Gravitational waves from an early matter era

Hooshyar Assadullahi and David Wands
Institute of Cosmology and Gravitation, University of Portsmouth,
Dennis Sciama Building, Burnaby Road, Portsmouth PO1 3FX,
United Kingdom

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Abstract

We investigate the generation of gravitational waves due to the gravitational instability of primordial density perturbations in an early matter-dominated era which could be detectable by experiments such as LIGO and LISA. We use relativistic perturbation theory to give analytic estimates of the tensor perturbations generated at second order by linear density perturbations. We find that large enhancement factors with respect to the naive second-order estimate are possible due to the growth of density perturbations on sub-Hubble scales. However very large enhancement factors coincide with a breakdown of linear theory for density perturbations on small scales. To produce a primordial gravitational wave background that would be detectable with LIGO or LISA from density perturbations in the linear regime requires primordial comoving curvature perturbations on small scales of order 0.02 for Advanced LIGO or 0.005 for LISA, otherwise numerical calculations of the non-linear evolution on sub-Hubble scales are required.

1 Introduction

Gravitational waves are a probe of the very early universe that go beyond electromagnetic signals such as the cosmic microwave background. Gravitons can propagate essentially unscattered from any energy scale below the Planck density. As a result they have been considered as probes of violent events in the early universe, such as bubble collisions [1], preheating after inflation [2, 3, 4, 5, 6, 7, 8, 9] or cosmic strings [10].

Much of the work to date has been based on the local generation of gravitational waves, e.g., due to time-varying quadrupole moment in flat spacetime [11]. But recent [12, 13, 14, 15, 16, 17, 18, 19] work has developed cosmological perturbation theory to deal with the generation of gravitational waves at second order from first-order density perturbations. Such a relativistic treatment is required for inhomogeneities on or above the Hubble scale in an expanding cosmology. It is possible to make a perturbative calculation of the gravitational waves inevitably generated at second order from the existence of linear density perturbations on all scales.

In the early radiation-dominated era, Ananda *et al* [17] found that gravitational waves are produced when density perturbations, which are overdamped at early times on super-Hubble scales ($k/a \ll H$) come inside the Hubble scale ($k/a = H$) and begin under-damped oscillations. The Hubble damping leads to the rapid decay of the amplitude of sub-Hubble metric perturbations, and an almost scale-invariant spectrum of gravitational waves is left on sub-Hubble scales to propagate freely, redshifted by the cosmological expansion.

Baumann *et al* [18] extended this calculation to follow the evolution of the gravitational waves generated at second order through into the matter-dominated era, and ultimately the present late-time acceleration using the numerical solution for the evolution of linear density perturbations. Their work showed a surprising behavior in the matter-dominated era, where tensor metric perturbations grow on large-scales until reaching a constant value once they come inside the Hubble scale, producing a larger amplitude on scales close to the Hubble scale today than the first-order gravitational waves generated by inflation [18]. Although intriguing, it seems very difficult to detect such extremely long-wavelength tensor perturbations which are only produced at late cosmic times [13, 20].

In this paper we will investigate the production of gravitational waves in an *early* matter-dominated era, preceding the standard radiation-dominated era, before primordial nucleosynthesis. A matter-dominated era is expected to occur in the very early universe after a period of inflation driven by overdamped scalar fields. At the end of inflation the Hubble damping, H, decreases and the scalar fields become massive ($m > H$). Oscillating massive scalar fields have an effectively pressureless equation of state [21]. The decay of the oscillating fields leads to the reheating of the universe and the start of the standard Hot Big Bang model [22]. If the decay is slow $\Gamma \ll H$ at the end of inflation then we have an extended early matter-dominated era. An early matter-dominated era might also occur if weakly-coupled massive scalar fields (or moduli) with non-zero vacuum expectation values come to dominate over the inflaton decay products sometime after inflation. In the curvaton scenario it is a weakly-coupled massive field, rather than the inflaton, whose inhomogeneous perturbations give rise to the primordial density perturbation [23, 24, 25, 26]. Gravitational waves generated at second-order have previously been studied in the curvaton scenario [27], in the limit where the curvaton field does not dominate the energy density of the universe, i.e., with no early matter era.

There are few constraints on an early matter era since the comoving Hubble scale must be less that a few parsecs. The primordial density perturbations on such small scales have long since been erased by Silk damping, and a stochastic background of gravitational waves might be one of the few ways we can probe the primordial power spectrum on such small scales (though primordial black holes might be another [28]). Gravitational waves produced in an early matter era would have wavelengths much smaller than the Hubble size at primordial nucleosynthesis and thus, depending on the reheating temperature at the end of the early matter era, could be directly directed by gravitational wave detectors currently in operation or being planned.

This paper is organised as follows. In section 2 we summarise the results we will need for the evolution of linear density perturbations in a matter-dominated era. In section 3 we give the evolution equation for second-order gravitational waves in a matter dominated era and present the resulting power spectrum for tensor metric perturbations at the end of the matter era. This leads to an effective density of gravitational waves at the present day,

$\Omega_{GW,0}$, presented in section 4. In section 5 we discuss the detectability of the gravitational wave background in possible early matter eras, and we conclude in section 6.

2 Density perturbations in an early matter era

In this paper we will work in the longitudinal [29] (or Poisson [12, 19]) gauge where the perturbed metric is

$$ds^2 = a^2(\eta)[-(1+2\Phi)d^2\eta + [(1-2\Psi)\delta_{ij} + 2F_{(i,j)} + h_{ij}]dx^i dx^j] \tag{1}$$

where η is conformal time, and the conformal Hubble rate is $\mathcal{H} \equiv a'/a$ where primes denote a derivative with respect to conformal time. F_i describes transverse vector perturbations and h_{ij} describes tensor metric perturbations which are transverse and trace-free.

Vector perturbations, F_i, must vanish at first order in the absence of any vector part of the fluid 3-velocity, e.g., during a period of inflation dominated by a scalar field. First-order tensor perturbations propagate as a free field, but are rapidly redshifted to large scales by an inflationary expansion. Gravitational waves on smaller scales are produced from initial quantum vacuum fluctuations, but their amplitude depends on the energy scale of inflation. In the following we assume any first-order vector or tensor metric perturbations are negligible and focus on the second-order tensor perturbations that are produced from first-order scalar perturbations which are known to exist on observable scales.

The scalar metric perturbations Φ and Ψ are supported by density and (curl-free) velocity perturbations of a fluid. In the absence of anisotropic stress we require $\Phi = \Psi$ [29], and the linear evolution equation for Φ is given by [19]

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\,\Phi = 4\,\pi\,G\,a^2\,\delta\,P \tag{2}$$

In a matter-dominated era with negligible pressure, $\delta P = 0$, we have $a \propto \eta^2$ and $\mathcal{H} = 2/\eta$, so the linear evolution equation for Φ is simply

$$\Phi'' + \frac{6}{\eta}\Phi' = 0\,, \tag{3}$$

with the general solution $\Phi = C + D/\eta^5$, where C and D may be spatially inhomogeneous, but are constant in time. Considering only regular solutions at early times ($\eta \to 0$) requires $D = 0$ and we then have Φ constant in time on all scales.

We will often find it convenient to use a Fourier transform

$$\Phi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k}\,\Phi_{\mathbf{k}}\,e^{i\mathbf{k}.\mathbf{x}}\,, \tag{4}$$

where the initial power spectrum of an isotropic distribution of scalar metric perturbations will be a function of $k \equiv |\mathbf{k}|$

$$\langle \Phi_{\mathbf{k}} \Phi_{\mathbf{k}'} \rangle = \frac{2\pi^2}{k^3} \delta^3(\mathbf{k} + \mathbf{k}')\,\mathcal{P}(k) \tag{5}$$

Note that the linear comoving density perturbation grows in time on all scales during a matter-dominated era and is given by [30, 19]

$$\frac{\delta\rho_m}{\rho} = \frac{2}{3\mathcal{H}^2}\nabla^2\Phi\,. \tag{6}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the comoving spatial Laplacian. Because we have assumed pressure is negligible, gravitational instability leads to a growing density perturbation on all scales. Eventually this will lead to a breakdown of the linear evolution on sub-Hubble scales when the density perturbations become of order one, corresponding to

$$k_{NL}^2(\eta) \sim \mathcal{P}^{-1/2}\mathcal{H}^2 \gg \mathcal{H}^2\,. \tag{7}$$

The above is an idealisation of any realistic model. In particular an oscillating massive scalar field with mass $m > H$ has a Compton wavelength $\lambda \sim m^{-1}$ and hence can only be described as pressureless matter on comoving scales $k \ll am$. Fourier modes with $k > am$ correspond to relativistic modes with non-negligible pressure and we expect Φ to be suppressed on these scales. For simplicity, and to remove any dependence on the preceding cosmological evolution, we will assume in our calculations that there are no density perturbations on sub-Hubble scales at the start of the early matter era

$$\mathcal{P}(k) = 0 \quad \text{for } k > k_{\rm dom} \tag{8}$$

where $k_{\rm dom} = \mathcal{H}_{\rm dom} < am$ is the comoving Hubble scale at the start of the matter-dominated era[1]. Any density perturbations on smaller scales will provide an additional, source for gravitational waves.

Even so, density perturbations on our smallest scale will become nonlinear, $k_{\rm dom} > k_{NL}(\eta)$, if the matter era lasts long enough. In what follows we will present results both extrapolating the linear results into the nonlinear regime, and also from imposing an abrupt cut-off on the scalar power spectrum at the non-linear scale, such that

$$\mathcal{P}(k) = 0 \quad \text{for } k > k_{\rm cut} \tag{9}$$

where $k_{\rm cut} = \min[k_{\rm dom}, k_{NL}(\eta)]$, which we expect to provide a conservative lower bound on the amplitude of gravitational waves generated at second-order.

On very large scales the power spectrum of the primordial scalar perturbation is commonly approximated by a power law

$$\mathcal{P}(k) = \frac{9}{25}\triangle^2_{\mathcal{R}}\left(\frac{k}{k_*}\right)^{n_s-1}\,. \tag{10}$$

where the numerical factor 9/25 comes from the relation between scalar curvature perturbation in the longitudinal and comoving gauges on large scales in a matter dominated era [29]. Observations of the cosmic microwave background (CMB) radiation [31] give the amplitude, $\triangle^2_{\mathcal{R}} \approx 2.4\times10^{-9}$, and spectral index, $n_s \approx 0.96$, for the primordial density perturbations on very large scales $2\pi a_0/k_* \sim 100$ Mpc today. We should note however that the primordial density perturbations could be very different on the much smaller scales relevant for the direct detection of gravitational waves, e.g., $2\pi a_0/k \sim 1$ km.

[1] In the late universe this corresponds to the turnover in the matter power spectrum on scales $k > k_{\rm eq}$ where $k_{\rm eq} = \mathcal{H}_{\rm eq}$ is the Hubble scale at matter-radiation equality.

3 Generation of tensor perturbations

The existence of first-order scalar perturbations inevitably leads to second-order vector and tensor perturbations [13, 15].

Analogous to Eq. (4) for scalar fields, we can give the Fourier transform of the tensor metric perturbations ([17, 18])

$$h_{ij}(x,\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}.\mathbf{x}} [h_{\mathbf{k}}(\eta) e_{ij}(\mathbf{k}) + \bar{h}_{\mathbf{k}} \bar{e}_{ij}(\mathbf{k})] \,, \tag{11}$$

where $e^{ij}(k)$ is the tensor polarization. The two polarization tensors $e_{ij}(k)$ and $\bar{e}_{ij}(k)$ can be calculated in terms of the orthonormal basis:

$$\begin{aligned} e_{ij}(k) &= \frac{1}{\sqrt{2}}[e_i(k)e_j(k) - \bar{e}_i(k)\bar{e}_j(k)] \\ \bar{e}_{ij}(k) &= \frac{1}{\sqrt{2}}[e_i(k)\bar{e}_j(k) + \bar{e}_i(k)e_j(k)] \end{aligned} \tag{12}$$

where $\mathbf{e}$ and $\bar{\mathbf{e}}$ are orthonormal transverse vectors, $\mathbf{e}.\mathbf{k} = \bar{\mathbf{e}}.\mathbf{k} = \mathbf{e}.\bar{\mathbf{e}} = 0$ and $\mathbf{e}.\mathbf{e} = \bar{\mathbf{e}}.\bar{\mathbf{e}} = 1$.

The power spectrum of the tensor perturbations is given by

$$\langle h_{\mathbf{k}}(\eta) h_{\mathbf{k}'}(\eta) \rangle = \frac{1}{2} \frac{2\pi^2}{k^3} \delta^3(\mathbf{k} + \mathbf{k}') \mathcal{P}_h(k,\eta) \,, \tag{13}$$

where the factor of 1/2 arises since, by convention, $\mathcal{P}_h$ includes the contributions from both polarisations ($\bar{h}_{\mathbf{k}}$ as well as $h_{\mathbf{k}}$).

The evolution of second-order tensor mode in (1) is given by the wave equation [17]:

$$h''_{ij} + 2\mathcal{H} h'_{ij} + k^2 h_{ij} = S^{TT}_{ij} \,, \tag{14}$$

where the S^{TT}_{ij} is a transverse and trace-free source term. If we include terms up to second order in the scalar perturbations, then S^{TT}_{ij} is the transverse and tracefree part of [17, 18]

$$\begin{aligned} S_{ij} &= 2\Phi\partial_i\partial_j\Phi - 2\Psi\partial_i\partial_j\Phi + 4\Psi\partial_i\partial_j\Psi + \partial_i\Phi\partial_j\Phi - \partial^i\Phi\partial_j\Psi - \partial^i\Psi\partial_j\Phi + 3\partial^i\Psi\partial_j\Psi \\ &\quad - \frac{4}{3(1+w)\mathcal{H}^2} \partial_i(\Psi' + \mathcal{H}\Phi)\partial_j(\Psi' + \mathcal{H}\Phi) \\ &\quad - \frac{2c_s^2}{3w\mathcal{H}} \left[3\mathcal{H}(\mathcal{H}\Phi - \Psi') + \nabla^2\Psi\right] \partial_i\partial_j(\Phi - \Psi) \end{aligned} \tag{15}$$

where $w = P/\rho$ is the equation of state and $c_s^2 = P'/\rho'$ is the adiabatic sound speed.

In the approximation of a pressureless matter-dominated era we have $w = c_s^2 = 0$ and $\Phi = \Psi = C(\mathbf{x})$ and the source term simplifies considerably. If we substitute Eq. (4) into Eq. (15) we obtain

$$S_{ij}(x) = \frac{1}{(2\pi)^3} \int d^3\widetilde{\mathbf{k}}\, d^3\widetilde{\mathbf{k}}' \,[-4\widetilde{k}'_i \widetilde{k}'_j - \frac{2}{3}\widetilde{k}'_i \widetilde{k}_j]\, \Phi_{\widetilde{\mathbf{k}}}\, \Phi_{\widetilde{\mathbf{k}}'}\, e^{i(\widetilde{\mathbf{k}}+\widetilde{\mathbf{k}}').\mathbf{x}} \tag{16}$$

Substituting Eqs. (16) and (11) in Eq. (14) we find the evolution of the amplitude of each tensor mode in Fourier space during the matter era

$$h''_{\mathbf{k}} + \frac{4}{\eta} h'_{\mathbf{k}} + k^2 h_{\mathbf{k}} = S_{\mathbf{k}} \,, \tag{17}$$

where

$$S_{\mathbf{k}} = \frac{40}{3} (2\pi)^{-\frac{3}{2}} \int d^3\widetilde{\mathbf{k}} \, e(\mathbf{k}, \widetilde{\mathbf{k}}) \, \Phi_{\mathbf{k}-\widetilde{\mathbf{k}}} \, \Phi_{\widetilde{\mathbf{k}}} \,, \tag{18}$$

and

$$e(\mathbf{k}, \widetilde{\mathbf{k}}) = e^{ij}(\mathbf{k}) \widetilde{k}_i \widetilde{k}_j = \widetilde{k}^2 \sin^2 \theta \,, \tag{19}$$

where θ is the angle between $\mathbf{k}$ and $\widetilde{\mathbf{k}}$,

$$\cos \theta = \frac{\mathbf{k}.\widetilde{\mathbf{k}}}{k \widetilde{k}} \,. \tag{20}$$

3.1 Evolution of the tensor mode

A striking feature of a matter dominated era is that the source term $S_{\mathbf{k}}$ in the wave equation (17) is constant for linear density perturbations. This contrasts with, for example, a radiation dominated era, where Φ, and hence the source for tensor modes, decays on sub-Hubble scales [17].

The general solution of the evolution equation (17) in the matter era is thus

$$\begin{aligned} h_{\mathbf{k}} &= \frac{S_{\mathbf{k}}}{k^2} + \sqrt{\frac{\pi}{2k^3\eta^3}} \left[C_{\mathbf{k}} J_{\frac{3}{2}}(k\eta) + D_{\mathbf{k}} Y_{\frac{3}{2}}(k\eta) \right] \\ &= \frac{S_{\mathbf{k}}}{k^2} + C_{\mathbf{k}} \left(\frac{\sin(k\eta) - k\eta \cos(k\eta)}{k^3\eta^3} \right) - D_{\mathbf{k}} \left(\frac{\cos(k\eta) + k\eta \sin(k\eta)}{k^3\eta^3} \right) \end{aligned} \tag{21}$$

where J and Y are the Bessel functions of the first and second kind and $C_{\mathbf{k}}$ and $D_{\mathbf{k}}$ are constants of integration.

The J and Y modes describe the damped, but source-free oscillations of the gravitational field and thus have the same form as the usual solution for first-order gravitational waves. The Y mode is singular as $\eta \to 0$ and rapidly decays at late times. J is regular at early times and gives oscillations whose amplitude redshifts with the expansion of the universe, $|h_{\mathbf{k}}| \propto a^{-1}$. However the constant second-order source term, S, supports a constant tensor part of the metric perturbation at late times. This behaviour is quite different from that usually associated with gravitational waves, reflecting the fact that this is no longer a freely propagating wave, but rather a metric distortion sourced by terms quadratic in first-order scalar perturbations.

If we impose the initial condition $h = h' = 0$ when $\eta = 0$, then $C_{\mathbf{k}} = -3S_{\mathbf{k}}/k^2$ and the singular term is absent, $D_{\mathbf{k}} = 0$, so we have the particular solution

$$h_{\mathbf{k}} = \frac{S_{\mathbf{k}}}{k^2} \left[1 + 3 \left(\frac{k\eta \cos(k\eta) - \sin(k\eta)}{k^3\eta^3} \right) \right] . \tag{22}$$

At early times, or equivalently in the large scale, super-Hubble limit ($k \ll \mathcal{H}$), we find a growing tensor perturbation

$$h_{\mathbf{k}} = \frac{S_{\mathbf{k}}}{10}\eta^2 \,. \tag{23}$$

We see that the large-scale tensor mode grows at the same rate as the comoving density perturbation (6) in the matter era

$$h \propto \frac{\delta\rho_m}{\rho} \propto \frac{1}{\mathcal{H}^2} \,. \tag{24}$$

The tensor amplitude grows until the mode enters the Hubble scale and at late times, or on sub-Hubble scales ($k \gg \mathcal{H}$), it becomes constant

$$h_{\mathbf{k}} = \frac{S_{\mathbf{k}}}{k^2} \tag{25}$$

Substituting Eq. (18) for $S_{\mathbf{k}}$ into Eq. (22) we will write the solution for the tensor perturbation as

$$h_{\mathbf{k}} = \frac{40 g(k\eta)}{3(2\pi)^{3/2}} k^{-2} \int_0^{k_{dom}} d^3\widetilde{\mathbf{k}}\, e(\mathbf{k}, \widetilde{\mathbf{k}})\, \Phi_{\mathbf{k}-\widetilde{\mathbf{k}}} \Phi_{\widetilde{\mathbf{k}}} \,, \tag{26}$$

where the growth function for the tensor modes is given by

$$g(k\eta) = 1 + 3\left(\frac{k\eta\cos(k\eta) - \sin(k\eta)}{k^3\eta^3}\right) \,, \tag{27}$$

which approaches unity at late times on sub-Hubble scales.

3.2 The power spectrum of the gravitational waves

From (26) we can immediately write down the two-point function for the tensor modes

$$\langle h_{\mathbf{k}}(\eta) h_{\mathbf{k}'}(\eta)\rangle = \left(\frac{40 g(k\eta)}{3(2\pi)^{3/2}}\right)^2 k^{-4} \int_0^{\mathbf{k}_{dom}} d^3\widetilde{\mathbf{k}}\, e(\mathbf{k}, \widetilde{\mathbf{k}})\, d^3\widetilde{\mathbf{k}}'\, e(\mathbf{k}', \widetilde{\mathbf{k}}')\, \langle \Phi_{\mathbf{k}-\widetilde{\mathbf{k}}}\, \Phi_{\widetilde{\mathbf{k}}}\, \Phi_{\mathbf{k}'-\widetilde{\mathbf{k}}'}\, \Phi_{\widetilde{\mathbf{k}}'}\rangle \tag{28}$$

where on the right-hand-side for a Gaussian distribution of scalar perturbations we have (for non-zero k and k')

$$\langle \Phi_{\mathbf{k}-\widetilde{\mathbf{k}}}\, \Phi_{\widetilde{\mathbf{k}}}\, \Phi_{\mathbf{k}'-\widetilde{\mathbf{k}}'}\, \Phi_{\widetilde{\mathbf{k}}'}\rangle = \langle \Phi_{\mathbf{k}-\widetilde{\mathbf{k}}}\, \Phi_{\mathbf{k}'-\widetilde{\mathbf{k}}'}\rangle \langle \Phi_{\widetilde{\mathbf{k}}}\, \Phi_{\widetilde{\mathbf{k}}'}\rangle + \langle \Phi_{\mathbf{k}-\widetilde{\mathbf{k}}}\, \Phi_{\widetilde{\mathbf{k}}'}\rangle \langle \Phi_{\widetilde{\mathbf{k}}}\, \Phi_{\mathbf{k}'-\widetilde{\mathbf{k}}'}\rangle \tag{29}$$

Substituting Eqs. (29) and (5) into Eq. (28) we obtain

$$\begin{aligned} \langle h_{\mathbf{k}}(\eta) h_{\mathbf{k}'}(\eta)\rangle &= \left(\frac{40 g(k\eta)}{3}\right)^2 \frac{\pi\delta^3(\mathbf{k}+\mathbf{k}')}{2k^4} \\ &\times \int d^3\widetilde{\mathbf{k}}\; e(\mathbf{k}, \widetilde{\mathbf{k}}) \left[e(\mathbf{k}', \widetilde{\mathbf{k}}) + e(\mathbf{k}', \mathbf{k}-\widetilde{\mathbf{k}})\right] \mathcal{P}(\mathbf{k}-\widetilde{\mathbf{k}})\, \mathcal{P}\,(\widetilde{\mathbf{k}}) \,, \end{aligned} \tag{30}$$

where $e(\mathbf{p}, \mathbf{q}) = q^2 \sin^2\theta$ is defined in Eq. (19).

For simplicity we will assume that the power spectrum of the primordial scalar perturbations given in Eq. (10) is effectively scale invariant $n_s = 1$ on scales $k < k_{\rm dom}$. The power spectrum (13) of gravitational waves generated is then

$$\mathcal{P}_h(k,\eta) = 2\left(\frac{24g(k\eta)}{5}\right)^2 \triangle_{\mathcal{R}}^4 \left(\frac{k_{\rm dom}}{k}\right) I_1(k/k_{dom})\,, \tag{31}$$

where the integral

$$I_1(k/k_{\rm dom}) = \frac{1}{2\pi k_{\rm dom}} \int d^3\widetilde{\mathbf{k}}\, \frac{[e(\mathbf{k},\widetilde{\mathbf{k}})]^2}{\widetilde{k}^3|\mathbf{k}-\widetilde{\mathbf{k}}|^3}\, \Theta(k_{\rm dom}-\widetilde{k})\, \Theta(k_{\rm dom}-|\mathbf{k}-\widetilde{\mathbf{k}}|)\,, \tag{32}$$

can be written as

$$I_1(x) = \int_{-1}^{1} d\mu \int_0^1 dy \frac{(1-\mu^2)^2 y^3}{(x^2+y^2-2xy\mu)^{\frac{3}{2}}}\, \Theta(1-x^2-y^2+2xy\mu)\,, \tag{33}$$

and we introduce the Heaviside step function, $\Theta(k_{\rm dom}-q)$, to cut-off the scalar spectrum, $\mathcal{P}(q)$, on small scales $q > k_{\rm dom}$.

In the limit $x \ll 1$ the step function is equal to one throughout the integral in Eq. (33) and we obtain the analytic expression

$$I_1(x) \approx \frac{16}{15} - \frac{4}{3}x + \frac{16}{35}x^2\,. \tag{34}$$

For any $x < 1$ we can take I_1 to be a numerical factor $\leq 16/15$.

Equation (31) thus gives a simple numerical estimate of the tensor perturbation generated at second order during a matter dominated era. In particular at the end of an early matter-dominated era we have

$$\mathcal{P}_h(k,\eta_{dec}) \simeq 2\left(\frac{24}{5}\right)^2 \triangle_{\mathcal{R}}^4 \times \left(\frac{k_{dom}}{k}\right) \times g^2(k/k_{\rm dec}) \times I_1(k/k_{\rm dom})\,, \tag{35}$$

where $k_{\rm dec} = \eta_{\rm dec}^{-1}$ is the Hubble scale at the start of the radiation-dominated era, when the matter decays into radiation. This power spectrum for tensor perturbations is shown by the solid line in Figure 1 for an example where $k_{\rm dec} = 10^3 k_{\rm dom}$.

In the super-Hubble limit the tensor amplitude grows and we have $g(k\eta) \simeq k^2\eta^2/10$ from Eq. (27). Thus on scales larger than the Hubble size at the end of the matter era ($k < k_{\rm dec}$) we have

$$\mathcal{P}_h(k,\eta_{\rm dec}) \simeq 0.5\triangle_{\mathcal{R}}^4 \left(\frac{k_{\rm dom}k^3}{k_{\rm dec}^4}\right)\,, \tag{36}$$

where we have taken $I_1(k/k_{\rm dom}) \simeq 16/15$ using Eq. (34). The tensor perturbations thus have a steep blue spectrum on large scales, and are strongly suppressed on super-Hubble scales at the end of the matter era.

On scales which enter the Hubble scale during the early matter dominated era $k_{\rm dec} < k < k_{\rm dom}$ then we find that the tensor amplitude has settled down to a constant value by the end of the matter era, $g(k\eta_{\rm dec}) \simeq 1$ and we have

$$\mathcal{P}_h(k,\eta_{\rm dec}) \simeq 46\triangle_{\mathcal{R}}^4 \left(\frac{k_{\rm dom}}{k}\right) I_1(k/k_{\rm dom})\,, \tag{37}$$

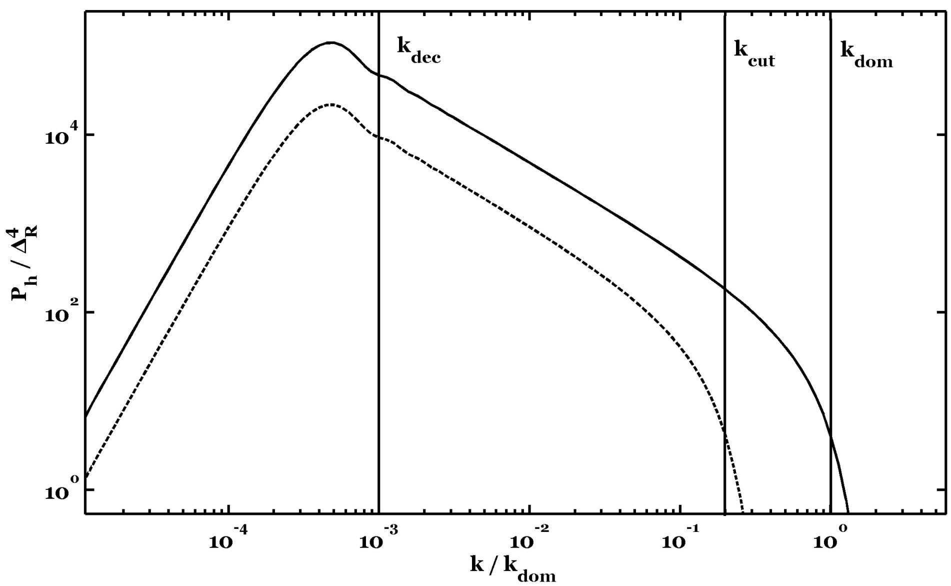

Figure 1: The power spectrum of gravitational waves, shown as a function of wavenumber k, generated from scalar perturbations during a matter dominated era, $\mathcal{P}_h(k,\eta_{\rm dec})$. The solid line shows the prediction using the linear matter power spectrum down to $k_{\rm dom} = 10^3 k_{\rm dec}$, the comoving Hubble scale at the start of matter domination. The dotted line shows the prediction when the matter power spectrum is truncated at $k_{\rm cut} = 200 k_{\rm dec}$. $k_{\rm dec}$ denotes the Hubble scale at the end of the matter era.

Thus we find a decreasing power spectrum for the tensor modes on small scales with $\mathcal{P}_h \propto k^{-1}$ if we take $I_1(k/k_{\rm dom})$ to be constant. In practice I_1 becomes small for $k \sim k_{\rm dom}$ leading to an additional suppression on the smallest scales.

Thus we find the maximum amplitude of tensor perturbations is generated at the scale just entering the Hubble scale at the end of the matter era, $k_{\rm dec}$, for which we have

$$\mathcal{P}_h^{\rm max} = \mathcal{P}_h(k_{\rm dec}, \eta_{\rm dec}) \simeq 50 \triangle_{\mathcal{R}}^4 \left(\frac{k_{\rm dom}}{k_{\rm dec}} \right) , \tag{38}$$

where we have taken $I_1(k_{\rm dec}/k_{\rm dom}) \simeq 16/15$ for $k_{\rm dec} \ll k_{\rm dom}$ using Eq. (34).

Although the tensor power spectrum generated at second order is necessarily proportional to the square of the first-order scalar power spectrum, we find that the constant source term from scalar perturbations, S_{ij} in Eq. (16), extending to sub-Hubble scales, leads to a additional factor $k_{\rm dom}/k_{\rm dec}$ which may be large depending upon the duration of the early matter dominated era. Thus we find that the tensor power spectrum may be significantly enhanced with respect to the naive estimate $\mathcal{P}_h \sim \triangle_{\mathcal{R}}^4$.

3.3 Nonlinear cut-off

The gravitational wave power spectrum (38) becomes largest when the comoving Hubble scale at the end of the matter era, $k_{\rm dec}$, becomes much larger than the smallest scale $k_{\rm dom}$. But as the Hubble scale grows, the comoving density contrast (6) becomes large on scales far inside the Hubble scale, signalling a breakdown of the linear results used thus far to estimate the source term in Eq. (17).

Below the non-linear scale, k_{NL} defined in Eq. (7), a perturbative analysis suggests that power will be rapidly transfered to smaller scales [32] leading to a suppression of $\Phi_{\mathbf{k}}$ and thus the source, $S_{\mathbf{k}}$, on scales $k > k_{NL}$. A full analysis of the nonlinear regime requires a full numerical treatment, such as a lattice field theory calculation, as has been performed in preheating models at the end of inflation, or an N-body simulation in the (small-scale) Newtonian regime.

In practice we will obtain a conservative lower limit on the amplitude of primordial gravitational waves by assuming the scalar power spectrum is vanishing on all scales $k > k_{NL}(\eta)$ instead of the fixed comoving cut-off, $k_{\rm dom}$. This implies a time-dependent source term (18) since the upper limit of the integral in k-space becomes time-dependent.

For $k < k_{NL}$ we find $S_{\mathbf{k}} k_{\rm NL}^2 \mathcal{P}$ and thus

$$S_{\mathbf{k}}' = k_{NL}' \frac{\partial}{\partial k_{NL}} S_{\mathbf{k}} \sim \mathcal{H} S_{\mathbf{k}} \, . \tag{39}$$

The rate of change of the source term is thus slow compared with the decay time for the transient part of the solution in Eq. (21) for $k > \mathcal{H}$ and thus we take the quasi-static generalisation of Eq. (25) for sub-Hubble scales

$$h_{\mathbf{k}}(\eta) \simeq \frac{S_{\mathbf{k}}[k_{NL}(\eta)]}{k^2} \, . \tag{40}$$

Thus a very conservative lower bound on the power spectrum of tensor perturbations generated on sub-Hubble scales at the end of an early matter-dominated era ($k_{\rm dec} < k < k_{\rm cut}$)

is given by the generalisation of (37)

$$\mathcal{P}_h(k, \eta_{\rm dec}) \simeq 46 \triangle_{\mathcal{R}}^4 \left(\frac{k_{\rm cut}}{k}\right) I_1(k/k_{\rm cut})\,, \tag{41}$$

where we have $k_{\rm cut} = \min\{k_{\rm dom}, k_{NL}(\eta_{\rm dec})\}$ and

$$k_{NL}(\eta_{\rm dec}) \simeq \mathcal{P}^{-1/4} k_{\rm dec} \sim 200\, k_{\rm dec}\,. \tag{42}$$

For $k \gg k_{NL}$ we find that either $\Phi_{\tilde{\mathbf{k}}}$ or $\Phi_{\mathbf{k}-\tilde{\mathbf{k}}}$ in the integrand in Eq. (18) vanishes for any $\tilde{\mathbf{k}}$ and the source term, $S_{\mathbf{k}}$, goes to zero. Equation (21) for $S_{\mathbf{k}} = 0$ reduces to the standard solution for a free gravitational wave in a matter-dominated era, whose amplitude is redshifted, $|h_{\mathbf{k}}| \propto a^{-1}$, on small scales, $k > k_{NL} \gg \mathcal{H}$. Thus we obtain

$$\mathcal{P}_h(k, \eta) \simeq 46 \triangle_{\mathcal{R}}^4 \left(\frac{k_{NL}(\eta)}{k}\right)^4 \tag{43}$$

Hence $\mathcal{P}_h(k, \eta_{\rm dec}) \propto k^{-4}$, and the power spectrum of the gravitational waves is strongly suppressed on small scales ($k \gg k_{NL}(\eta)$) if non-linear evolution suppresses the scalar perturbation on these scales.

Assuming the scalar perturbations rapidly decay to effectively zero on non-linear scales may be unduly pessimistic as in fact gravitational instability continues on small scales and non-linearity transfers power to smaller scales. However eventually the scalar metric perturbation must decay on the smallest scales, where the velocity of the matter becomes non-negligible. We leave a detailed numerical calculation for future work and henceforth present the predicted gravitational wave background using both the linear result on all scales and the linear result cut-off at the non-linear scale.

4 Present density of gravitational waves

The effective energy density of a stochastic background of gravitational waves, on scales much smaller than the Hubble scale, is given by [33]

$$\rho_{GW} = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle = \frac{k^2}{32\pi G a^2}\int d(\ln k)\ \mathcal{P}_h(k, \eta)\,. \tag{44}$$

Note that our second-order tensor modes (22) produced by linear scalar perturbations become constant on sub-Hubble scales in the early matter dominated era and therefore have negligible energy density at that time. They behaves quite differently from the conventional view of gravitational waves. But as the Newtonian potential Φ decays on sub-Hubble scales either due to non-linear evolution on small scales or in the radiation era on sub-Hubble scales, the source term $S_{\mathbf{k}}$ for tensor modes also decays on sub-Hubble scales and we are left with freely oscillating gravitational waves.

The fraction of the critical energy density in gravitational waves per logarithmic range of wavenumber k in the radiation era is

$$\Omega_{GW}(k, \eta) = \frac{1}{12}\left(\frac{k}{\mathcal{H}}\right)^2 \mathcal{P}_h(k, \eta)\,. \tag{45}$$

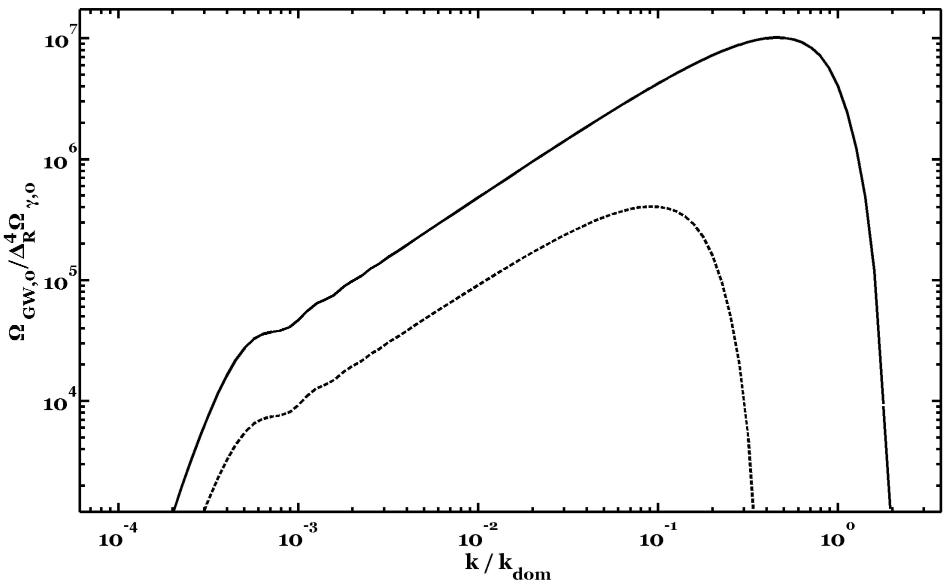

Figure 2: The present energy density of gravitational waves, $\Omega_{GW,0}$, generated during a matter dominated era shown as a function of wavenumber k. In this example $F = (k_{\rm dom}/k_{\rm dec})^2 = 10^6$. The solid line shows the result predicted using the linear matter power perturbation for $k < k_{\rm dom}$, while the dotted line shows the result using the matter power spectrum truncated at $k > k_{\rm cut} = 200 k_{\rm dec}$.

During and after the radiation-dominated era, the density of gravitational waves on sub-Hubble scales then redshifts exactly as any non-interacting relativistic particles and in the present day we have

$$\Omega_{GW,0}(k) \simeq \frac{\Omega_{\gamma,0}}{12} \left(\frac{k}{k_{\rm dec}} \right)^2 \mathcal{P}_h(k, \eta_{\rm dec}) \,, \tag{46}$$

where the present density of photons is $\Omega_{\gamma,0} \simeq 1.2 \times 10^{-5}$, and we neglect additional numerical factors due to the detailed thermal history, such as the heating of photons by the annihilation of other relativistic particle species.

4.1 Linear scalar perturbations

If we take Eq. (37) for the amplitude of tensor perturbations for $k < k_{\rm dec}$ at the start of the radiation era, when $\mathcal{H} = k_{\rm dec}$, we have

$$\Omega_{GW}(k, \eta) \simeq \frac{23}{12} \triangle_{\mathcal{R}}^4 \left(\frac{k_{\rm dom} k}{k_{\rm dec}^2} \right) I_1(k/k_{\rm dom}) \,, \tag{47}$$

and this remains constant (assuming no further production of gravitational waves on sub-Hubble scales) during the radiation era. The present day density of second-order gravitational waves produced due to first-order scalar perturbations is thus given by

$$\Omega_{GW,0}(k) \simeq \frac{23}{12} \triangle_{\mathcal{R}}^4 \Omega_{\gamma,0} \left(\frac{k_{\rm dom} k}{k_{\rm dec}^2} \right) I_1(k/k_{\rm dom}) \,, \tag{48}$$

The density as a function of wavenumber, k, is shown in Figure 2.

Whereas the power spectrum (37) at the end of the early matter dominated era has a maximum on the Hubble scale at the start of the radiation era, $k_{\rm dec}$, we find that the present density of gravitational waves is largest on comoving scales of order the Hubble size at the start of the early matter era, $k_{\rm dom}$,

$$\Omega_{GW,0}(k_{\rm dom}) \approx \triangle_{\mathcal{R}}^4 \, \Omega_{\gamma,0} \left(\frac{k_{\rm dom}}{k_{\rm dec}}\right)^2 . \tag{49}$$

We find that the maximum density of gravitational waves generated from linear density perturbations during an early matter dominated era is enhanced with respect to the naive expectation, $\Omega_{GW,0}(k_{\rm dom}) \sim \triangle_{\mathcal{R}}^4 \, \Omega_{\gamma,0} \sim 3 \times 10^{-22}$, by a factor

$$F^2 = \left(\frac{k_{\rm dom}}{k_{\rm dec}}\right)^2 . \tag{50}$$

This enhancement factor represents enhanced amplitude with respect to the gravitational waves that would be produced by scalar metric perturbations $\mathcal{P}_h(k_{\rm dom}, \eta_{\rm dom}) \sim \triangle_{\mathcal{R}}^4$ for modes which re-entered the Hubble scale at the start of the early matter era but then redshifted once inside the Hubble scale. Instead we find that the tensor perturbations remain constant even on sub-Hubble scales while they are supported by constant linear metric perturbations during the matter era, leading to $\mathcal{P}_h(k_{\rm dom}, \eta_{\rm dec}) \sim \triangle_{\mathcal{R}}^4$. The energy density of such modes during the radiation dominated era is proportional to their frequency, k^2, and we find $\Omega_{GW,0}(k_{\rm dom}) \approx F^2 \triangle_{\mathcal{R}}^4 \Omega_{\gamma,0}$.

The enhancement factor calculated by continuing to use the linear result for scalar metric perturbations, $\dot{\Phi}_{\mathbf{k}} = 0$, down to very small scales, $k_{\rm dom} \gg k_{\rm dec}$, could be very large indeed. It is determined by the duration of the early matter-dominated era, for which $\mathcal{H} = aH \propto t^{-1/3} \propto H^{1/3}$, and we have

$$F^2 = \left(\frac{H_{\rm dom}}{H_{\rm dec}}\right)^{2/3} . \tag{51}$$

If the early matter-dominated era arises due to a massive scalar field displaced from the minimum of its potential in the very early universe, then we require $H_{\rm dom} < m$, the mass of the field, and the matter era will end when the field decays, $H_{\rm dec} \sim \Gamma$, and thus

$$F^2 < \left(\frac{m}{\Gamma}\right)^{2/3} . \tag{52}$$

In addition we require that the matter fields decay before primordial nucleosynthesis and thus $\Gamma > H_{\rm BBN} \sim {\rm MeV}^2/M_{\rm Pl}$ where $M_{\rm Pl}^2 = G^{-1}$. In practice there is a tighter bound on the Hubble rate at decay for particles with mass $m > 10$ TeV, if we require that their decay rate should not be suppressed by more that the Planck scale, and thus $\Gamma > m^3/M_{\rm Pl}^2$. In the optimal case where the field is precisely 10 TeV we find the weakest bound

$$F^2 < \left(\frac{M_{\rm Pl}}{m}\right)^{4/3} \sim 10^{20} . \tag{53}$$

The bound is more restrictive for lighter fields, for which $H_{\rm dom}$ must be smaller, or more massive fields, which must decay earlier, unless their decay rate is suppressed with respect to gravitational strength decay. Nonetheless large enhancement factors would be possible in early matter-dominated eras that are sufficiently long-lived.

4.2 Non-linear cut-off

In practice we have seen that the validity of linear results for the scalar perturbations is restricted to a limited range of scales due to non-linear effects on scales $k > k_{NL}$ where k_{NL} is given by Eq. (7). Thus the maximum value of F^2 for which we can reliably use the calculation based on linear scalar perturbations is when $k_{\rm dom} = k_{NL}(\eta_{\rm dec})$ and hence

$$F^2 = \left(\frac{k_{NL}(\eta_{\rm dec})}{k_{\rm dec}}\right)^2 \simeq \mathcal{P}^{-1/2} \sim 3\times 10^4 \,. \tag{54}$$

If we assume that non-linear effects give a rapid suppression of the scalar metric perturbations on scales $k > k_{NL}$ then this non-linear cut-off suppresses the amplitude of gravitational waves, especially below the cut-off scale. For $k_{NL} < k_{\rm dom}$ we find, from Eqs. (41) and (46), that for $k_{\rm dec} < k < k_{NL}$

$$\Omega_{GW,0}(k) \simeq 4\triangle_{\mathcal{R}}^4\,\Omega_{\gamma,0}\left(\frac{k_{NL}k}{k_{\rm dec}^2}\right) \simeq 5\Delta_{\mathcal{R}}^{3.5}\,\Omega_{\gamma,0}\left(\frac{k}{k_{\rm dec}}\right) . \tag{55}$$

The maximum value of the energy density of gravitational waves at the present time is reached for $k \simeq k_{NL}$:

$$\Omega_{GW,0}(k_{NL}) \simeq 4\triangle_{\mathcal{R}}^4\left(\frac{k_{NL}}{k_{\rm dec}}\right)^2 \sim 6\triangle_{\mathcal{R}}^3\Omega_{\gamma,0} \,. \tag{56}$$

For primordial density perturbations with the same amplitude on small scales as seen on CMB scales today, $\triangle_{\mathcal{R}}^2 \simeq 2\times 10^{-9}$, this gives $\Omega_{GW,0}(k_{NL}) \sim 10^{-17}$. This result neglects any source for the gravitational waves coming from non-linear scales, $k > k_{NL}$, and thus gives a very conservative lower bound on the density of gravitational waves produced during an early matter era.

5 Detectability

5.1 Frequency ranges

The comoving Hubble scale during a radiation-dominated era is given by $k = aH$ where

$$H^2 = \frac{8\pi G}{3}\left(g_*\frac{\pi^2}{30}T^4\right) . \tag{57}$$

where g_* is the effective number of degrees of freedom at temperature T.

At the present time this corresponds to a physical scale

$$\lambda_0 = \frac{2\pi a_0}{k} \approx 2\times 10^{16}\, g_*^{-1/6}\left(\frac{\rm GeV}{T}\right) {\rm m}\,, \tag{58}$$

and a frequency

$$\nu = \frac{c}{\lambda_0} \approx 1.2\times 10^{-8}\, g_*^{1/6}\left(\frac{T}{\rm GeV}\right) {\rm Hz}\,. \tag{59}$$

The standard radiation dominated hot big bang must be restored before the epoch of primordial nucleosynthesis, and thus the temperature after the decay of an early matter-dominated era, $T_{\rm dec}$, must be above 1 MeV and thus

$$\nu_{\rm dec} > 10^{-11}\ {\rm Hz}\,. \tag{60}$$

Note that ratio between the present frequency of Hubble-scale modes at the start of the matter era and Hubble-scale modes at the end of the matter era corresponds to the enhancement factor in Eq. (50)

$$F = \frac{\nu_{\rm dom}}{\nu_{\rm dec}}\,. \tag{61}$$

Thus a large enhancement factor, F, also implies a wide range of frequencies over which gravitational waves are generated during an early matter era.

Successful, standard, big-bang nucleosynthesis (BBN) [34] places an upper bound on the density of primordial gravitational waves,

$$\Omega_{GW,0} < 0.1\Omega_{\gamma,0} \simeq 10^{-6}\,, \tag{62}$$

on any scales smaller than the Hubble scale at that time, corresponding to $\nu > \nu_{BBN} \simeq 10^{-11}$ Hz. Gravitational waves generated during a early matter era are only present on LIGO scales, $f_{\rm LIGO} \sim 100$ Hz, if $T_{\rm dec} < 10^{10}$ GeV, and on LISA scales, $\nu_{\rm LISA} \sim 10^{-3}$ Hz, if $T_{\rm dec} < 10^{5}$ GeV. Current limits from the Laser Interferometer Gravitational-wave Observatory (LIGO) bound $\Omega_{GW,0} < 6\times 10^{-5}$ in the frequency range $\nu_{\rm LIGO} \sim 100$ Hz with Advanced LIGO sensitive to $\Omega_{\rm GW,0} \sim 10^{-9}$ in the future [35]. LISA should be able to detect a primordial GW background $\Omega_{GW,0} \sim 10^{-11}$ at frequencies $\nu_{\rm LISA} \sim 10^{-3}$ Hz [36], and future experiments such as Big Bang Observer (BBO) may be able to detect a primordial background $\Omega_{GW,0} \sim 10^{-17}$ at frequencies $\nu_{\rm BBO} \sim 1$ Hz [37].

5.2 Reheating after GUT-scale inflation

Most inflationary models of the early universe incorporate an early matter dominated period immediately after inflation when the energy density of the universe is dominated by oscillating, massive scalar fields. If the fields decay perturbatively to radiation this corresponds to a period of reheating. Current upper bounds from the CMB on first-order gravitational waves produced from vacuum fluctuations during inflation [38] place a bound on the maximum value of the energy density during inflation, $\rho_{\rm inf}^{1/4} = M_{\rm inf} < 10^{16}$ GeV, and thus the Hubble scale at the start of the early matter era.

Models such as chaotic inflation driven by a massive scalar field come close to saturating this bound and thus correspond to GUT-scale inflation. On the other hand constraints from the thermal production of gravitinos in supersymmetric models suggest that the maximum reheat temperature, $T_{\rm dec}$, should be less than about 10^9 GeV [34][2]. This implies, from Eq. (51), a lower bound on the enhancement factor, $F^2 > 10^8$, suggesting that the power spectrum of second-order gravitational waves generated during reheating after GUT-scale

[2]Note that in the case of reheating at the end of chaotic inflation, the mass of the inflaton is required to be $m \sim 10^{13}$ GeV, and delaying reheating so that $T_{\rm dec} < 10^9$ GeV then actually violates the bound given in Eq. (53) for this mass.

inflation could be larger than that from first-order gravitational waves at frequencies $\nu \sim \nu_{\rm dom}$, and possibly much larger.

Taking Eq. (10) for linear density perturbations on all scales $k < k_{\rm dom}$, we have, from Eqs. (49) and (51),

$$\Omega_{GW,0}(\nu_{\rm dom}) \approx 3\times 10^9 g_*^{-1/3} \Delta_{\mathcal{R}}^4 \Omega_{\gamma,0} \left(\frac{M_{\rm inf}}{10^{16}\ {\rm GeV}}\right)^{4/3} \left(\frac{T_{\rm dec}}{10^9\ {\rm GeV}}\right)^{-4/3}, \quad (63)$$

$$\approx 10^{-12} g_*^{-1/3} \left(\frac{M_{\rm inf}}{10^{16}\ {\rm GeV}}\right)^{4/3} \left(\frac{T_{\rm dec}}{10^9\ {\rm GeV}}\right)^{-4/3}, \quad (64)$$

where in the second line we set $\Delta_{\mathcal{R}}^2 \simeq 2.4\times 10^{-9}$. In this case the nucleosynthesis limit on the density of primordial gravitational waves (62) places a lower limit on the minimum allowed reheating temperature, $T_{\rm dec} > 10^3$ GeV, assuming $M_{\rm inf} \sim 10^{16}$ GeV. This value of $\Omega_{GW,0}$ is obtained at high frequencies

$$\nu_{\rm dom} \approx 7\times 10^5\ {\rm Hz} \left(\frac{T_{\rm dec}}{10^9\ {\rm GeV}}\right)^{1/3} \left(\frac{M_{\rm inf}}{10^{16}\ {\rm GeV}}\right)^{2/3}, \quad (65)$$

which is always above the range of detectors such as LIGO.

At the lower end of the range of generated frequencies, we have from Eq. (59) that $\nu_{\rm dec} \sim 10 (T_{\rm dec}/10^9\ {\rm GeV})$ Hz, which is always below the LIGO range if $T_{\rm dec} < 10^{10}$ GeV. Using the linear perturbations (10) for all $k < k_{\rm dom}$, the density of gravitational waves at LIGO frequencies is then, from Eq. (46),

$$\Omega_{GW,0}(\nu_{\rm LIGO}) \approx 5\times 10^5 g_*^{-1/3} \Delta_{\mathcal{R}}^4 \Omega_{\gamma,0} \left(\frac{T_{\rm dec}}{10^9\ {\rm GeV}}\right)^{-5/3} \left(\frac{M_{\rm inf}}{10^{16}\ {\rm GeV}}\right)^{2/3} \left(\frac{\nu_{\rm LIGO}}{100{\rm Hz}}\right). \quad (66)$$

Taking $\Delta_{\mathcal{R}}^2 \simeq 2\times 10^{-9}$, this would be within the sensitivity of Advanced LIGO if the reheat temperature takes a low value, $T_{\rm dec} < 10^5$ GeV:

$$\Omega_{GW,0}(\nu_{\rm LIGO}) \sim 10^{-9} \left(\frac{T_{\rm dec}}{10^5\ {\rm GeV}}\right)^{-5/3} \left(\frac{M_{\rm inf}}{10^{16}\ {\rm GeV}}\right)^{2/3} \left(\frac{\nu_{\rm LIGO}}{100{\rm Hz}}\right), \quad (67)$$

However Eqs. (63–67) assume large enhancement factors, extrapolating the linear power spectrum (10) well into the non-linear regime on small scales, $k_{\rm dom} \ll k_{NL}$ defined in Eq.(7). If we assume non-linear effects cut-off the matter power spectrum below $k_{NL} \simeq 200 k_{\rm dec}$ for $\Delta_{\mathcal{R}}^2 \simeq 10^{-9}$, then from Eq. (55) we have for $\nu_{\rm dec} < \nu_{\rm LIGO} < \nu_{NL}$

$$\Omega_{GW,0}(\nu_{\rm LIGO}) \approx 50 g_*^{-1/6} \Delta_{\mathcal{R}}^{3.5} \Omega_{\gamma,0} \left(\frac{T_{\rm dec}}{10^9\ {\rm GeV}}\right)^{-1} \left(\frac{\nu_{\rm LIGO}}{100{\rm Hz}}\right) \quad (68)$$

$$\approx 5\times 10^{-19} \left(\frac{T_{\rm dec}}{10^9\ {\rm GeV}}\right)^{-1} \left(\frac{\nu_{\rm LIGO}}{100{\rm Hz}}\right). \quad (69)$$

The amplitude increases for lower $T_{\rm dec} < 10^9$ GeV but it reaches a maximum value, $\Omega_{GW,0} \sim 10^{-17}$, at

$$\nu_{NL} \approx 2\times 10^3\ {\rm Hz} \left(\frac{T_{\rm dec}}{10^9\ {\rm GeV}}\right), \quad (70)$$

and this drops below the LIGO range of frequencies for $T_{\rm dec} < 5\times 10^7$ GeV. Advanced LIGO would thus only be able to detect a primordial gravitational wave background generated by density perturbations in the linear regime during reheating if the density perturbation $\Delta^2_{\mathcal{R}} \sim 10^{-3}$ and $T_{\rm dec}\sim 10^{10}$ GeV. However future experiments such as BBO could detect primordial gravitational waves generated during reheating even for $\Delta^2_{\mathcal{R}}\sim 10^{-9}$ if $T_{\rm dec}\sim 10^6$ GeV.

5.3 Reheating after intermediate-scale inflation

In many models of inflation the energy scale of inflation is set by the intermediate scale $\rho_{\rm inf}^{1/4} = M_{\rm inf}\simeq 10^{10}$ GeV [39]. Such models include hybrid inflation [40, 41, 42] in which case the inflaton mass and Hubble-scale at the end of inflation is of order TeV. Tachyonic instability leads to the end of inflation when the inflaton reaches a critical value and the vacuum energy density driving inflation is rapidly converted to oscillating scalar fields. The spatially coherent oscillations of the inflaton fields rapidly fragment [43], but if a significant fraction of the energy density goes into weakly coupled massive fields which decay perturbatively then we may have an extended period of reheating after inflation.

The lower energy scale during inflation, compared with GUT-scale inflation, leads to a lower frequency of gravitational waves produced at the end of inflation. From Eq. (65) we have

$$\nu_{\rm dom} \approx 0.7 \text{ Hz } \left(\frac{T_{\rm dec}}{10^3 \text{ GeV}}\right)^{1/3} \left(\frac{M_{\rm inf}}{10^{10}\text{GeV}}\right)^{2/3} , \tag{71}$$

If $T_{\rm dec} < 10^9$ GeV then this is below the frequency range for LIGO but could be detectable by BBO if $T_{\rm dec}\sim 10^3$ GeV. Indeed if the inflaton or other moduli field excited at the end of inflation are weakly coupled and decay only just before big bang nucleosynthesis, $T_{\rm dec}\sim 1$ MeV, we have $\nu_{\rm dom}\sim\nu_{\rm LISA}$ and a maximal enhancement factor $F^2\sim 10^{17}$ for $M_{\rm inf}\sim 10^{10}$ GeV, which would be large enough to violate the BBN bound in Eq. (62).

Such large enhancement factors imply that the small scale density perturbations are far into the non-linear regime. If non-linear evolution suppresses the scalar perturbations on scales $k > k_{NL}$ defined in Eq. (7), then the gravitational waves generated are suppressed for $\nu > \nu_{NL}$ where

$$\nu_{NL} \approx 2\times 10^{-3} \text{ Hz } \left(\frac{T_{\rm dec}}{10^3 \text{ GeV}}\right) , \tag{72}$$

and we have set $\Delta^2_{\mathcal{R}}\approx 2.4\times 10^{-9}$. Thus we find $\nu_{NL} < \nu_{\rm LISA}$ for $T_{\rm dec} < 10^3$ GeV. For $T_{\rm dec}\sim 10^4$ GeV, LISA could detect this background if $\Delta^2_{\mathcal{R}}\approx 10^{-5}$ on these scales.

5.4 Moduli or curvaton domination

Any scalar field, with mass m, displaced from the minimum of its effective potential after inflation begins to oscillate with an initial amplitude $\chi_{\rm osc}$ either at the end of inflation, when the Hubble rate is $H_{\rm osc}\simeq H_{\rm inf}$ if $m > H_{\rm inf}$, or once the Hubble rate drops below the mass, $H_{\rm osc}\simeq m$, if $m < H_{\rm inf}$. The fractional energy density of the field at this time is

$$\Omega_{\chi,{\rm osc}} = \frac{4\pi}{3}\left(\frac{m}{H_{\rm osc}}\right)^2 \left(\frac{\chi_{\rm osc}}{M_{\rm Pl}}\right)^2 . \tag{73}$$

Note that $\Omega_{\chi,\rm osc} < 1$ since $\chi_{\rm osc} < M_{\rm Pl}$, otherwise the field would slow-roll and drive a period of inflation. During a radiation dominated era following inflation, the density of the oscillating field grows relative to the radiation, eventually coming to dominate the energy density and drive an early matter era when the Hubble rate drops below [44]

$$H_{\rm dom} \simeq \Omega^2_{\chi,\rm osc} H_{\rm osc} \simeq \left(\frac{4\pi}{3}\right)^2 \left(\frac{m}{H_{\rm osc}}\right)^3 \left(\frac{\chi_{\rm osc}}{M_{\rm Pl}}\right)^4 m\,. \tag{74}$$

The duration of the early matter era depends on the decay time of the field, $H_{\rm dec} = \Gamma$, which must be before primordial nucleosynthesis. The duration is described by the factor F^2 defined in Eq. (51)

$$F^2 \simeq 10^{20} g_*^{-1/3} \left(\frac{\chi_{\rm osc}}{M_{\rm Pl}}\right)^{8/3} \left(\frac{m}{H_{\rm osc}}\right)^2 \left(\frac{m}{10 \text{ TeV}}\right)^{2/3} \left(\frac{T_{\rm dec}}{\rm MeV}\right)^{-4/3} , \tag{75}$$

which can be very large if $\chi_{\rm osc} \sim M_{\rm Pl}$, becoming largest for fields which decay just before nucleosynthesis, $T_{\rm dec} \sim 1$ MeV. Equation (75) is consistent with Eq. (63) for the amplitude of gravitational waves produced from reheating at the end of inflation if we set $\chi_{\rm osc} \sim M_{\rm Pl}$ and $H_{\rm osc} \sim m \sim H_{\rm inf}$.

If a moduli field is light during inflation, $m < H_{\rm inf}$, then it acquires a spectrum of perturbations on large, super-Hubble scales, from initial vacuum fluctuations on small, sub-Hubble scales. In the curvaton scenario [23, 24, 25, 26] these inhomogeneities about the background expectation value of the field is supposed to give rise to the primordial density perturbation, and hence in the simplest curvaton model in which the curvaton comes to dominate the total energy density of the universe before decaying, observations require [25]

$$\Delta^2_{\mathcal{R}} \simeq \left(\frac{H_{\rm inf}}{3\pi\chi_{\rm osc}}\right)^2 . \tag{76}$$

This determines the initial amplitude of the curvaton oscillations in terms of the inflationary energy scale, $\chi_{\rm osc} \sim \Delta^{-1}_{\mathcal{R}} H_{\rm inf}$, and hence from Eqs. (49) and (75) we can calculate the peak amplitude of gravitational waves generated from linear density perturbations

$$\Omega_{GW,0}(\nu_{\rm dom}) \simeq 6 \times 10^{-14} g_*^{-1/3} \left(\frac{H_{\rm inf}}{10^7 \text{ GeV}}\right)^{8/3} \left(\frac{m}{10 \text{ TeV}}\right)^{2/3} \left(\frac{T_{\rm dec}}{\rm MeV}\right)^{-4/3} , \tag{77}$$

where we have taken $\Delta^2_{\mathcal{R}} \simeq 2.4 \times 10^{-9}$ and $m = H_{\rm osc} < H_{\rm inf}$, and we note that there is a lower bound on the Hubble rate during inflation, $H_{\rm inf} > 10^7$ GeV [45] in the simplest curvaton models. Note also that if the curvaton comes to dominate the energy density of the universe before it decays, the primordial density perturbation is close to Gaussian (the non-linearity parameter $f_{\rm NL} \simeq -5/4$ [46, 47, 48, 49, 50]).

Thus we see that an early matter dominated era may have a long duration in the curvaton scenario leading to a large enhancement factor in our calculation based on the linear density perturbation. However, as in the case of reheating after intermediate-scale inflation, the frequency is below LIGO scales if the decay temperature is low. The highest frequency GW generated in the curvaton-dominated era would be at frequency

$$\nu_{\rm dom} = F \nu_{\rm dec} \simeq 10^{-7} \left(\frac{H_{\rm inf}}{10^7 \text{ GeV}}\right)^{4/3} \left(\frac{m}{10 \text{ TeV}}\right)^{1/3} \left(\frac{T_{\rm dec}}{\rm MeV}\right)^{1/3} \text{ Hz}\,, \tag{78}$$

where the lowest frequencies would be, from Eq. (59),

$$\nu_{\rm dec} \simeq 10^{-11} \left(\frac{T_{\rm dec}}{\rm MeV}\right) \; {\rm Hz}\,. \tag{79}$$

If non-linearities suppress the density perturbation on sub-Hubble scales in an early matter era then the only gravitational waves that are not reshifted away will be on the lowest frequencies, $\nu < \nu_{\rm NL}$ and we recover the same results as in the case of reheating after inflation, given in sub-sections (5.2) and (5.3) determined by the final decay temperature, $T_{\rm dec}$.

6 Conclusions

In a matter-dominated era the behaviour of the linearised density perturbations is particularly simple. The gravitational instability of pressureless matter leads to the growth of the comoving density perturbation, while the Newtonian metric potential, Φ, remains constant on all scales. This provides a constant source term, $S_{\mathbf{k}}$, at second-order in the wave equation for tensor metric perturbations, leading to the general solution given in Eq. (21). At late times this approaches the constant solution, given in Eq. (25), on sub-Hubble scales.

This constant tensor mode during the matter era is quite different from the behaviour we would expect for a free gravitational wave on sub-Hubble scales. Indeed the free part of the tensor perturbation, given by the oscillating terms in Eq. (21), are redshifted away $|h_{\mathbf{k}}| \propto a^{-1}$, diluting any pre-existing gravitational waves on sub-Hubble scales at the start of a matter-era. But the constant amplitude of tensor metric perturbations in the Poisson gauge, supported by first-order metric perturbations in an early matter era, $h_{\mathbf{k}} \propto S_{\mathbf{k}}/k^2$, becomes the initial condition for the amplitude of freely oscillating gravitational waves in the subsequent radiation era when Φ rapidly decays on sub-Hubble scales.

We have seen that a constant amplitude of scalar metric perturbations, $\mathcal{P}_\Phi \sim \Delta^2_{\mathcal{R}}$, extending down to sub-Hubble scales can lead to a large enhancement in the amplitude of gravitational waves. On the smallest scales which we considered, $k_{\rm dom}$ corresponding to the comoving Hubble scale at the start of matter domination, the enhancement factor is easily understood due to the amplitude of tensor perturbations on this scale remaining constant during the matter dominated era instead of being redshifted on sub-Hubble scales. As a result the energy density of gravitational waves in the late universe is enhanced with respect to the naive estimate $\Omega_{GW,0} \sim \triangle^4_{\mathcal{R}}\Omega_{\gamma,0}$, by a factor $F^2 = (k_{\rm dom}/k_{\rm dec})^2$, where $k_{\rm dec}$ is the comoving Hubble scale at the end of the matter era. For $F^2 > r\Delta^{-2}_{\mathcal{R}} \sim 4 \times 10^8 r$, where $r \ll 1$ is the usual tensor-scalar ratio at first-order [38], the second-order gravitational waves generated from linear perturbations during an early matter dominated era would dominate over those produced directly from vacuum fluctuations in the gravitational field during inflation. Large enhancement factors occur when there is a sufficiently long matter dominated era in the early universe, preceding the standard radiation dominated era required for successful primordial nucleosynthesis. This is certainly possible given our state of ignorance of the thermal history of the universe before nucleosynthesis. The spectrum of gravitational waves could provide one of the few constraints on the universe before nucleosynthesis.

However, a sufficiently long-lived matter era also predicts a breakdown of linear perturbation theory for density perturbations on sub-Hubble scales. Predictions based on linear theory can only be trusted for $F^2 < (k_{NL}/k_{\rm dec})^2 \sim \Delta_{\mathcal{R}}^{-1}$. A constant scalar metric perturbation in linear theory in the longitudinal gauge, Φ, implies a growing comoving density perturbation, $\delta\rho/\rho$. If non-linear evolution of the matter perturbations suppresses the scalar source term on scales $k > k_{NL}$ then the peak in the gravitational wave density is shifted from $k_{\rm dom}$ to k_{NL} and the maximum amplitude at this peak is reduced to $\Omega_{GW,0} \sim 6\Delta_{\mathcal{R}}^3\Omega_{\gamma,0} \sim 10^{-17}$ for $\Delta_{\mathcal{R}}^2 \simeq 2\times 10^{-9}$ which would only be detectable by experiments in the far future such as the Big Bang Observer.

In summary we have seen that growing density perturbations during an early matter era could lead to a detectable background of primordial gravitational waves. But this is only detectable with planned experiments such as LIGO and LISA if the primordial metric perturbations are significantly larger on small scales ($\Delta_{\mathcal{R}}^2 \sim 6\times 10^{-4}$ for LIGO, or $\Delta_{\mathcal{R}}^2 \sim 3\times 10^{-5}$ for LISA) than currently observed on very large scales, where $\Delta_{\mathcal{R}}^2 \simeq 2\times 10^{-9}$, or if significant non-linear density perturbations survive on sub-Hubble scales during the early matter era. This latter case requires non-linear (and hence probably numerical) solutions for the density perturbations. Because these perturbations are on sub-Hubble scales, Newtonian theory may be sufficient on these scales. There has already been investigation of gravitational waves generated by nonlinear density perturbations on sub-Hubble scales during preheating after inflation [2, 3, 4, 5, 7, 8, 9], and our work suggests that similar numerical calculations will also be required to investigate whether a detectable level of gravitational waves could be produced from gravitational instability in an early matter era.

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