Singularities in loop quantum cosmology

Thomas Cailleteau [1,2,][*] Antonio Cardoso [2,][†] Kevin Vandersloot [2,][‡] and David Wands [2§]

[1] *Physique Fondamentale et appliquée, Universite Paris-Sud 11, 91405 Orsay Cedex, France*

[2] *Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth P01 2EG, United Kingdom*

(Dated: October 20, 2008)

We show that simple scalar field models can give rise to curvature singularities in the effective Friedmann dynamics of Loop Quantum Cosmology (LQC). We find singular solutions for spatially flat Friedmann-Robertson-Walker cosmologies with a canonical scalar field and a negative exponential potential, or with a phantom scalar field and a positive potential. While LQC avoids big bang or big rip type singularities, we find sudden singularities where the Hubble rate is bounded, but the Ricci curvature scalar diverges. We conclude that the effective equations of LQC are not in themselves sufficient to avoid the occurrence of curvature singularities.

The absence of singularities may be taken to be a pre-requisite for any fundamental theory of nature, and in particular we expect a quantum theory of gravity to resolve the curvature singularities found in Einstein's theory of general relativity. Loop quantum cosmology (LQC) [1] has been shown to cure the big bang singularity and replace it with a big bounce for Friedmann-Robertson-Walker (FRW) spacetimes with a massless canonical scalar field [2]. The bounce occurs when the matter energy density reaches a Planckian value and quantum gravity effects behave repulsively. The results have been generalized by considering the approximate modifications to the Friedmann equation implied by LQC. In particular, the models are non-singular for more general forms of the scalar field potential [3], the inclusion of anisotropies in the Bianchi I model [4], the Schwarzschild black hole interior [5], and big rip models with constant equation of state [6]. These results provide hope that maybe a general singularity resolution theory exists for the full theory of loop quantum gravity (LQG), of which LQC is a specialized model.

In this letter we study both canonical and phantom scalar fields with exponential potentials using the effective Friedmann equations of LQC. We find a class of singular cosmologies characterized by a bounded Hubble parameter, but diverging time derivative of the Hubble parameter occurring in finite proper time. While the energy density is bounded, the pressure diverges, a type of singularity known in general relativity as a sudden singularity [7].

The energy density and pressure of the scalar field are, respectively, given by

$$\rho = \pm\frac{1}{2}\dot{\phi}^2 + V, \qquad p = \pm\frac{1}{2}\dot{\phi}^2 - V, \qquad (1)$$

where the plus sign indicates a canonical scalar field and the minus corresponds to a phantom field. The local conservation of energy-momentum leads to the Klein-Gordon equation governing the dynamics of the scalar field

$$\ddot{\phi} + 3H\dot{\phi} \pm \frac{\partial V(\phi)}{\partial \phi} = 0, \qquad (2)$$

with $V(\phi)$ representing the scalar field potential. We will consider exponential potentials of the form

$$V = V_0 e^{\lambda\kappa\phi}, \qquad (3)$$

where without loss of generality we take $\lambda > 0$. Scalar fields with exponentials commonly arise in effective theories derived from dimensional reductions of higher dimensional models, and appear, for example, in the four-dimensional effective theory in the ekpyrotic scenario [8]. We have found qualitatively similar results for simple polynomial potentials.

In general relativity the Friedmann equations for a spatially flat, isotropic universe are given by

$$H^2 = \frac{\kappa^2}{3}\rho, \qquad \dot{H} = -\frac{\kappa^2}{2}(\rho + p), \qquad (4)$$

where $H \equiv \dot{a}/a$ is the Hubble rate, a the scale factor, and $\kappa^2 = 8\pi G_N$.

---

[*]Electronic address: thomas.cailleteau@gmail.com

[†]Electronic address: antonio.cardoso@port.ac.uk

[‡]Electronic address: kevin.vandersloot@port.ac.uk

[§]Electronic address: david.wands@port.ac.uk

General relativistic and spatially flat cosmologies with canonical scalar fields and exponential potentials are always singular. With a positive potential [9, 10, 11], $V_0 > 0$, the scale factor is monotonic with a big bang singularity where $H \to \infty$ as $a \to 0$ at a finite proper time in the past of an expanding universe, or the time reverse (a big crunch singularity where $H \to -\infty$ as $a \to 0$ at a finite time in the future). With a negative potential [12], $V_0 < 0$, there exist turning points where $H = 0$ but these are always maxima of the scale factor where $\dot{H} < 0$. Solutions have either a big bang singularity in the past or a big crunch singularity in the future, or both. A phantom scalar field with a positive exponential potential, $V_0 > 0$, can avoid a big bang singularity, but suffers a big rip [13] singularity where $H \to \infty$ and $a \to \infty$ at a finite proper time in the future (or the time reverse in the past) [14, 15]. Note that one cannot have a phantom scalar field with negative potential as the total energy density must be non-negative in a spatially flat Friedmann cosmology.

To incorporate the quantum effects due to LQC, we consider effective Friedmann equations with corrections of the form [2, 3]

$$H^2 = \frac{\kappa^2 \rho}{3} \left[1 - \frac{\rho}{\rho_c} \right], \tag{5}$$

$$\dot{H} = -\frac{\kappa^2 (\rho + p)}{2} \left[1 - 2\frac{\rho}{\rho_c} \right]. \tag{6}$$

where ρ_c is a constant parameter measuring the loop quantum effects. The value of ρ_c is on the order of the Planck density, κ^{-4}. Note that these equations are approximations to the true quantum dynamics. They have been validated from more rigorous constructions of the quantum dynamics for simpler models with a massless scalar field in [2]. Some additional quantum state dependant corrections are suggested in [16, 17], but they have yet to be compared to rigorous quantum constructions. Furthermore, the effective Friedmann equation of [17] has been derived assuming the LQC effects occur in kinetic dominated regimes, which will not be the case for our solutions. The classical GR limit can be recovered by letting $\rho_c \to \infty$, whence it is easy to see that the GR Friedmann equations (4) are recovered. The loop effects do not modify the form of the scalar field equations and thus the Klein-Gordon equation (2) is unchanged.

The prediction of a bounce in LQC with canonical scalar fields and positive potentials can be understood from the modified Friedmann equation (5). When the matter energy density reaches the critical value ρ_c, the Hubble rate becomes zero triggering the bounce. Therefore, in the case of the canonical scalar field with positive exponential potential ($V_0 > 0$), the past big bang or future big crunch singularity of the GR solutions is replaced by a bounce and the cosmology becomes non-singular.

Numerical results for scalar field cosmologies with the loop quantum equations of motion are shown in figure 1 for the canonical scalar field with negative potential and figure 2 for the phantom scalar. For the numerics we have taken $\lambda = 1$, $V_0 = \mp \kappa^{-4}$, $\rho_c = \kappa^{-4}$ and set $\kappa = 1$. Initial conditions are chosen such that the scalar field is zero and the universe is at a turn-around with $H(0) = 0$ and energy density equal to ρ_c. This implies a bouncing (re-collapsing) phase for the canonical (phantom) case. The qualitative results do not depend on the choice of initial conditions.

Both the canonical and phantom scalar fields show similar behavior. The solutions are characterized by repeated bouncing and collapsing phases with the period of oscillations decreasing to zero in finite proper time. As the period decreases, the expanding and collapsing phases become shorter in duration and the scale factor approaches a constant at the point when the evolution terminates. The scalar field rolls down (up) the potential for the canonical (phantom) case reaching infinity at some final value of t. The Hubble rate is bounded, as expected from the LQC Friedmann equation (5), while the time derivative of the Hubble rate diverges. Thus the solutions have a "quiescent" [18] or "sudden" [7] singularity where a and H are both bounded but $\dot{H}$ diverges. The Ricci curvature scalar, $R = 6\dot{H} + 12H^2$, diverges at a finite proper time.[1]

We note that the case of a phantom scalar field with positive exponential potential in LQC was previously studied in [21]. Our numerical results are in qualitative agreement, but we find that the solutions are in fact singular.

In order to verify that the singular nature of the solutions is not a numerical artifact, we can find an approximate analytical solution for the regime where $a(t)$ is approximately constant, which becomes increasingly accurate as the singularity is approached. The total density, ρ, remains bounded as the kinetic energy, $\dot{\phi}^2/2$, and potential energy, $V(\phi)$, separately diverge. Setting $\rho = 0$ we can use the definition of the energy density (1) and solve for $\phi(t)$ to get

$$\phi(t) = -\frac{2}{\lambda\kappa} \ln \left[e^{-\lambda\kappa\phi_0/2} - \lambda\kappa\sqrt{\frac{|V_0|}{2}}(t - t_0) \right]. \tag{7}$$

---

[1] We note that although the curvature diverges, these singularities are weak in that they may be geodesically complete [19, 20].
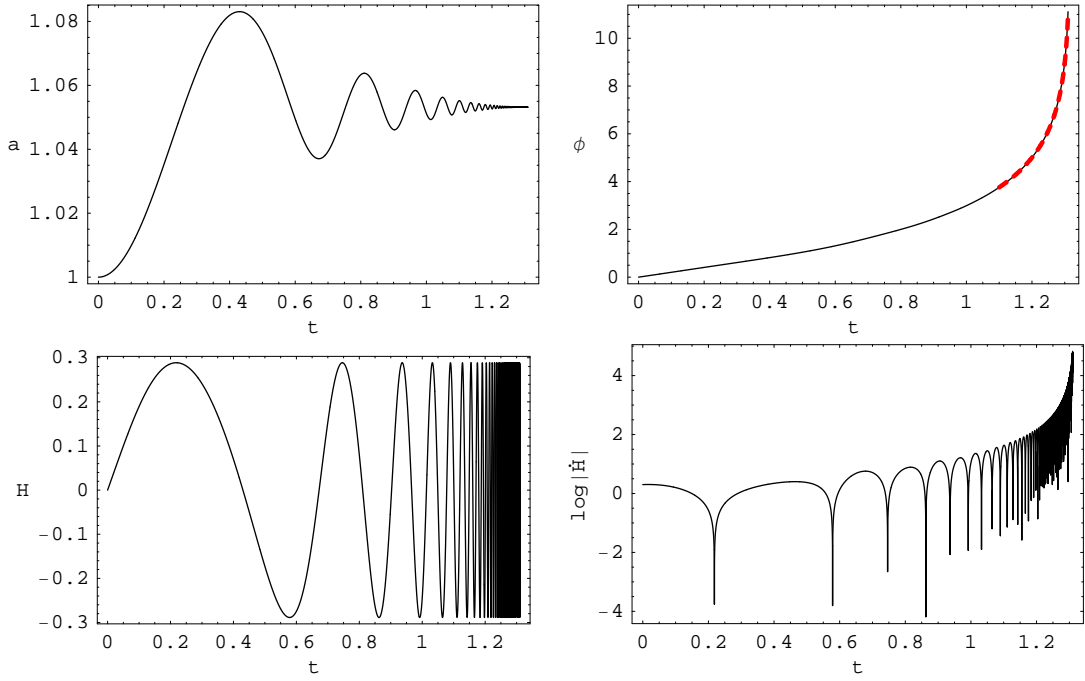
FIG. 1: Canonical scalar field with negative exponential potential. Plots of a, ϕ, H and $\dot{H}$ as a function of time, obtained from numerical simulations for $\kappa = 1$, $V_0 = -1$, $\lambda = 1$ and $\rho_c = 1$. As initial conditions we set $\phi(0) = 0$, $\dot{\phi}(0) = \sqrt{2(\rho_c - V_0)}$, and $H(0) = 0$. The dashed curve in the plot of $\phi(t)$ is the prediction from the approximate analytical solution (7).
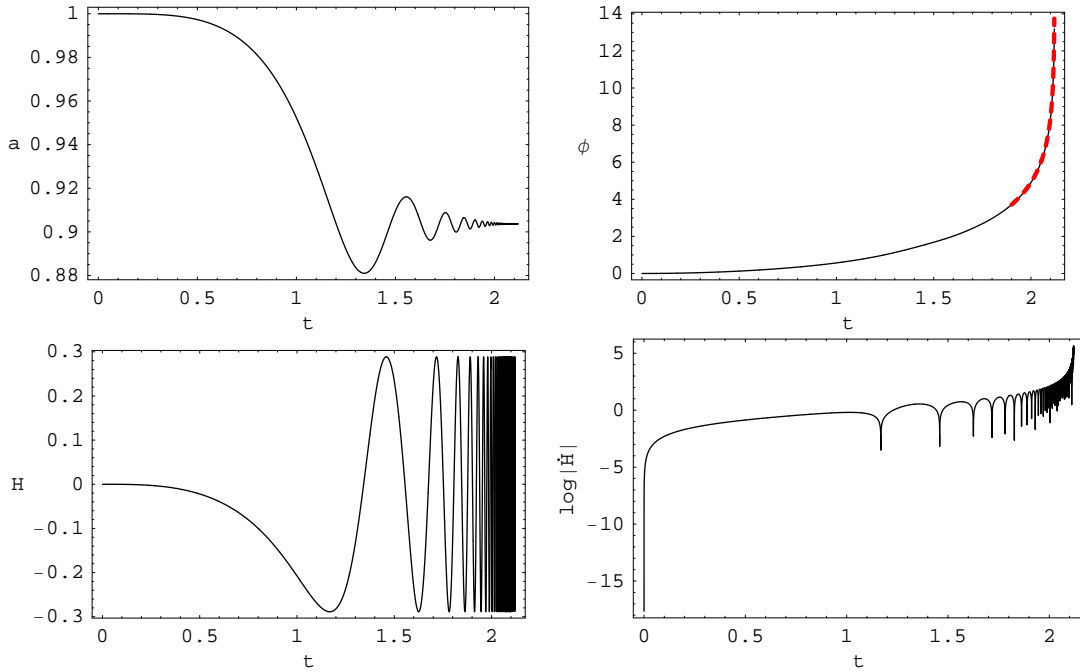

FIG. 2: Phantom scalar field with positive exponential potential. Plots of a, ϕ, H and $\dot{H}$ as a function of time, obtained from numerical simulations, for $\kappa = 1$, $V_0 = 1$, $\lambda = 1$ and $\rho_c = 1$. As initial conditions we use $\phi(0) = 0$, $\dot{\phi}(0) = \sqrt{2(V_0 - \rho_c)}$, and $H(0) = 0$. The dashed curve in the plot of $\phi(t)$ is the prediction from the approximate analytical solution (7).

Here t_0 refers to a reference time near the singularity where the scale factor approaches its constant value and $\phi_0 = \phi(t_0)$. Note that this solution applies to both the canonical and phantom scalar fields approaching the singularity. In addition, because the energy density is bounded and the pressure grows, the oscillating solutions occur in a regime where the kinetic energy of the field approximately equals the potential energy. The analytical solution diverges at a critical value of t given by

$$t_c = t_0 + \frac{1}{\lambda\kappa}\sqrt{\frac{2}{|V_0|}}e^{-\lambda\kappa\phi_0/2}\,. \tag{8}$$

Both ϕ and $\dot\phi$ diverge at this time, and we see that also the pressure and $\dot H$ diverge, indicating a curvature singularity at a finite time, in agreement with the numerical results. A comparison of the numerical solution to the analytic approximation shows that the two are in close agreement and the approximation becomes better as the critical time is approached. This can be seen in figures 1 and 2 where the red dashed line is the analytic solution (7) which matches well the numerical solution approaching the singularity at t_c. Thus the singularity is a genuine feature of the LQC equations and not a numerical instability.

We have shown that the class of exponential potentials studied here gives rise to a singularity in LQC despite the quantum corrections in equation (5). We can show that this is a consequence of the potential being unbounded. The effective Friedmann equation (5) requires that ρ is bounded, and for a canonical field, with non-negative kinetic energy, the upper bound on $\rho \leq \rho_c$ also requires that $V \leq \rho_c$, but the potential is not bounded from below by the effective Friedmann equation (5). We can write the evolution equation (6) as

$$\dot H = -\kappa^2(\rho - V)\left[1 - 2\frac{\rho}{\rho_c}\right]\,, \tag{9}$$

Thus for a canonical field with non-negative kinetic energy, $\dot H$ remains bounded if $V(\phi)$ is bounded from below. Conversely, $\dot H$ is unbounded only if $V(\phi)$ is unbounded from below. Similarly for a phantom field with negative kinetic energy one can show that $\dot H$ is unbounded only if $V(\phi)$ is unbounded from above.

If we modify the exponential potential (3) to introduce a lower bound, the sudden singularity disappears. In figure 3 we show numerical results for a canonical scalar field with a potential of the form

$$V(\phi) = V_0 e^{\lambda\kappa\phi} + V_2 e^{\lambda_2\kappa\phi}\,, \tag{10}$$

where $V_0 < 0$, $\lambda > 0$, as before, but $V_2 > 0$ and $\lambda_2 > \lambda$. These conditions ensure that the potential is bounded from below and grows as a positive exponential for large values of ϕ. In the numerical run, the parameter values were chosen to be $V_0 = \kappa^{-4}$, $\lambda = 1$, $V_2 = e^{-6}\kappa^{-4}$, and $\lambda_2 = 2$, and we have chosen the same initial conditions as the single exponential case. With these parameter values the potential minimum is located at

$$\phi_{min} = \frac{1}{\kappa(\lambda_2 - \lambda)}\ln\left(-\frac{V_0\lambda}{V_2\lambda_2}\right) \approx 5.3\kappa^{-1}\,. \tag{11}$$

The numerical run indicates that the solution behaves as in the single exponential case with oscillatory expansion/collapse phases with decreasing period until the potential minimum is reached, after which the period of oscillations grows. The solution is non-singular as is shown by the boundedness of both the Hubble rate H and its time derivative $\dot H$. The scalar field passes through the potential minimum and rolls up the potential only to turn around at a point where the potential is positive which is roughly given by $\phi \approx 6\kappa^{-1}$ in figure 3. Similar behavior occurs in the phantom case for a double exponential potential with $V_0 > 0$ and $V_2 < 0$ where the potential is now bounded from above, and the behavior is also non-singular.

We have shown that singular solutions exist to the LQC effective Friedmann equations for potentials that are not bounded from below (above) for a canonical (phantom) scalar field. It is interesting that sudden singularities, which may appear to be somewhat contrived in general relativity [7], appear in quite simple models in LQC. While this may indicate that generic singularity resolution is not a feature of loop quantum gravity, it may simply indicate that the effective equations of LQC break-down in some specific cases. The scalar field Lagrangians which give rise to singular behavior also may themselves be regarded as sufficiently pathological that this need not be considered as a significant limitation of LQC.
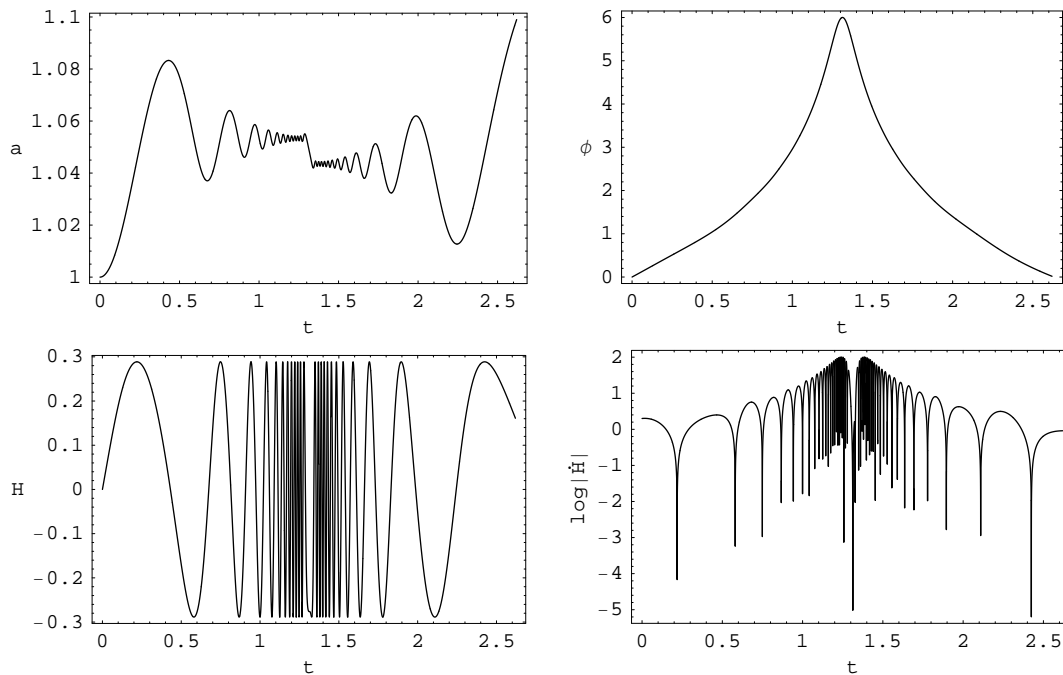
FIG. 3: Canonical scalar field with bounded (double exponential) potential. Plots of a, ϕ, H and $\dot{H}$ as a function of time, obtained from our numerical simulations, for $\kappa = 1$, $V_0 = -1$, $\lambda = 1$, $V_2 = e^{-6}$, $\lambda_2 = 2$. As initial conditions we set $\phi(0) = 0$, $\dot{\phi}(0) = \sqrt{2(\rho_c - V_0)}$, and $H(0) = 0$.

[1] M. Bojowald, Living Rev. Rel. **8**, 11 (2005) [arXiv:gr-qc/0601085].
[2] A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. Lett. **96**, 141301 (2006) [arXiv:gr-qc/0602086]. A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. D **73**, 124038 (2006) [arXiv:gr-qc/0604013]. A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. D **74**, 084003 (2006) [arXiv:gr-qc/0607039]. A. Ashtekar, T. Pawlowski, P. Singh and K. Vandersloot, Phys. Rev. D **75**, 024035 (2007) [arXiv:gr-qc/0612104]. K. Vandersloot, Phys. Rev. D **75**, 023523 (2007) [arXiv:gr-qc/0612070].
[3] P. Singh, K. Vandersloot and G. V. Vereshchagin, Phys. Rev. D **74**, 043510 (2006) [arXiv:gr-qc/0606032].
[4] D. W. Chiou and K. Vandersloot, Phys. Rev. D **76**, 084015 (2007) [arXiv:0707.2548 [gr-qc]].
[5] C. G. Boehmer and K. Vandersloot, Phys. Rev. D **76**, 104030 (2007) [arXiv:0709.2129 [gr-qc]]. L. Modesto, Class. Quant. Grav. **23**, 5587 (2006) [arXiv:gr-qc/0509078]. M. Campiglia, R. Gambini and J. Pullin, Class. Quant. Grav. **24**, 3649 (2007) [arXiv:gr-qc/0703135].
[6] M. Sami, P. Singh and S. Tsujikawa, Phys. Rev. D **74**, 043514 (2006) [arXiv:gr-qc/0605113].
[7] J. D. Barrow, Class. Quant. Grav. **21**, L79 (2004) [arXiv:gr-qc/0403084].
[8] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. D **64**, 123522 (2001) [arXiv:hep-th/0103239].
[9] J. J. Halliwell, Phys. Lett. B **185**, 341 (1987).
[10] A. B. Burd and J. D. Barrow, Nucl. Phys. B **308**, 929 (1988).
[11] E. J. Copeland, A. R. Liddle and D. Wands, Phys. Rev. D **57**, 4686 (1998) [arXiv:gr-qc/9711068].
[12] I. P. C. Heard and D. Wands, Class. Quant. Grav. **19**, 5435 (2002) [arXiv:gr-qc/0206085].
[13] R. R. Caldwell, Phys. Lett. B **545** (2002) 23 [arXiv:astro-ph/9908168].
[14] J. g. Hao and X. z. Li, Phys. Rev. D **67**, 107303 (2003) [arXiv:gr-qc/0302100].
[15] L. A. Urena-Lopez, JCAP **0509**, 013 (2005) [arXiv:astro-ph/0507350].
[16] M. Bojowald, H. Hernandez and A. Skirzewski, Phys. Rev. D **76**, 063511 (2007) [arXiv:0706.1057 [gr-qc]].
[17] M. Bojowald, [arXiv:0801.4001 [gr-qc]].
[18] Y. Shtanov and V. Sahni, Class. Quant. Grav. **19**, L101 (2002) [arXiv:gr-qc/0204040].
[19] L. Fernandez-Jambrina and R. Lazkoz, Phys. Rev. D **70**, 121503 (2004) [arXiv:gr-qc/0410124].
[20] L. Fernandez-Jambrina and R. Lazkoz, Phys. Rev. D **74**, 064030 (2006) [arXiv:gr-qc/0607073].
[21] D. Samart and B. Gumjudpai, Phys. Rev. D **76**, 043514 (2007) [arXiv:0704.3414 [gr-qc]].