Quick Completeness for the Evidential Conditional

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1 Introduction

Conditionals are notoriously difficult to analyse. Conditionals are natural language sentences of the form 'if A then C', where A is called the antecedent and C the consequent of the conditional. A standard account has however emerged in the 70'ies, the so-called possible world account (Lewis, 1973b; Stalnaker, 1968). This account even spread into the fields of linguistics and formal semantics in the work of Kratzer (1979), and, in some form or another, into the domain of the psychology of reasoning (Over, 2009). According to this account, a conditional $A > C$ is true in the actual world if and only if the closest A-worlds to the actual world are C-worlds. However, recent reflections and analysis suggest that the defining clause is not strong enough and that we may want to add some additional conditions. What these conditions is not settled. Different approaches argue for different conditions (Crupi & Iacona, 2019; Krzyżanowska, Wenmackers, & Douven, 2013; Raidl, 2018; Rott, ms; Spohn, 2015). Some of these logics are not worked out yet, or they are only worked out for specific models. To get a grasp to compare them, we need to know what kind of logics they generate depending on the underlying closeness analysis. This article proposes a general method which generates completeness results for such strengthened conditionals. In particular, the article proves completeness for the evidential conditional introduced by Crupi and Iacona (2019).

The *general problem* is this: Imagine that you have a strengthened conditional of the form

- $\varphi \rhd \psi$ in world w iff closest φ-worlds are ψ-worlds and X.

Suppose additionally that X is also formulated in terms of the closeness apparatus. It thus seems that one can rephrase the conditional $\varphi \rhd \psi$ in the language for $>$, namely as $(\varphi > \psi) \wedge Y$, where Y is the sentence expression corresponding to the semantic condition X. The *main question* is this:

- Can we use known completeness results for $>$ to obtain completeness results for $\rhd$?

The answer is yes and the paper provides a general method. The idea is roughly this. First redefine $>$ in terms of $\rhd$. This backtranslation of $\varphi > \psi$ will roughly yield a sentence Z in the language with $\rhd$. If everything is well behaved, and bracketing some details, one can use this backtranslation to translate axioms for $>$ into axioms for $\rhd$. In other words, the back-translation is a looking glass which provides a distorted picture of the logic for $>$, in terms of $\rhd$. The method can be applied to Lewis' (1973a) causal dependency, to Spohn's (2015) sufficient and necessary reason relation, to Rott's (ms) difference making conditional and dependency conditional, to Raidl's (2018) neutral conditional, doxastic conditional and metaphysical conditional, to Crupi and Iacona's (2019) evidential conditional and even to counterpossible conditionals in the style of Berto, French, Pries, and Ripley (2018). And the list probably continues. The hope is indeed, that more complex conditionals could be treated as well, such as Spohn's (2015) supererogatory and insufficient reason as well as the plain reason relation, or the probabilistic conditional of van Rooij and Schulz (2019). Here I will implement the method for the evidential conditional.

The plan of the paper is this: §2 introduces my framework for the basic conditional and the definition of the evidential conditional. §3 introduces the general method to generate completeness for new conditionals. §4 recalls axioms for the basic conditional and states the axioms for the evidential conditional. §5 proves soundeness and §6 completeness for the evidential conditional.

2 The Evidential Conditional

This section briefly rehearses the standard analysis of the conditional and then introduces the evidential conditional.

The usual analysis of the conditional (Stalnaker 1968, Lewis 1973b) is as follows

Basic Conditional: $\varphi > \psi$ is true in world w iff closest φ-worlds are ψ-worlds.

I will call this the *basic conditional*.[1] There are several ways to analyse closeness. The central ones, being either in terms of a Lewisean similarity relation, or in terms of systems

[1] The term is borrowed from Chellas (1975) and here refers to the general idea that more complex conditionals can be based on the basic conditional, making them definable.

of spheres, or in terms of a selection function. Under the right assumptions these analysis yield the same logic (Lewis 1971). I will here use the more flexible world selection function analysis.

A *selection function* $F : W \times \wp(W) \longrightarrow \wp(W)$, associates to each world and each proposition (i.e., subset of W) a proposition. Additionally, we assume F has the following properties:

- $F(w, A) \subseteq A$. (id)
- If $F(w, A) \subseteq C$ and $F(w, B) \subseteq C$ then $F(w, A \cup B) \subseteq C$. (ca)
- If $F(w, A) \subseteq C$ and $F(w, A) \subseteq B$, then $F(w, A \cap B) \subseteq C$. (cmon)
- If $F(w, A) \subseteq C$ and $F(w, A) \nsubseteq \overline{B}$, then $F(w, A \cap B) \subseteq C$. (cv)
- If $w \in A$, then $F(w, A) = \{w\}$. (cs)

I will call a selection function *Lewisean* if it satisfies (id), (ca), (cmon) and (cv), and *strongly Lewisean* if it also satisfies (cs). Our ground propositional language $\mathcal{L}$ is generated from propositional variables Var, negation $\neg$, conjunction $\wedge$, disjunction $\vee$ and the material conditional $\rightarrow$. We extend $\mathcal{L}$ in two ways: $\mathcal{L}_{>}$ extends $\mathcal{L}$ by the conditional $>$, intended to represent the basic conditional and $\mathcal{L}_{\rhd}$ extends $\mathcal{L}$ by the conditional $\rhd$, intended to represent the evidential conditional.

A *frame* $\mathfrak{F} = \langle W, F \rangle$ is given by a set of worlds W and a selection function F. A *model* $\mathfrak{M} = \langle W, F, V \rangle$ is given by a frame $\langle W, F \rangle$ and a valuation $V : \text{Var} \longrightarrow \wp(W)$ which associates to every propositional variable p of $\mathcal{L}$ a set of worlds to be interpreted as the worlds in which p is true. A frame (or model) is called *Lewisean* (*strongly Lewisean*) if F is a Lewisean (strongly Lewisean) selection function.

The *truth clauses for* $\mathcal{L}$ in a model are as follows:

- $w \vDash p$ iff $p \in V(p)$,
- $w \vDash \neg\varphi$ iff $w \nvDash \varphi$,
- $w \vDash \varphi \wedge \psi$ iff $w \vDash \varphi$ and $w \vDash \psi$,
- $w \vDash \varphi \vee \psi$ iff $w \vDash \varphi$ or $w \vDash \psi$,
- $w \vDash \varphi \rightarrow \psi$ iff if $w \vDash \varphi$ then $w \vDash \psi$ [alternatively: $w \nvDash \varphi$ or $w \vDash \psi$].

In what follows, we will consider two augmented versions of $\mathfrak{M}$. The first, denoted $\mathfrak{M}^{>}$, is $\mathfrak{M}$ where the conditional is interpreted as the basic conditional $>$. The second, denoted $\mathfrak{M}^{\rhd}$, is $\mathfrak{M}$ where the conditional is interpreted as the evidential conditional $\rhd$.[2]

Let us start with $\mathfrak{M}^{>}$ and the basic conditional. The truth clause encoding the above intuitive definition of the basic conditional in terms of the selection function is:

[2]Another way to see it is that we have one model $\mathfrak{M}$, but since we have two languages $\mathcal{L}_{>}$ and $\mathcal{L}_{\rhd}$, we have two sets of truth clauses, which one could denote $\vDash^{>}$ and $\vDash^{\rhd}$, which differ in the evaluation of $>$ and $\rhd$. The option adopted here is however to consider the truth clauses part of the interpreted model.

BC: $w \vDash \varphi > \psi$ iff $F(w, [\varphi]) \subseteq [\psi]$,

where $[\varphi]$ abbreviates $[\varphi]^{\mathfrak{M}^{>}} = \{w \in W : w \vDash \varphi\}$ which is the proposition expressed by φ in the model $\mathfrak{M}$ where $>$ is interpreted as the basic conditional as above. BC encodes the above idea of the basic conditional.

A recent trend suggests that (indicative) conditionals might follow another idea. Namely that $\varphi \triangleright \psi$ holds if one can justifiably infer ψ from φ, or if φ is a reason for ψ, or if φ makes a difference to ψ, or if φ supports ψ. Several analysis and motivations in this direction have been suggested, for example by Krzyżanowska et al. (2013), Rott (ms), Spohn (2015). Since this article concentrates on proving a completeness result, I will not rehearse the different motivations and refer the reader to the mentioned authors.

In this article I will consider an account, recently proposed by Crupi and Iacona (2019). Their conditional is called the *evidential conditional*. The definition is roughly:

Evidential Conditional: $\varphi \triangleright \psi$ is true in world w iff the closest φ-worlds are ψ worlds and the closest non-ψ-worlds are non-φ-worlds.

They analyse closeness based on a system of spheres, which they assume to be strongly centred (and satisfying the limit assumption). It is known that we can restate this equivalently in terms of a strongly Lewisean selection function F (Lewis, 1971). The truth clause encoding the above intuitive definition of the evidential conditional in terms of the selection function is:

EC: $w \vDash \varphi \triangleright \psi$ iff $F(w, [\varphi]) \subseteq [\psi]$ and $F(w, [\neg\psi]) \subseteq [\neg\varphi]$.

where $[\varphi]$ is an abbreviation of $[\varphi]^{\mathfrak{M}^{\triangleright}} = \{w \in W : w \vDash_{\mathfrak{M}^{\triangleright}} \varphi\}$, where now $\mathfrak{M}^{\triangleright}$ is the model $\mathfrak{M}$ but with the conditional interpreted as the evidential conditional, as given above. Intuitively, we have thus one model $\mathfrak{M}$, which splits into two interpreted versions, which can each be seen as a different interpreted models: $\mathfrak{M}^{>}$ is $\mathfrak{M}$ with the conditional interpreted as the basic conditional and $\mathfrak{M}^{\triangleright}$ is $\mathfrak{M}$ with the conditional interpreted as the evidential conditional. The only thing that changes is the truth clause for the conditional. In $\mathfrak{M}^{>}$ it is the usual clause BC for the basic conditional. In $\mathfrak{M}^{\triangleright}$ it is the clause EC for the evidential conditional. The difference between $[\varphi]^{\mathfrak{M}^{\triangleright}}$ and $[\varphi]^{\mathfrak{M}^{>}}$ only appears when φ contains as subformula a conditional. Thus if one restricts to the non-nested fragment where conditionals don't contain conditionals as proper subformulas and conditionals do not enter into boolean combinations, the difference only appears when φ is itself a conditional.

Just looking at the above definition (EC), it becomes visible, that $\varphi \triangleright \psi$ is of the form $(\varphi > \psi) \wedge (\neg\psi > \neg\varphi)$, if we had $>$ in the language $\mathcal{L}_{\triangleright}$. However we don't. This 'definability' feature will be of central importance, to my analysis. Crupi and Iacona (2019) prove some essential validites for $\triangleright$ in strong Lewisean models. What is missing is a completeness result. Hence we do not know which logic characterises the evidential conditional in the mentioned models. This is the fundamental question I will address here:

- What is the sound and complete logic for $\triangleright$ in strong Lewisean models?

The way I answer this question has a twist. I use the known soundness and completeness results for the basic conditional $>$ to obtain soundness and completeness for $\triangleright$. The method can be depicted as a knowledge transfer. In fact the method is rather general, and can be applied to any conditional $\blacktriangleright$ 'definable' from a basic conditional $>$ provided one can do the converse, namely define $>$ in terms of $\blacktriangleright$.[3] The method is here applied to the evidential conditional. The resulting logic for $\triangleright$ is of course not the same as the one for $>$. It is a distorted image of the logic for $>$. The distortion comes from the above converse definability of $>$ in terms of $\triangleright$, what I will call more precisely the 'backtranslation' of $>$ into $\triangleright$. This backtranslation is the looking-glass which creates the distorted image and generates axioms for $\triangleright$ from axioms for $>$.

3 The Method

This section develops a theory to derive soundness and completeness results of a defined conditional $\triangleright$, from known soundness and completeness of the defining basic conditional $>$, using the way $\triangleright$ is 'defined' from $>$ *and* the way $>$ can be 'redefined' from $\triangleright$. Since $>$ is not in the language $\mathcal{L}_\triangleright$ and $\triangleright$ not in the language $\mathcal{L}_>$, we can not speak of interdefinability properly. This is why I use another terminology. Instead of saying improperly that $\triangleright$ is defined from $>$, I make this precise by a translation, denoted $\circ$, mapping sentences from $\mathcal{L}_\triangleright$ to sentences in $\mathcal{L}_>$ which capture their meaning in terms of the basic conditional $>$. Conversely, instead of saying improperly that $>$ is definable from $\triangleright$, I make this precise by a back-translation, denoted $*$, mapping sentences from $\mathcal{L}_>$ to sentences in $\mathcal{L}_\triangleright$ which captures their meaning in terms of the defined conditional $\triangleright$. To be precise and because of the above remarks, 'defined conditional' is also used improperly (unless one restricts to the non-nested fragment). The method is generally developed in Raidl (2019ms), but for the purpose of self-containement of the article, it is reproduced here in a cooked down version.

Definition 1. Let $\mathcal{L}_\triangleright$ and $\mathcal{L}_>$ be conditional languages. A *conditional translation* is a total function $\circ\colon \mathcal{L}_\triangleright \longrightarrow \mathcal{L}_>$ such that $p^\circ = p, (\neg\varphi)^\circ = \neg\varphi^\circ, (\varphi \bullet \psi)^\circ = (\varphi^\circ \bullet \psi^\circ)$ for $\bullet \in \{\wedge, \vee, \rightarrow\}$, $\top^\circ = \top$, $\bot^\circ = \bot$, and there is a sentence $\theta[p,q] \in \mathcal{L}_>$ such that $(\varphi \triangleright \psi)^\circ = \theta[\varphi^\circ/p, \psi^\circ/q]$.

For the evidential conditional, the conditional translation is given by

- $(\varphi \triangleright \psi)^\circ = (\varphi^\circ > \psi^\circ) \wedge (\neg\psi^\circ > \neg\varphi^\circ)$

The translation maps sentences of $\mathcal{L}_\triangleright$ to sentences of $\mathcal{L}_>$. This mapping is such that the sentence in $\mathcal{L}_\triangleright$ is mapped to the sentence in $\mathcal{L}_>$ which expresses its meaning using $>$ instead of $\triangleright$. The term 'translation' is to highlight the fact that we replace expressions from $\mathcal{L}_\triangleright$ by expressions in $\mathcal{L}_>$ – as if we were translating english to denglish where, say, we only replace some of the english words by german words. The qualifier 'conditional' is to specify that in the recursive form of the translation, it is only the conditional which makes translating necessary – in the above analogy, we are only replacing one particular

[3] In its generality, the method is developed in Raidl (2019ms) and applied to several other examples.

english sentence form (the conditional form) by a german one, which thus creates recursive echoes.

For the method to work, it will be essential to have a converse translation of $>$ into $\rhd$. I call this a *backtranslation*. We will later show that the following provides such a back-translation (in our underlying models):

- $(\varphi > \psi)^* = (\varphi^* \wedge \psi^*) \vee (\varphi^* \rhd (\varphi^* \wedge \psi^*))$

As a reminder, we consider two languages $\mathcal{L}_>$ and $\mathcal{L}_\rhd$. Given a model $\mathfrak{M}$, we denote $\mathfrak{M}^>$ the version where the conditional is interpreted basically $>$ and $\mathfrak{M}^\rhd$ where it is interpreted (intuitively) in a semantically augmented way by a new conditional $\rhd$ (ultimately as the evidential conditional). Similarly for model classes which we denote $M, M^>$ and $M^\rhd$. By this assumption, we already have that each model $\mathfrak{M}^>$ in $M^>$ corresponds to a model $\mathfrak{M}^\rhd$ in $M^\rhd$ and conversely, furthermore the worlds of $M^>$ and $M^\rhd$ are exactly the same.

Definition 2. Let $\circ\colon \mathcal{L}_\rhd \longrightarrow \mathcal{L}_>$ be a translation. The model class $M^\rhd$ is $\circ$-*isomorphic* to the model class $M^>$, written $M^\rhd \overset{\circ}{\approx} M^>$ iff for all $\varphi \in \mathcal{L}_\rhd$, all $\mathfrak{M}^\rhd$ in $M^\rhd$, and all $w \in W(\mathfrak{M}^\rhd) = W(\mathfrak{M}^>)$: $w \vDash_{\mathfrak{M}^\rhd} \varphi$ iff $w \vDash_{\mathfrak{M}^>} \varphi^\circ$.[4]

An intuitive way of phrasing this is that $M^\rhd$ is isomorphic to $M^>$ modulo $\circ$ and in fact each model in $\mathfrak{M}^\rhd$ in $M^\rhd$ is isomorphic modulo $\circ$ to the corresponding model $\mathfrak{M}^>$ in $M^>$.

Definition 3. Let $\circ\colon \mathcal{L}_\rhd \longrightarrow \mathcal{L}_>$ and $\Sigma_>$, $\Sigma_\rhd$ systems in $\mathcal{L}_>$ and $\mathcal{L}_\rhd$ respectively. $\Sigma_>$ *simulates* $\Sigma_\rhd$ *modulo* $\circ$, $\Sigma_\rhd \overset{\circ}{\propto} \Sigma_>$, iff for every $\varphi \in \mathcal{L}_\rhd$, $\Sigma_\rhd \vdash \varphi$ implies $\Sigma_> \vdash \varphi^\circ$.

The easiest way to understand this notion is that $\Sigma_>$ simulates $\Sigma_\rhd$ if every proof in $\Sigma_\rhd$ can be simulated by a corresponding proof in $\Sigma_>$, the conclusion of the proof is the same modulo the translation $\circ$.

The following should be clear (proof omitted):

Lemma 1. *For* $\circ\colon S(\mathcal{L}_\rhd) \longrightarrow S(\mathcal{L}_>)$ *a translation and* $M^\rhd \overset{\circ}{\approx} M^>$*:* $M^> \vDash \varphi^\circ$ *iff* $M^\rhd \vDash \varphi$.

We are now ready for our two central results. First, with a $\circ$-isomorphism and a simulation, we can transfer a known soundness result for the defining conditional $>$ to the defined conditional $\rhd$.

Theorem 1. *$Ax(M^\rhd), Ax(M^>)$ systems in $\mathcal{L}_\rhd, \mathcal{L}_>$ respectively. Assume*

(1) $Ax(M^>)$ is sound for $M^>$,
(2) $\circ\colon \mathcal{L}_\rhd \longrightarrow \mathcal{L}_>$ is a translation,
(3) $M^\rhd \overset{\circ}{\approx} M^>$,
(4) $Ax(M^\rhd) \overset{\circ}{\propto} Ax(M^>)$.

Then $Ax(M^\rhd)$ is sound for $M^\rhd$.

Proof. Suppose $Ax(M^\rhd) \vdash \varphi$. Thus $Ax(M^>) \vdash \varphi^\circ$ (4). Hence $M^> \vDash \varphi^\circ$ (1). Therefore $M^\rhd \vDash \varphi$ (3). □

[4] Raidl (2019ms) uses a more general notion of 'embedding', where $M^\rhd$ and $M^>$ are not necessarily based on the same underlying structure.

This theorem allows to transfer a known soundness result from a basic conditional $>$ to a defined conditional $\rhd$. In most cases, (1) will be known or easy to figure out, and the form of the translation $\circ$ in (2) will be a simple corollary to the way $\rhd$ is defined semantically as a strengthening of $>$. Furthermore, (3) will easily be verified, so that it suffices to check (4). For this, it suffices to check that each of the rules and axioms of $Ax(M^{\rhd})$ can be simulated by the rules and axioms of $Ax(M^{>})$. This is a purely mechanical task. The real work is to figure out $Ax(M^{\rhd})$. I will come back to this problem below.

Second, with a $\circ$-isomorphism and a simulation for a back-translation $*$ of $\circ$, we can transfer a known completeness result for the defining conditional to the defined conditional:

Theorem 2. *With $Ax(M^{\rhd}), Ax(M^{>})$ as in the previous Theorem. Assume*

(1) $Ax(M^{>})$ is complete for $M^{>}$,
(2) $\circ : \mathcal{L}_{\rhd} \to \mathcal{L}_{>}$, $* : \mathcal{L}_{>} \to \mathcal{L}_{\rhd}$ are translations,
(3) $M^{\rhd} \overset{\circ}{\approx} M^{>}$,
(4) $Ax(M^{>}) \overset{*}{\propto} Ax(M^{\rhd})$,
(5) $*$ inverts $\circ$ in $Ax(M^{\rhd})$, i.e., $Ax(M^{\rhd}) \vdash \varphi^{\circ *} \leftrightarrow \varphi$.

Then $Ax(M^{\rhd})$ is complete for $M^{\rhd}$.

Proof. Suppose $M^{\rhd} \vDash \varphi$ then $M^{>} \vDash \varphi^{\circ}$ (3). Thus $Ax(M^{>}) \vdash \varphi^{\circ}$ (1). Therefore $Ax(M^{\rhd}) \vdash \varphi^{\circ *}$ (4). Hence $Ax(M^{\rhd}) \vdash \varphi$ (5). □

Again, (1) will be known or easy to figure out, and the form of $\circ$ in (2) follow from the the semantic definition of $\rhd$. (3) will easily be verified and the only work will be to suitably choose $Ax(M^{\rhd})$ and prove (4) and (5). Furthermore, some work will be needed to figure out the right backtranslation * in (2). The above results are quite general. They allow to generate a sound and complete logic $Ax(M^{\rhd})$ for a defined conditional $\rhd$ in a model class $M^{\rhd}$, based on the sound and complete logic $Ax(M^{>})$ for the defining conditional $>$ in the model class $M^{>}$. Although the two theorems appear as trivial as 2+2=4, they provide a powerful method for generating new knowledge on new semantics using old knowledge on known semantics.

One central question which remains unanswered by the above results is how to find $Ax(M^{\rhd})$. I only have a general heuristics to suggest, which should work in most (if not all) cases. First, find a backtranslation $*$. Second: figure out simple $\rhd$-axioms which are sufficient to prove the above (5) in the completeness result. Third: counter-check, whether these $\rhd$-axioms are sound and what kind of $>$-axioms are needed to simulate them. Fourth: check which $\rhd$-axioms correspond to, or simulate these $>$-axioms, using $*$. Fifth: counter-check, whether the available $>$-axioms suffice to simulate the new $\rhd$-axioms from step 4. If yes, you are done; else, repeat until you reach a fixed point. I am confident that this heuristics can be turned into a provable result. In fact, I have evidence for such a claim, since for the five conditionals analysed in Raidl (2019ms) as for the evidential conditional analysed here, it is always this heuristics which I used to find $Ax(M^{\rhd})$.

The careful reader has probably noted that the mentioned heuristics starts with finding a backtranslation $*$ of $\circ$. Thus another central question remains: how do we find $*$? And is there always such a backtranslation? I do not have any answers to these questions. For the first, I can only provide a heuristics: Consider the definition of $\triangleright$ in terms of $>$, say $\varphi \triangleright \psi$ is of the form $(\varphi > \psi) \wedge \alpha$ (modulo translation). To define $>$ in terms of $\triangleright$ (modulo translation) you could proceed as follows: Obviously $\varphi > \psi$ could have the form $(\varphi \triangleright \psi) \vee \beta$, where β should express $(\varphi > \psi) \wedge \neg\alpha$ but in the language $\triangleright$. Sometimes finding such a β directly is not possible. You can then try to strengthen or weaken the component $(\varphi \triangleright \psi)$ in the disjunction, for example to $\varphi \triangleright (\varphi \wedge \psi)$ or to $(\varphi \vee \neg\psi) \triangleright \psi$ or something else (provided these imply $\varphi > \psi$), and repeat the procedure, i.e., find the β which expresses the complementary case when $\varphi > \psi$ holds. More generally, given the axioms for $>$, figure out disjoint or covering cases when $\varphi > \psi$ holds. Express them in terms of $>$ and then in terms of $\triangleright$. You see that this remains a rather vague heuristics, and I must admit that I cannot give you more than this. In brief: be creative. As for the existence of the backtranslation, I do not know the answer to this question. I do not even have an idea how to attack it. If I had an idea, I could eventually either prove that Spohn's supererogatory reason relation is not axiomatisable or give you its complete axiomatics, to mention one example. Luckily, $*$ exists for the evidential conditional (as for many other strengthened conditionals), and I know how it looks like. Because of this, sections 5 and 6 are straight forward implementations of the two above theorems and provide the sound and complete logic for the evidential conditional defined in strong Lewisean selection models.

4 Axioms

Before introducing the axioms for the evidential conditional, let me briefly introduce those axioms for the basic conditional $>$ which we will need here.

Lewis' logic VC can be axiomatised by all substitution instances of classical propositional tautologies (PT), Modus Ponens (MoPo) for the material conditional $\rightarrow$, and for the conditional $>$ we add the additional rules RCEA, RCEC, and the axioms RCM, ID, CC, CA, CMon, CV, MP, CS as given below:

$$\frac{\vdash \varphi \leftrightarrow \varphi'}{\vdash (\varphi > \chi) \leftrightarrow (\varphi' > \chi)} \quad (\mathsf{RCEA}, \text{LLE})^{5} \qquad \frac{\vdash \varphi \leftrightarrow \varphi'}{\vdash (\chi > \varphi) \leftrightarrow (\chi > \varphi')} \quad (\mathsf{RCEC}, \text{RLE})$$

$$(\varphi > \psi) \rightarrow (\varphi > (\psi \vee \chi)) \qquad (\mathsf{RCM}, \text{RW})^{6}$$

$$\varphi > \varphi \qquad (\mathsf{ID}, \text{Refl})$$

$$((\varphi > \psi) \wedge (\varphi > \chi)) \rightarrow (\varphi > (\psi \wedge \chi)) \qquad (\mathsf{CC}, \text{AND})$$

$$((\varphi > \chi) \wedge (\psi > \chi)) \rightarrow ((\varphi \vee \psi) > \chi) \qquad (\mathsf{CA}, \text{OR})$$

[5] In (X, Y), X refers to Chellas' (1975) notation used here and Y to the KLM-tradition (Kraus, Lehmann, & Magidor, 1990).

[6] The usual form is the rule: from $\vdash \psi \rightarrow \chi$ infer $\vdash (\varphi > \psi) \rightarrow (\varphi > \chi)$. In the presence of this rule, RCEC is redundant. The rule also implies the mentioned axiom. Conversely, the axiom together with RCEC implies the rule.

$$((\varphi > \chi) \wedge (\varphi > \psi)) \to ((\varphi \wedge \psi) > \chi) \tag{CMon}$$

$$(\varphi > \chi) \wedge \neg(\varphi > \neg\psi)) \to ((\varphi \wedge \psi) > \chi) \tag{CV, RM}$$

$$(\varphi > \psi) \to (\varphi \to \psi) \tag{MP, If-to-Or}$$

$$(\varphi \wedge \psi) \to (\varphi > \psi) \tag{CS}$$

It is known that VC is sound and complete for strong Lewisean selection models (Lewis, 1971). Thus VC is the logic henceforth assumed for the basic conditional $>$.

Some derivable principles in VC are (proof omitted):

$$\varphi > \top \tag{CN}$$

$$\varphi > (\varphi \vee \chi) \tag{SC}$$ [7]

$$(\varphi > \bot) \to (\varphi > \psi) \tag{IA}$$ [8]

$$(\neg\psi > \bot) \to (\varphi > \psi) \tag{NC}$$ [9]

$$((\varphi \vee \psi) > \chi) \to ((\varphi > \chi) \vee (\psi > \chi)) \tag{DR}$$ [10]

$$\neg(\top > \bot) \tag{P}$$ [11]

The (conjectured) axiomatics EC for the evidential conditional $\rhd$ in strong Lewisean selection models is: PT, MoPo, RCEA, RCEC, ID, CC, CMon, augmented by the following additional axioms:

$$(\varphi \rhd \psi) \to (\neg\psi \rhd \neg\varphi) \tag{Contraposition}$$

$$(\varphi \rhd (\varphi \wedge \psi)) \to (\varphi \to \psi) \tag{MP*}$$

$$(\varphi \rhd (\varphi \wedge \psi)) \to (\varphi \vee (\varphi \rhd (\varphi \wedge (\psi \vee \chi)))) \tag{RCM*}$$

$$((\varphi \rhd (\varphi \wedge \chi)) \wedge (\psi \rhd (\psi \wedge \chi))) \to (\varphi \vee \psi \vee ((\varphi \vee \psi) \rhd ((\varphi \vee \psi) \wedge \chi))) \tag{CA*}$$

$$((\varphi \rhd (\varphi \wedge \chi)) \wedge \neg\varphi \wedge \neg(\varphi \rhd (\varphi \wedge \neg\psi))) \to ((\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi)) \tag{CV*}$$

$$(\varphi \wedge \psi \wedge ((\psi \vee \varphi) \rhd \psi)) \to (\varphi \rhd (\varphi \wedge \psi)) \tag{CS*}$$

$$((\varphi \rhd (\varphi \wedge \psi)) \wedge (\neg\psi \rhd (\neg\psi \wedge \neg\varphi))) \to (\varphi \rhd \psi) \tag{E}$$ [12]

[7]For Super Classicality.

[8]For Impossible Antecedent.

[9]For Necessary Consequent.

[10]For Disjunctive Rationality.

[11]For Probabilistic Consistency. MP is needed here.

[12]I denote it 'E' for 'evidential conditional', since, in addition to Contraposition, I consider it to be essentially linked to the way the evidential conditional is defined in terms of the basic conditional $>$. See (5) in the completeness proof, which is the only place where E is needed.

The $*$-axioms are ugly, and it would be nice to simplify them.Let me make sense of them. By ID and CC, $\varphi \rhd \psi$ implies $\varphi \rhd (\varphi \wedge \psi)$. Thus MP* implies the usual MP. By this fact, non-negated conditionals in the form $\varphi \rhd (\varphi \wedge \psi)$ appearing in the antecedent of $*$-axioms can be replaced by $\varphi \rhd \psi$ to yield a weaker axiom, i.e., with stronger antecedent. This remark applies to MP*, RCM*, CA* and CV*. Additionally, consequents of the type $X \vee (X \rhd XY)$ in $*$-axioms can also be strengthened into $X \vee (\neg X \wedge (X \rhd XY))$. The resulting axiom X** is equivalent to the original X*. This applies to RCM* and CA*. Also, instead of having a consequent of type $X \vee \theta$, we can have as consequent θ provided we put $\neg X$ as one of the conjuncts in the antecedent.[13] This applies to RCM* and CA*. For the moment, I do not know, how to further simplify the axiomatics. Furthermore, given ID, CC and Contraposition, we also have the reverse implication of E and thus an equivalence restating $\varphi \rhd \psi$ in terms of two conditionals $\varphi \rhd (\varphi \wedge \psi)$ and $\neg\psi \rhd (\neg\psi \wedge \neg\varphi)$. Finally, the original CA can be obtained by CC and Contraposition. Some of these facts are proven below.

A few remarks on the notation and terminology are also necessary. The star ($*$) appearing after an axiom term (X^*) refers to the fact that these new axioms (eg. CA*) are required to simulate the original axiom for the basic conditional $>$ (CA($>$)), except for CS*. Note that each $*$-axiom (X^*) is derivable from the original axiom (X) in the logic VC for the basic conditional $>$. This is essentially due to the fact that the conjunct $\varphi > (\varphi \wedge \psi)$ appearing in the antecedent implies $\varphi > \psi$ (RCM) for the basic conditional, and thus (the second disjunct of) the consequent of the $*$-axiom will (roughly) directly be obtained by the original axiom, since $\varphi > \psi$ also implies $\varphi > (\varphi \wedge \psi)$ (given ID, CC).[14] In this sense, the $*$-axioms are weaker than the original ones (given RCM). This is not so for the defined conditional $\rhd$, since although $\varphi \rhd \psi$ implies $\varphi \rhd (\varphi \wedge \psi)$, the reverse is not true: $\varphi \rhd (\varphi \wedge \psi)$ *does not* imply $\varphi \rhd \psi$. Thus the argument is blocked because of the absence of RCM.

To collect what has been said, the following things are important to note, when we move from the basic conditional $>$ to the defined evidential conditional $\rhd$:

(1) We loose exactly three axioms: right weakening (RCM), rational monotonicity (CV) and and-to-if or conjunctive sufficiency (CS). CA and MP are not lost, since we can recover them (MP from MP* and CA from CC and Contraposition).
(2) We gain new axioms: all $*$-axioms, Contraposition and E.
(3) Contraposition does not hold for the original conditional.
(4) All other 'new' axioms hold for the original conditional and are derivable in VC. Modulo RCM, the $*$-axioms are weakenings of the original axioms.

Let me now prove some of the above mentioned facts, as well as certain derivable principles.

Lemma 2. *In* EC*, the following are derivable:*

$$\varphi \rhd (\varphi \vee \psi) \tag{SC}$$

[13] This holds because $Z \to (X \vee Y)$ is equivalent to $Z \to (\neg X \to Y)$ which is equivalent to $(Z \wedge \neg X) \to Y$.
[14] For CV*, the additional ID, CC are needed.

$(\varphi \wedge \psi) \rhd \psi$ (-SC)[15]

$(\varphi \rhd \psi) \to (\varphi \to \psi)$ (MP)

$((\varphi \rhd \chi) \wedge (\psi \rhd \chi)) \to ((\varphi \vee \psi) \rhd \chi)$ (CA)

$(\varphi \rhd \psi) \to ((\varphi \rhd (\varphi \wedge \psi)) \wedge (\neg\psi \rhd (\neg\psi \wedge \neg\varphi)))$ (E-)[16]

$\neg(\top \rhd \bot)$ (P)

$(\varphi \rhd \bot) \to (\varphi \rhd \chi)$ (IA)

$(\neg\psi \rhd \bot) \to (\varphi \rhd \psi)$ (NC)

$\varphi \rhd \top$ (CN)

$\bot \rhd \varphi$ (-CN)

$((\varphi \rhd \psi) \wedge (\chi \rhd \psi)) \to (\varphi \rhd (\psi \vee \chi))$ (RCMon)[17]

$(\neg(\varphi \wedge \psi) \rhd \bot) \to (\neg\psi \rhd \bot)$ (UC)[18]

$(\varphi \wedge \psi \wedge ((\psi \vee \varphi) \rhd \psi)) \to (\varphi \rhd \psi)$ (CS**)

Proof. Using freely RCEA, RCEC.

SC: We have $\varphi \rhd \varphi$ (ID), thus $\varphi \rhd ((\varphi \wedge \varphi) \vee (\varphi \wedge \psi))$ (RCEC). Hence $\varphi \rhd (\varphi \wedge (\varphi \vee \psi))$ (RCEC). We also have $(\neg\varphi \wedge \neg\psi) \rhd (\neg\varphi \wedge \neg\psi)$ (ID). But $\neg\varphi \wedge \neg\psi$ is equivalent to $\neg(\varphi \vee \psi)$ as well as to $\neg(\varphi \vee \psi) \wedge \neg\varphi$. Thus by (RCEA, RCEC) $\neg(\varphi \vee \psi) \rhd (\neg(\varphi \vee \psi) \wedge \neg\varphi)$. Therefore $\varphi \rhd (\varphi \vee \psi)$ by (E), where $(\varphi \vee \psi)$ plays the role of ψ.

-SC: We have $\neg\psi \rhd (\neg\psi \vee \neg\varphi)$ (SC). Thus $(\psi \wedge \varphi) \rhd \psi$ (Contraposition).

MP: Suppose $\varphi \rhd \psi$. Thus $\varphi \rhd (\varphi \wedge \psi)$ (ID, CC). Hence $\varphi \to \psi$ (MP*).

CA: assume $\varphi \rhd \chi$ and $\psi \rhd \chi$. Thus $\neg\chi \rhd \neg\varphi$ and $\neg\chi \rhd \neg\psi$ (Contraposition). Therefore (CC): $\neg\chi \rhd (\neg\varphi \wedge \neg\psi)$, that is $\neg\chi \rhd \neg(\varphi \vee \psi)$ (RCEC) and hence $(\varphi \vee \psi) \rhd \chi$ (Contraposition).

E-: Suppose $\varphi \rhd \psi$. Thus $\varphi \rhd (\varphi \wedge \psi)$ (ID, CC). We also have $\neg\psi \rhd \neg\varphi$ (Contraposition). Thus $\neg\psi \rhd (\neg\psi \wedge \neg\varphi)$ (ID, CC).

P: We have $\neg(\top \to \bot)$. Thus $\neg(\top \rhd \bot)$, contraposing MP($\rhd$).

[15]-X for X transformed by Contraposition.
[16]E- for reverse implication of E.
[17]RCMon for a weakened version of RCM, obtained by CMon.
[18]UC for $\Box X = \neg X \rhd \bot$ is upwards closed.

IA: Suppose $\varphi \rhd \bot$. Thus $\neg\varphi$ (MP) as well as $\varphi \rhd (\bot \wedge \varphi)$ (ID, CC). Hence by RCM* $\varphi \vee (\varphi \rhd (\varphi \wedge (\bot \vee \chi)))$. Therefore, since $\neg\varphi$, we obtain $\varphi \rhd (\varphi \wedge (\bot \vee \chi))$. This is equivalent to $\varphi \rhd (\varphi \wedge \chi)$ (RCEC). If we prove $\neg\chi \rhd (\neg\chi \wedge \neg\varphi)$, we are done, since using (E), we obtain $\varphi \rhd \chi$. So let us prove the mentioned formula. Given the assumption $\varphi \rhd \bot$, we can also prove $\varphi \rhd (\varphi \wedge \neg\chi)$ (same reasoning as above, since χ was arbitrary). Thus $(\varphi \wedge \neg\chi) \rhd \bot$. That is $(\neg\chi \wedge \varphi) \rhd (\neg\chi \wedge \varphi \wedge \neg\varphi)$ (RCEC). We also have $(\neg\chi \wedge \neg\varphi) \rhd (\neg\chi \wedge \neg\varphi \wedge \neg\varphi)$ (ID, RCEC). Thus, by CA* (and RCEC, RCEA), we obtain $\neg\chi$ or $\neg\chi \rhd (\neg\chi \wedge \neg\varphi)$. Thus, if χ, we obtain $\neg\chi \rhd (\neg\chi \wedge \neg\varphi)$. Let us show that we obtain the same desired conclusion, if $\neg\chi$: $\varphi \rhd \bot$ implies $\varphi \rightarrow \bot$ (MP) and thus $\neg\varphi$. Hence $\neg\chi$ and $\neg\varphi$. But we also have $(\neg\varphi \vee \neg\chi) \rhd \neg\varphi$ (contraposing $\varphi \rhd (\varphi \wedge \chi)$). Thus by (CS*) we can conclude $\neg\chi \rhd (\neg\chi \wedge \neg\varphi)$.

NC: Suppose $\neg\psi \rhd \bot$. Thus (IA) $\neg\psi \rhd \neg\varphi$. Hence $\varphi \rhd \psi$ (Contraposition).

CN: We have $\bot \rhd \bot$ (ID). Thus $\bot \rhd \neg\varphi$ (IA). Hence $\varphi \rhd \top$ (Contraposition).

-CN: We have $\neg\varphi \rhd \top$ (CN) thus $\bot \rhd \varphi$ (Contraposition).

RCMon: Suppose $\varphi \rhd \psi$ and $\chi \rhd \psi$. Thus $\neg\psi \rhd \neg\varphi$ and $\neg\psi \rhd \neg\chi$. Therefore $(\neg\psi \wedge \neg\chi) \rhd \neg\varphi$ (CMon). Hence $\neg(\psi \vee \chi) \rhd \neg\varphi$ (RCEA). Thus $\varphi \rhd (\psi \vee \chi)$ (Contraposition).

UC: Suppose $\neg(\varphi \wedge \psi) \rhd \bot$. Thus $(\neg\varphi \vee \neg\psi) \rhd \bot$ (RCEA). Hence $(\neg\varphi \vee \neg\psi) \rhd \neg\psi$ (IA). Therefore $\neg\psi \rhd \bot$ (CMon, RCEA).

CS:** From CS* and E.

□

The reader may have remarked that contraposition makes IA and NC, as well as CN and -CN be duals to each other. This is also the case for CC and CA:

Lemma 3. *The following are inter-derivable given Contraposition.*

(1) IA, NC.
(2) CN, -CN.
(3) CC, CA.

As a consequence, in the above axiomatics, we could replace CC by CA.

Proof. (1) Above we have shown that IA+Contraposition implies NC. Suppose now NC and assume $\varphi \rhd \bot$. Thus $\neg\psi \rhd \neg\varphi$ (NC). Therefore $\varphi \rhd \psi$ (Contraposition).

(2) $\varphi \rhd \top$ iff $\bot \rhd \neg\varphi$ (Contraposition).

(3) CC to CA: see above. CA to CC: assume $\varphi \rhd \psi$ and $\varphi \rhd \chi$. Thus $\neg\psi \rhd \neg\varphi$ and $\neg\chi \rhd \neg\varphi$ (Contraposition). Therefore $(\neg\psi \vee \neg\chi) \rhd \neg\varphi$ (CA). Hence $\neg(\psi \wedge \chi) \rhd \neg\varphi$ (RCEC) and thus $\varphi \rhd (\psi \wedge \chi)$ (Contraposition).

□

5 Soundness

We consider the conditional translations:

- $(\varphi \rhd \psi)^\circ = (\varphi^\circ > \psi^\circ) \wedge (\neg\psi^\circ > \neg\varphi^\circ)$
- $(\varphi > \psi)^* = (\varphi^* \wedge \psi^*) \vee (\varphi^* \rhd (\varphi^* \wedge \psi^*))$

The translation $\circ$ obviously arises from the way $\rhd$ is defined semantically and is mentioned in Crupi and Iacona (2019) after we exchanged on these matters. The backtranslation $*$ is non-obvious. I obtained it as a simplification of an initial proposal by Crupi and Iacona[19] which was of the form:

- $(\varphi > \psi)^* = (\varphi^* \rhd \psi^*) \vee (\varphi^* \wedge \psi^*) \vee ((\varphi^* \vee \neg\psi^*) \rhd (\varphi^* \wedge \psi^*))$

The simplified backtranslation simplifies the axioms for $\rhd$. The message is this: you can have more than one backtranslation, but the simpler your backtranslation, the nicer your generated axioms. This is not astonishing, given that the backtranslation is a looking-glass. If the looking-glass is blurred, the image will be blurred or distorted. If the looking-glass is clean, the image will be as well.

For the soundness result, the following equivalence will be important:

Lemma 4. *In* VC, $\varphi > \psi$ *is equivalent to* $(\varphi \wedge \psi) \vee ((\varphi > (\varphi \wedge \psi)) \wedge ((\neg\varphi \vee \neg\psi) > \neg\varphi))$.

Proof. $\varphi \wedge \psi$ implies $\varphi > \psi$ (CS). And $\varphi > (\varphi \wedge \psi)$ also implies $\varphi > \psi$ (RCM(>)). Conversely, assume $\varphi > \psi$. Either φ or $\neg\varphi$. In the first case, we obtain ψ (MP) and thus $\varphi \wedge \psi$. In the second case, we have $\varphi > (\varphi \wedge \psi)$ (ID, CC) and since $\neg\varphi$, also $\neg\varphi \vee \neg\psi$. Thus $(\neg\varphi \vee \neg\psi) > \neg\varphi$ (CS). □

This also yields that $\varphi > \psi$ is equivalent to $(\varphi > \psi)^{*\circ}$.[20] That is, the translation $\circ$ inverses $*$ in VC.[21] We will need the above Lemma in the form $\varphi^\circ > \psi^\circ$ is equivalent to $(\varphi^\circ \wedge \psi^\circ) \vee (\varphi \rhd (\varphi \wedge \psi))^\circ$. We will later drop $\circ$ and just write $(\varphi \wedge \psi) \vee (\varphi \rhd (\varphi \wedge \psi))$.

Theorem 3 (Soundness)**.** EC *is sound for* $\rhd$ *in Lewisean strongly centred selection models.*

Proof. We know that VC is sound for $M^>$ the class of strongly centred Lewisean selection models. We also know that $\circ$ is a conditional translation $\circ\colon \mathcal{L}_\rhd \longrightarrow \mathcal{L}_>$. To prove that the conjectured axiomatics for $\rhd$ is sound for the class of models $M^\rhd$ ($M^>$ where $\rhd$ is defined as evidential conditional instead of being defined as basic conditional), it suffices to prove the following two facts (cf. Theorem 1):

3. $M^\rhd \overset{\circ}{\approx} M^>$,
4. $Ax(M^\rhd) \overset{\circ}{\propto} Ax(M^>)$.

[19] Private communication. Their initial proposal was of great help to finding the new one.
[20] See appendix for the recursive proof.
[21] In the completeness part we show that $*$ inverses $\circ$ in EC.

3. is clear: For all $\varphi \in \mathcal{L}_{\triangleright}$ we have $w \vDash_{\mathfrak{M}^{\triangleright}} \varphi$ iff $w \vDash_{\mathfrak{M}^{>}} \varphi^{\circ}$. This can be shown by induction using the above conditional translation. It suffices to verify it for $\varphi = (\psi \triangleright \chi)$, using the induction hypohtesis for ψ and χ:

$$\begin{array}{lll} & w \vDash_{\mathfrak{M}^{\triangleright}} \psi \triangleright \chi & \\ \text{iff} & F(w, [\psi]^{\mathfrak{M}^{\triangleright}}) \subseteq [\chi]^{\mathfrak{M}^{\triangleright}} \;\;\&\;\; F(w, [\neg\chi]^{\mathfrak{M}^{\triangleright}}) \subseteq [\neg\psi]^{\mathfrak{M}^{\triangleright}} & (\text{Def.}) \\ \text{iff} & F(w, [\psi^{\circ}]^{\mathfrak{M}^{>}}) \subseteq [\chi^{\circ}]^{\mathfrak{M}^{>}} \;\;\&\;\; F(w, [\neg\chi^{\circ}]^{\mathfrak{M}^{>}}) \subseteq [\neg\psi^{\circ}]^{\mathfrak{M}^{>}} & (\text{IH}, \circ) \\ \text{iff} & w \vDash_{\mathfrak{M}^{>}} (\psi^{\circ} > \chi^{\circ}) \wedge (\neg\chi^{\circ} > \neg\psi^{\circ}) & (\text{Def.}) \end{array}$$

4. We now show that the Logic $Ax(M^{\triangleright}) = \mathsf{EC}$ for $\triangleright$ can be simulated by the Axioms $Ax(M^{>}) = \mathsf{VC}$ for $>$, modulo the translation $\circ$. We will do this, without restating the translation every time. That is, we will often write $\varphi \triangleright \psi$ where in fact we should write $(\varphi \triangleright \psi)^{\circ}$ or equivalently $(\varphi^{\circ} > \psi^{\circ}) \wedge (\neg\psi^{\circ} > \neg\varphi^{\circ})$. In agreement with this abbreviation, we write $\varphi > \psi$ where we should write $\varphi^{\circ} > \psi^{\circ}$. Also note that we extensively use the fact that in VC, $\varphi > \psi$ is equivalent to the disjunction $(\varphi \wedge \psi) \vee (\varphi \triangleright (\varphi \wedge \psi))$ (see Lemma 4 above). Thus, based on the left to right implication, $\varphi > \psi$ also implies the weaker $\varphi \vee (\varphi \triangleright (\varphi \wedge \psi))$.

RCEA($\triangleright$): Suppose $\vdash \varphi \leftrightarrow \psi$. And $\varphi \triangleright \chi$. Thus $\varphi > \chi$ and $\neg\chi > \neg\varphi$. Hence, by RCEA($>$) and RCEC($>$), we have $\psi > \chi$ and $\neg\chi > \neg\psi$. Thus $\psi \triangleright \chi$.

RCEC($\triangleright$): Suppose $\vdash \psi \leftrightarrow \chi$. And $\varphi \triangleright \psi$. Thus $\varphi > \psi$ and $\neg\psi > \neg\varphi$. Hence by RCEC($>$) and RCEA($>$), we have $\varphi > \chi$ and $\neg\chi > \neg\varphi$. Thus $\varphi > \chi$.

ID($\triangleright$): Since $\varphi > \varphi$ and $\neg\varphi > \neg\varphi$ (ID($>$)).

CC($\triangleright$): Assume $\varphi \triangleright \psi$ and $\varphi \triangleright \chi$. Thus $\varphi > \psi$ and $\varphi > \chi$. Thus $\varphi > (\psi \wedge \chi)$ (CC). But we also have $\neg\psi > \neg\varphi$ and $\neg\chi > \neg\varphi$. Thus $(\neg\psi \vee \neg\chi) > \neg\varphi$ (CA), that is $\neg(\psi \wedge \chi) > \neg\varphi$ (RCEA). And thus $\varphi \triangleright (\psi \wedge \chi)$.

CMon($\triangleright$): Assume $\varphi \triangleright \chi$ and $\varphi \triangleright \psi$. That is, $\varphi > \chi$ and $\varphi > \psi$, from which we obtain $(\varphi \wedge \psi) > \chi$ (CMon). Additionally, we also have $\neg\chi > \neg\varphi$. From this it follows already that $\neg\chi > (\neg\varphi \vee \neg\psi)$ (RCM). Thus $\neg\chi > \neg(\varphi \wedge \psi)$ (RCEC). Therefore $(\varphi \wedge \psi) \triangleright \chi$.

Contraposition: Suppose $\varphi \triangleright \psi$ thus $\varphi > \psi$ and $\neg\psi > \neg\varphi$. Hence $\neg\psi \triangleright \neg\varphi$.

RCM*: $(\varphi \triangleright (\varphi \wedge \psi)) \rightarrow (\varphi \vee (\varphi \triangleright (\varphi \wedge (\psi \vee \chi))))$: $\varphi \triangleright (\varphi \wedge \psi)$ implies $\varphi > \psi$ (Def., RCM) which implies $\varphi > (\psi \vee \chi)$ (RCM) which implies $\varphi \vee (\varphi \triangleright (\varphi \wedge (\psi \vee \chi)))$ (remark above).

MP*: $(\varphi \triangleright (\varphi \wedge \psi)) \rightarrow (\varphi \rightarrow \psi)$: $\varphi \triangleright (\varphi \wedge \psi)$ implies $\varphi > \psi$. Hence $\varphi \rightarrow \psi$ (MP).

CA*: $((\varphi \triangleright (\varphi \wedge \chi)) \wedge (\psi \triangleright (\psi \wedge \chi))) \rightarrow (\varphi \vee \psi \vee ((\varphi \vee \psi) \triangleright ((\varphi \vee \psi) \wedge \chi)))$: The first antecedent conjunct implies $\varphi > \chi$, the second implies $\psi > \chi$, thus $(\varphi \vee \psi) > \chi$ (CA) which implies the consequent disjunction (remark above).

CV*: First note that $(\neg X \wedge Y) \to Z$ is equivalent to $((\neg X \vee D) \wedge Y) \to (X \vee Z)$. Thus

$$((\varphi \rhd (\varphi \wedge \chi)) \wedge \neg\varphi \wedge \neg(\varphi \rhd (\varphi \wedge \neg\psi))) \to ((\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi))$$

is equivalent to

$$((\varphi \rhd (\varphi \wedge \chi)) \wedge (\neg\varphi \vee \psi) \wedge \neg(\varphi \rhd (\varphi \wedge \neg\psi))) \to (\varphi \vee ((\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi))).$$

The first antecedent conjunct implies $\varphi > \chi$. The two remaining antecedent conjuncts imply (modulo logical transformations) $\neg(\varphi > \neg\psi)$. Thus $(\varphi \wedge \psi) > \chi$ (CV($>$)). But this implies $(\varphi \wedge \psi) \vee ((\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi))$ (above Lemma) which implies the weaker $\varphi \vee ((\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi))$ (compare the above remark).

CS*: Assume $\varphi \wedge \psi$ and $(\psi \vee \varphi) \rhd \psi$. Thus $\varphi > (\varphi \wedge \psi)$ (CS). But we also have $\neg\psi > \neg(\psi \vee \varphi)$, that is $\neg\psi > (\neg\psi \wedge \neg\varphi)$ (RCEC). Hence $\neg\psi > \neg\varphi$ (RCM). But $\neg\varphi > \neg\varphi$ (ID). Hence $(\neg\psi \vee \neg\varphi) > \neg\varphi$ (CA). Thus $\neg(\varphi \wedge \psi) > \neg\varphi$ (RCEA). Therefore $\varphi \rhd (\varphi \wedge \psi)$.

E: Assume $\varphi \rhd (\varphi \wedge \psi)$ and $\neg\psi \rhd (\neg\psi \wedge \neg\varphi)$. Thus $\varphi > (\varphi \wedge \psi)$ and $\neg\psi > (\neg\psi \wedge \neg\varphi)$. Hence $\varphi > \psi$ and $\neg\psi > \neg\varphi$ (RCM($>$)). Therefore $\varphi \rhd \psi$.

□

6 Completeness

We consider the same conditional translations $\circ$ and $*$.

Theorem 4 (Completeness)**.** EC *is complete for* $\rhd$ *defined as evidential conditional in strong Lewisean selection models.*

Proof. We know that $\mathsf{VC} = Ax(M^{>})$ is complete for $M^{>}$ the class of strongly Lewisean selection models. Additionally $\circ$ and $*$ are conditional translations $\circ\colon \mathcal{L}_{\rhd} \longrightarrow \mathcal{L}_{>}$ and $*\colon \mathcal{L}_{>} \longrightarrow \mathcal{L}_{\rhd}$. And we have already proven $M^{\rhd} \overset{\circ}{\approx} M^{>}$ (Theorem 3). Thus, to prove completeness (by Theorem 2), it suffices to show:

4. $Ax(M^{>}) \overset{*}{\propto} Ax(M^{\rhd})$,
5. $*$ *inverts* $\circ$ in $Ax(M^{\rhd})$, i.e., $Ax(M^{\rhd}) \vdash \varphi^{\circ *} \leftrightarrow \varphi$.

We show **5** and then **4**:

5: By the translation, the induction hypothesis of invertibility (IH), RCEA and RCEC for $\rhd$, we obtain:

$$\begin{aligned} & (\varphi \rhd \psi)^{\circ *} \\ \equiv\ & (\varphi^\circ > \psi^\circ)^* \wedge (\neg\psi^\circ > \neg\varphi^\circ)^* && (\circ) \\ \equiv\ & ((\varphi \wedge \psi) \vee (\varphi \rhd (\varphi \wedge \psi))) \wedge ((\neg\psi \wedge \neg\varphi) \vee (\neg\psi \rhd (\neg\psi \wedge \neg\varphi))) && (*, \text{IH, RCEA, RCEC}) \end{aligned}$$

The big conjunction in the last line is equivalent to the following disjunctive cases

(1) $\varphi, \psi, \neg\psi, \neg\varphi$,
(2) $\varphi, \psi, \neg\psi \rhd (\neg\psi \wedge \neg\varphi)$,
(3) $\varphi \rhd (\varphi \wedge \psi), \neg\psi, \neg\varphi$,
(4) $\varphi \rhd (\varphi \wedge \psi), \neg\psi \rhd (\neg\psi \wedge \neg\varphi)$.

It suffices to show that this big disjunction is equivalent to $\varphi \rhd \psi$ (in EC). It is clear that $\varphi \rhd \psi$ implies this disjunction: since $\varphi \rhd \psi$ implies $\varphi \rhd (\varphi \wedge \psi)$ (ID, CC) and $\neg\psi \rhd \neg\varphi$ (Contraposition), thus also $\neg\psi \rhd (\neg\psi \wedge \neg\varphi)$ (ID, CC). This is the last disjunct of the above disjunction.

Conversely: we need to show that each disjunction implies $\varphi \rhd \psi$. (1) is a contradiction and thus implies $\varphi \rhd \psi$. (2) Assume φ, ψ and $\neg\psi \rhd (\neg\psi \wedge \neg\varphi)$. Thus $\neg(\neg\psi \wedge \neg\varphi) \rhd \psi$ (Contraposition). Hence $(\psi \vee \varphi) \rhd \psi$ (RECA). But together with $\varphi \wedge \psi$ this implies $\varphi \rhd (\varphi \wedge \psi)$ (CS*). This, together with $\neg\psi \rhd (\neg\psi \wedge \neg\varphi)$ implies $\varphi \rhd \psi$ (E). (3) same reasoning (exchanging φ with $\neg\psi$ and simultaneously ψ with $\neg\varphi$). (4) implies $\varphi \rhd \psi$ by E.

4: Now we prove that the axioms and rules for $>$, i.e., VC, can be simulated by those for $\rhd$, i.e., EC, modulo the backtranslation $*$. We translate relatively freely and use RCEA($\rhd$) and RCEC($\rhd$) often without mentioning it.

MoPo($\to$): is simulated by MoPo($\to$).

RCEA($>$): Its translate is: From $\vdash \varphi \leftrightarrow \psi$ infer that $(\varphi \wedge \chi) \vee (\varphi \rhd (\varphi \wedge \chi))$ implies $(\psi \wedge \chi) \vee (\psi \rhd (\psi \wedge \chi))$. It suffices to show the following two inferences, under the assumption (+) that $\vdash \varphi \leftrightarrow \psi$.

- $\vdash (\varphi \wedge \chi) \to ((\psi \wedge \chi) \vee (\psi \rhd (\psi \wedge \chi)))$: From $\varphi \wedge \chi$, we obtain $\psi \wedge \chi$ (by +)
- $\vdash (\varphi \rhd (\varphi \wedge \chi)) \to ((\psi \wedge \chi) \vee (\psi \rhd (\psi \wedge \chi)))$: From $\varphi \rhd (\varphi \wedge \chi)$ we obtain $\psi \rhd (\psi \wedge \chi)$, (+, RECA($\rhd$), RCEC($\rhd$)).

RCEC($>$): Its translate is: From $\vdash \psi \leftrightarrow \chi$ infer that $(\varphi \wedge \psi) \vee (\varphi \rhd (\varphi \wedge \psi))$ implies $(\varphi \wedge \chi) \vee (\varphi \rhd (\varphi \wedge \chi))$. Using RECA($\rhd$) and RCEC($\rhd$) a similar argument as above shows this.

RCM($>$): The translate is that $(\varphi \wedge \psi) \vee (\varphi \rhd (\varphi \wedge \psi))$ implies the disjunction $(\varphi \wedge (\psi \vee \chi)) \vee (\varphi \rhd (\varphi \wedge (\psi \vee \chi)))$. It suffices to show the following principles:

- $\varphi \wedge \psi$ implies the disjunction: since $\varphi \wedge \psi$ implies $\varphi \wedge (\psi \vee \chi)$
- $\varphi \rhd (\varphi \wedge \psi)$ implies the disjunction: By RCM* $\varphi \vee (\varphi \rhd (\varphi \wedge (\psi \vee \chi)))$. By MP* we also have $\varphi \to \psi$ thus $\varphi \to (\varphi \wedge \psi)$ as well as $\varphi \to (\varphi \wedge (\psi \vee \chi))$. Thus we have $(\varphi \wedge (\psi \vee \chi)) \vee (\varphi \rhd (\varphi \wedge (\psi \vee \chi)))$.

ID($>$): The translate is (equivalent to) $\varphi \vee (\varphi \rhd \varphi)$. But this follows from ID($\rhd$).

CC($>$): The translation is that $(\varphi \wedge \psi) \vee (\varphi \rhd (\varphi \wedge \psi))$ together with $(\varphi \wedge \chi) \vee (\varphi \rhd (\varphi \wedge \chi))$ imply the disjunction $(\varphi \wedge \psi \wedge \chi) \vee (\varphi \rhd (\varphi \wedge \psi \wedge \chi))$. We prove 4 principles, which jointly entail this:

- $\varphi \wedge \psi, \varphi \wedge \chi$ implies the disjunction: Since they imply $\varphi \wedge \psi \wedge \chi$.
- $\varphi \wedge \psi, \varphi \rhd (\varphi \wedge \chi)$ implies the disjunction: MP* yields $\varphi \wedge \psi \wedge \chi$.
- $\varphi \rhd (\varphi \wedge \psi), \varphi \wedge \chi$ implies the disjunction: MP* yields $\varphi \wedge \psi \wedge \chi$.
- $\varphi \rhd (\varphi \wedge \psi), \varphi \rhd (\varphi \wedge \chi)$ implies the disjunction: Since this implies $\varphi \rhd (\varphi \wedge \psi \wedge \chi)$ (CC($\rhd$)).

CS($>$): The translation of CS, $(\varphi \wedge \psi) \rightarrow ((\varphi \wedge \psi) \vee (\varphi \rhd (\varphi \wedge \psi)))$, is trivially true.

MP($>$): The translation of MP is $((\varphi \wedge \psi) \vee (\varphi \rhd (\varphi \wedge \psi))) \rightarrow (\varphi \rightarrow \psi)$. $(\varphi \wedge \psi) \rightarrow (\varphi \rightarrow \psi)$ is trivially true and $(\varphi \rhd (\varphi \wedge \psi)) \rightarrow (\varphi \rightarrow \psi)$ is MP*.

CA($>$): The translation of CA is that $(\varphi \wedge \chi) \vee (\varphi \rhd (\varphi \wedge \chi))$ together with $(\psi \wedge \chi) \vee (\psi \rhd (\psi \wedge \chi))$ implies the disjunction: $((\varphi \vee \psi) \wedge \chi) \vee ((\varphi \vee \psi) \rhd ((\varphi \vee \psi) \wedge \chi))$. We do not have to bother about the first terms $\varphi \wedge \chi$ or $\psi \wedge \chi$, since each of them implies $(\varphi \vee \psi) \wedge \chi$. Thus it suffices to prove that $\varphi \rhd (\varphi \wedge \chi)$ together with $\psi \rhd (\psi \wedge \chi)$ implies the disjunction. CA* yields $(\varphi \vee \psi) \vee ((\varphi \vee \psi) \rhd ((\varphi \vee \psi) \wedge \chi))$. But note that under our assumption of $\varphi \rhd (\varphi \wedge \chi)$ and $\psi \rhd (\psi \wedge \chi)$, we have $\varphi \rightarrow \chi$ and $\psi \rightarrow \chi$ (MP*), thus $(\varphi \vee \psi) \rightarrow \chi$. Therefore we can strengthen the disjunct $\varphi \vee \psi$ to $(\varphi \vee \psi) \wedge \chi$. That is, we obtain $((\varphi \vee \psi) \wedge \chi) \vee ((\varphi \vee \psi) \rhd ((\varphi \vee \psi) \wedge \chi))$.

CMon($>$): The translation is $(\varphi \wedge \chi) \vee (\varphi \rhd (\varphi \wedge \chi))$ together with $(\varphi \wedge \psi) \vee (\varphi \rhd (\varphi \wedge \psi))$ implies the disjunction $(\varphi \wedge \psi \wedge \chi) \vee ((\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi))$. Thus it suffices to prove the following principles:

- $\varphi \wedge \chi, \varphi \wedge \psi$, implies the disjunction: since it implies $\varphi \wedge \psi \wedge \chi$.
- $\varphi \wedge \chi, \varphi \rhd (\varphi \wedge \psi)$, implies the disjunction: MP* yields $\varphi \wedge \psi \wedge \chi$.
- $\varphi \rhd (\varphi \wedge \chi), \varphi \wedge \psi$, implies the disjunction: MP* yields $\varphi \wedge \psi \wedge \chi$.
- $\varphi \rhd (\varphi \wedge \chi), \varphi \rhd (\varphi \wedge \psi)$, implies the disjunction: By CMon($\rhd$) $(\varphi \wedge \psi) \rhd (\varphi \wedge \chi)$. But $(\varphi \wedge \psi) \rhd (\varphi \wedge \psi)$ (ID($\rhd$)). Thus $(\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi)$ (CC($\rhd$)).

CV($>$): The translation is: $(\varphi \wedge \chi) \vee (\varphi \rhd (\varphi \wedge \chi))$ together with $\neg((\varphi \wedge \neg\psi) \vee (\varphi \rhd (\varphi \wedge \neg\psi)))$ implies the disjunction $(\varphi \wedge \psi \wedge \chi) \vee ((\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi))$. The antecedent conjunctive assumption is equivalent to $(\varphi \wedge \chi) \vee (\varphi \rhd (\varphi \wedge \chi))$ conjoined with the conjunction of $\varphi \rightarrow \psi$ and $\neg(\varphi \rhd (\varphi \wedge \neg\psi))$. Thus it suffices to show the following two principles:

- $\varphi \wedge \chi$, $\varphi \rightarrow \psi, \neg(\varphi \rhd (\varphi \wedge \neg\psi))$ implies the disjunction: since $\varphi, \chi, \varphi \rightarrow \psi$ implies $\varphi \wedge \psi \wedge \chi$.
- $\varphi \rhd (\varphi \wedge \chi)$, $\varphi \rightarrow \psi, \neg(\varphi \rhd (\varphi \wedge \neg\psi))$ implies the disjunction: First note, if we have $\neg\varphi$ then CV* yields $(\varphi \wedge \psi) \rhd (\varphi \wedge \psi \wedge \chi)$. Else, if φ, then ψ (since $\varphi \rightarrow \psi$). Furthermore $\varphi \rhd (\varphi \wedge \chi)$ yields $\varphi \rightarrow \chi$ (MP*). Thus, under our assumption that φ, we have $\varphi \wedge \psi \wedge \chi$.

□

7 Conclusion

This article proved completeness for the evidential conditional $\rhd$ which can be thought of as a strengthening of a basic $>$ conditional. For this a general method was developed which allows to transfer completeness results from $>$ to $\rhd$, provided one has a back-translation of $>$ in terms of $\rhd$. The careful reader will have recognised, that the assumption of cautious monotonicity (CMon) and rational monotonicity (CV) for the basic conditional are not 'essential', in the sense of the heuristics to figure out the axioms for $\rhd$ (given on p. 7). Thus, the conjecture is that the whole procedure could be carried out for a weaker base logic, without CMon and without CV. This would yield a weaker evidential conditional logic, presumably without CMon and without CV*. For this, one would need to consider selection models which are weaker than the Lewisean ones, i.e., without the assumption of (cmon) and (cv).

A Invertibility in VC

Lemma 5. $\mathsf{VC} \vdash \chi^{*\circ} \leftrightarrow \chi$.[22]

Proof. By induction. It suffices to verify this for $\chi = \varphi > \psi$, assuming the induction hypothesis $\varphi^{*\circ} \equiv \varphi$ and $\psi^{*\circ} \equiv \psi$ (in VC). Using the translations, RCEA and RCEC, as well as the IH for φ, ψ, we obtain

$$
\begin{array}{lll}
& (\varphi > \psi)^{*\circ} & \\
= & ((\varphi^* \wedge \psi^*) \vee (\varphi^* \rhd (\varphi^* \wedge \psi^*)))^\circ & (*) \\
\equiv & (\varphi^* \wedge \psi^*)^\circ \vee (\varphi^* \rhd (\varphi^* \wedge \psi^*))^\circ & (\circ) \\
\equiv & (\varphi^{*\circ} \wedge \psi^{*\circ}) \vee ((\varphi^{*\circ} > (\varphi^{*\circ} \wedge \psi^{*\circ})) \wedge (\neg(\varphi^{*\circ} \wedge \psi^{*\circ}) > \neg\varphi^{*\circ})) & (\circ) \\
\equiv & (\varphi^{*\circ} \wedge \psi^{*\circ}) \vee ((\varphi^{*\circ} > (\varphi^{*\circ} \wedge \psi^{*\circ})) \wedge ((\neg\varphi^{*\circ} \vee \neg\psi^{*\circ}) > \neg\varphi^{*\circ})) & (\mathsf{RCEA}) \\
\equiv & (\varphi \wedge \psi) \vee ((\varphi > (\varphi \wedge \psi)) \wedge ((\neg\varphi \vee \neg\psi) > \neg\varphi)) & (\mathrm{IH}, \mathsf{RCEC}, \mathsf{RCEA})
\end{array}
$$

We show that $\varphi > \psi$ is equivalent to this last disjunction.

Right to Left: $\varphi \wedge \psi$ implies $\varphi > \psi$ (CS). $\varphi > (\varphi \wedge \psi)$ also implies $\varphi > \psi$ (RCM, RCEC).

Left to Right: Suppose $\varphi > \psi$. If φ, we obtain $\varphi \wedge \psi$ by MP. Thus let us assume $\neg\varphi$. Thus $\neg\varphi$. Hence $\neg\varphi \vee \neg\psi$. Thus $(\neg\varphi \vee \neg\psi) > \neg\varphi$ (CS). Yet $\varphi > \psi$ implies $\varphi > (\varphi \wedge \psi)$ (ID, CC). This proves the second disjunct.

□

Note that one should read $\varphi \rhd (\varphi \wedge \psi)$ as meaning (1) $\varphi > \psi$ and (2) $(\neg\varphi \vee \neg\psi) > \neg\varphi$, where (2) really means that $\neg\psi$ does not come before $\neg\varphi$ and if $\neg\psi$ and $\neg\varphi$ are equally close then $\neg\psi > \neg\varphi$. Else, whether or not $\neg\psi > \neg\varphi$ is left undetermined by (2).

[22] A monotone logic (RCEA, RCEC, RCM), with ID, CC, CS suffices.

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