second order momentum constraint previously. We then find

$$\chi = \frac{3}{2}\partial^{-4}\partial^j F_j, \quad \chi_j = \partial^{-2}F_j - \partial_j\partial^i\partial^{-4}F_i, \tag{B3}$$

where $F_i \equiv \partial^j T_{ij} - \frac{1}{3}\partial_i T$. And

$$\chi_{ij} = T_{ij} - \frac{T}{3}\delta_{ij} - D_{ij}\chi - \partial_i\chi_j - \partial_j\chi_i. \tag{B4}$$

In conclusion, given a tensor T_{ij}, Eq. (B4) defines its transverse and traceless part.

Let us see how this works at the action level for the particular case of the tensor perturbations described in the main text. In the action (58), the source for $\tilde{\gamma}_{ij}$ is of the form

$$S = \int dt d^3x \tilde{\gamma}^{ij} T_{ij}. \tag{B5}$$

with T_{ij} being quadratic in $\delta\phi$. We can see that because $\tilde{\gamma}_{ij}$ is transverse and traceless we are allowed to replace T_{ij} in the previous action with χ_{ij} defined in (B4). If we calculate the equations of motion by varying the resulting action we get as a source χ_{ij} and not simply T_{ij} (see Eq. (59)), ensuring that both sides of the equations of motion are transverse and traceless.

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