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Standard Bayes logic is not finitely axiomatizable

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Abstract

In the paper [2] a hierarchy of modal logics have been defined to capture the logical features of Bayesian belief revision. Elements in that hierarchy were distinguished by the cardinality of the set of elementary propositions. By linking the modal logics in the hierarchy to Medvedev's logic of (in)finite problems it has been shown that the modal logic of Bayesian belief revision determined by probabilities on a finite set of elementary propositions is not finitely axiomatizable. However, the infinite case remained open. In this paper we prove that the modal logic of Bayesian belief revision determined by standard Borel spaces (these cover probability spaces that occur in most of the applications) is also not finitely axiomatizable.

Keywords: Modal logic, Bayesian inference, Bayes learning, Bayes logic, Medvedev frames, Non finite axiomatizability.

1 Introduction and background

Bayes logics have been introduced in the recent paper [2] in order to investigate (modal) logical properties of statistical inference (Bayesian belief revision). If $(X, \mathcal{B}, p)$ is a probability space, then elements of $\mathcal{B}$ can be regarded as propositions or possible statements about the world, and the probability measure p describes knowledge of statistical information (or say, it represents degree of beliefs in the truth of these propositions). Learning proposition $A \in \mathcal{B}$ to be true means revising the probability measure p on the basis of the evidence A, and replacing p by some other probability q in certain ways. The transition from p to q is statistical inference: This new probability measure q can be regarded as the probability measure that one *infers* from p on the basis of the information (evidence) that A is true. A fundamental model of statistical inference is the standard Bayes model which relies on Bayes conditionalization of probabilities: given a prior probability measure p and an evidence $A \in \mathcal{B}$ the inferred measure q is defined by conditionalizing p upon A using Bayes' rule:

$$q(B) \doteq p(B \mid A) = \frac{p(B \cap A)}{p(A)} \quad (\forall B \in \mathcal{B}) \tag{1}$$

provided $p(A) \neq 0$. The paper [2] aimed at studying the logical aspects of this type of inference from the perspective of modal logic.[1]

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[1] Bayesian inference is just a particular type of statistical inference: Various rules replacing the Bayes's rule have been considered in the literature (e.g. Jeffrey conditionalization, maxent principle; see [13] and [5]). Questions

It is very natural to regard the move from p to q in terms of modal logic: The core idea is to view A in the Bayes' rule (1) as a variable and say that a probability measure q *can* be inferred from p if there exits an A in $\mathcal{B}$ such that $q(\cdot) = p(\cdot \mid A)$. Equivalently, we will say in this situation that "q *can* be (Bayes) learned from p". That "it is *possible* to obtain/learn q from p" is clearly a modal talk and calls for a logical modeling in terms of concepts of modal logic. For further motivation and background we suggest [2].

We will use the standard unimodal language given by the grammar

$$a \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \tag{2}$$

defining formulas φ, where a belongs to a nonempty countable set Φ of propositional letters. As usual $\Box$ abbreviates $\neg\Diamond\neg$. We refer to the books [1, 4] concerning basic notions in modal logic and we mostly follow notation of the book [1].

Let $\langle X, \mathcal{B}\rangle$ be a measurable space and denote by $M(X, \mathcal{B})$ the set of all probability measures over $\langle X, \mathcal{B}\rangle$. Bayes accessibility relation has been defined in [2] as follows: For $v, w \in M(X, \mathcal{B})$ we say that w is *Bayes accessible* from v if there is an $A \in \mathcal{B}$ such that $w(\cdot) = v(\,\cdot \mid A)$. We denote the Bayes accessibility relation on $M(X, \mathcal{B})$ by $R(X, \mathcal{B})$.

Definition 1.1 (Bayes frames)**.** A Bayes frame is a Kripke frame $\langle W, R\rangle$ that is isomorphic, as a directed graph, to $\mathcal{F}(X, \mathcal{B}) = \langle M(X, \mathcal{B}), R(X, \mathcal{B})\rangle$ for a measurable space $\langle X, \mathcal{B}\rangle$. $\Box$

From the point of view of applications the most important classes of Bayes frames $\mathcal{F}(X, \mathcal{B})$ are Bayes frames determined by standard Borel spaces $\langle X, \mathcal{B}\rangle$. A measurable space $\langle X, \mathcal{B}\rangle$ is *standard Borel* if X can be endowed with a metric which makes it a complete, separable metric space in such a way that $\mathcal{B}$ is the Borel σ-algebra (the smallest σ-algebra containing the open sets). According to the Borel isomorphism theorem (Section 17 in [7]) a standard Borel space is always isomorphic to one of $\langle\{1, \dots, n\}, \wp(\{1, \dots, n\})\rangle$, or $\langle\mathbb{N}, \wp(\mathbb{N})\rangle$, or $\langle[0, 1], \mathcal{B}\rangle$, where $\mathcal{B}$ is the set of Borel subsets of $[0, 1]$ ($[0, 1]$ endowed with the Euclidean topology). If $\langle X, \mathcal{B}\rangle$ is a standard Borel space and w is a probability measure on $\mathcal{B}$, then the completion of $\langle X, \mathcal{B}, w\rangle$ is a standard probability space (see Section 17.F in [7]). Thus any standard probability space is isomorphic (modulo zero) to either $\langle[0, 1], \mathcal{L}, \lambda\rangle$, where $\mathcal{L}$ is the Lebesgue σ-algebra and λ is the Lebesgue measure; or to an atomic probability space with countable (possibly finitely) many atoms; or the combination (disjoint union) of both. Let $\langle X, \mathcal{A}, p\rangle$ and $\langle Y, \mathcal{B}, q\rangle$ be two probability spaces and $r \in (0, 1)$. Denote the the disjoint union of the two spaces by $\langle X + Y, \mathcal{A} + \mathcal{B}, p + q\rangle$, where $X + Y$ is the disjoint union of X and Y; $\mathcal{A} + \mathcal{B}$ is the σ-algebra generated by $\mathcal{A} \cup \mathcal{B}$ and $p + q$ is the probability measure $r \cdot p + (1 - r) \cdot q$. (If we would like to make r explicit, we write $p +_r q$). It follows that any standard probability space is isomorphic (modulo zero) to $\langle\mathbb{N} + [0, 1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda\rangle$, where $\langle\mathbb{N}, \wp(\mathbb{N}), p\rangle$ is an arbitrary probability space (as p might not be faithful, this covers the finite case as well).

Let us introduce the following notation

$$\mathcal{F}_n = \langle M(X, \wp(X)),\ R(X, \wp(X))\rangle, \quad \text{where } X = \{1, \dots, n\} \tag{3}$$

$$\mathcal{F}_\omega = \langle M(X, \wp(X)),\ R(X, \wp(X))\rangle, \quad \text{where } X = \mathbb{N} \tag{4}$$

similar to what we consider here can be (and have been) asked regarding these other types of inference rules (cf. [2, 3]), but taking the first steps we stick to the basic case of Bayesian inference here.

Note that if the measurable space $\langle X, \mathcal{B} \rangle$ is finite or countable, then $\mathcal{B}$ is the full powerset algebra $\wp(X)$ (we rely on the convention that elementary events $\{x\}$ for $x \in X$ always belong to the algebra $\mathcal{B}$).

Definition 1.2 (Bayes logics)**.** A family of normal modal logics have been defined in [2] based on finite or countable or countably infinite or all Bayes frames as follows.

$$\begin{aligned}
\mathbf{BL}_n &= \{\phi : \mathcal{F}_n \Vdash \phi\} &(5)\\
\mathbf{BL}_{<\omega} &= \{\phi : (\forall n \in \mathbb{N})\mathcal{F}_n \Vdash \phi\} &(6)\\
\mathbf{BL}_\omega &= \{\phi : \mathcal{F}_\omega \Vdash \phi\} &(7)\\
\mathbf{BL}_{\leq\omega} &= \mathbf{BL}_{<\omega} \cap \mathbf{BL}_\omega &(8)\\
\mathbf{BL} &= \{\phi : (\forall \text{ Bayes frames } \mathcal{F})\ \mathcal{F} \Vdash \phi\} &(9)
\end{aligned}$$

We call $\mathbf{BL}_{<\omega}$ (resp. $\mathbf{BL}_{\leq\omega}$) the logic of finite (resp. countable) Bayes frames; however, observe that the set of possible worlds $M(X, \mathcal{B})$ of a Bayes frame $\mathcal{F}(X, \mathcal{B})$ is finite if and only if X is a one-element set, otherwise it is at least of cardinality continuum. $\square$

$\mathbf{BL}_{<\omega}$ is the set of general laws of Bayesian learning based on all finite Bayes frames. The *general laws of Bayesian learning* independent of the particular representation $\langle X, \mathcal{B} \rangle$ of the events is then the modal logic $\mathbf{BL}$.

The following theorem has been proved in [2] (see Proposition 3.1 and Theorem 4.1 therein).

Theorem 1.3. $\mathbf{S4} \subseteq \mathbf{BL} \subseteq \mathbf{S4.1} \subsetneq \mathbf{BL}_\omega = \mathbf{BL}_{\leq\omega} \subsetneq \mathbf{S4.1} + \mathbf{Grz} \subsetneq \mathbf{BL}_{<\omega}$.

The finite Bayes frame case has been completely described in [2] and, in particular, it has been shown that $\mathbf{BL}_{<\omega}$ has the finite frame property and is *not* finitely axiomatizable (see Propositions 5.8, 5.9 in [2]), but the infinite case remained almost completely open.

In this paper we deal with Bayes frames $\mathcal{F}(X, \mathcal{B})$, where $\langle X, \mathcal{B} \rangle$ is a standard Borel space (such frames we will refer to as standard Borel Bayes frames). This covers the finite and countably infinite cases (e.g. $\mathbf{BL}_{<\omega}$ and $\mathbf{BL}_\omega$) but is more general because the uncountably infinite space $\langle [0,1], \mathcal{B}, \lambda \rangle$ is also considered. Let us write $\mathbf{BL}_{st}$ for the standard Bayes logic

$$\mathbf{BL}_{st} = \{\phi : (\forall \text{Standard Borel Bayes frames } \mathcal{F})\ \mathcal{F} \Vdash \phi\}.$$

By relating standard Borel Bayes frames and standard Bayes logics to generalized Medvedev frames (the definitions will be recalled later on) and continuing Shehtman's [11] analysis we will prove the following two main theorems.

Theorem 3.3. Let $\langle X, \mathcal{B} \rangle$ be a standard Borel space and $\mathcal{F}(X, \mathcal{B})$ be the corresponding Bayes frame. The logic $\Lambda(\mathcal{F}(X, \mathcal{B}))$ is not finitely axiomatizable.

Theorem 3.4. $\mathbf{BL}_{st}$ is not finitely axiomatizable.

We will also see the containments

$$\mathbf{S4} \subseteq \mathbf{BL} \subseteq \mathbf{BL}_{st} \subsetneq \mathbf{S4.1} \subsetneq \mathbf{BL}_{\omega} = \mathbf{BL}_{\leq\omega} \subsetneq \mathbf{S4.1Grz} \subsetneq \mathbf{BL}_{<\omega}$$

but the general case **BL** remains open:

Problem 1.4. Is $\mathbf{BL} = \mathbf{BL}_{st}$? If not, is **BL** finitely axiomatizable? Is $\mathbf{BL} = \mathbf{S4}$?

Structure of the paper. In the remaining part of the Introduction we recall useful facts from modal logic that we will make use of many times. To obtain not finite axiomatizability results we employ the techniques presented in Shehtman [11] which we recall in Section 2. In Subsections 2.1 and 2.2 we prove that the logics corresponding to finitary Boolean frames and measure algebra frames are not finitely axiomatizable. These are the key preliminary theorems that we will apply in Section 3, where we prove that standard Bayes logic $\mathbf{BL}_{st}$ is not finitely axiomatizable (Theorem 3.4).

Useful preliminaries. By a frame we always understand a Kripke frame, that is, a structure of the form $\mathcal{F} = \langle W, R\rangle$, where W is a non-empty set (of possible worlds) and $R \subseteq W \times W$ a binary relation (accessibility). Kripke models are tuples $\mathfrak{M} = \langle W, R, [\,\cdot\,]\rangle$ based on frames $\mathcal{F} = \langle W, R\rangle$, and $[\,\cdot\,] : \Phi \to \wp(W)$ is an evaluation of propositional letters. Truth of a formula φ at world w is defined in the usual way by induction:

- $\mathfrak{M}, w \Vdash p \iff w \in [\,p\,]$ for propositional letters $p \in \Phi$.
- $\mathfrak{M}, w \Vdash \varphi \wedge \psi \iff \mathfrak{M}, w \Vdash \varphi$ AND $\mathfrak{M}, w \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \neg\varphi \iff \mathfrak{M}, w \nVdash \varphi$.
- $\mathfrak{M}, w \Vdash \Diamond\varphi \iff$ there is v such that wRv and $\mathfrak{M}, v \Vdash \varphi$.

Formula φ is valid over a frame $\mathcal{F}$ ($\mathcal{F} \Vdash \varphi$ in symbols) if and only if it is true at every point in every model based on the frame. For a class C of frames the modal logic of C is the set of all modal formulas that are valid on every frame in C:

$$\Lambda(\mathsf{C}) = \big\{\phi :\ (\forall \mathcal{F} \in \mathsf{C})\ \mathcal{F} \Vdash \phi\big\} \tag{10}$$

$\Lambda(\mathsf{C})$ is always a normal modal logic. Let us recall the most standard list of modal axioms (frame properties) that are often considered in the literature (cf. [1] and [4]).

Basic frame properties		
Name	Formula	Corresponding frame property
T	$\Box\phi \to \phi$	accessibility relation R is reflexive
4	$\Box\phi \to \Box\Box\phi$	accessibility relation R is transitive
M	$\Box\Diamond\phi \to \Diamond\Box\phi$	2nd order property not to be covered here
Grz	$\Box(\Box(\phi \to \Box\phi) \to \phi) \to \phi$	$\mathbf{T} + \mathbf{4} + \neg\exists P(\forall w \in P)(\exists v\ wRv)(v \neq w\ \wedge\ P(v))$
S4	$\mathbf{T} + \mathbf{4}$	preorder
S4.1	$\mathbf{T} + \mathbf{4} + \mathbf{M}$	preorder having endpoints

For two frames $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{G} = \langle W', R'\rangle$ we write $\mathcal{F} \trianglelefteq \mathcal{G}$ if $\mathcal{F}$ is (isomorphic as a frame to) a generated subframe of $\mathcal{G}$. We recall that if $\mathcal{F} \trianglelefteq \mathcal{G}$, then $\mathcal{G} \Vdash \phi$ implies $\mathcal{F} \Vdash \phi$, whence $\Lambda(\mathcal{G}) \subseteq \Lambda(\mathcal{F})$ (see Theorem 3.14 in [1]). If $w \in W$, then we write $\mathcal{F}^w$ to denote the subframe of $\mathcal{F}$ generated by w, and we call such subframes point-generated subframes. Further, let $\mathcal{F} \twoheadrightarrow \mathcal{G}$ denote a surjective, bounded morphism (sometimes called p-morphisms). Such morphisms preserve the accessibility relation and have the zig-zag property (see [1]). Recall that if $\mathcal{F} \twoheadrightarrow \mathcal{G}$, then $\mathcal{F} \Vdash \phi$ implies $\mathcal{G} \Vdash \phi$, hence $\Lambda(\mathcal{F}) \subseteq \Lambda(\mathcal{G})$ (see Theorem 3.14 in [1]).

2 A method for non-finite axiomatizability

We start by recalling definitions and theorems from [11] and [2]. Medvedev's logic of finite problems and its extension to infinite problems by Skvortsov originate in intuitionistic logic. (For an overview we refer to the book [4] and to Shehtman [11]; Medvedev's logic of finite problems is covered in the papers [9, 12, 10, 11, 8, 6]).

Definition 2.1 (Medvedev frame)**.** A *Medvedev frame* is a frame that is isomorphic (as a directed graph) to $\mathcal{P}^0(X) = \langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq\rangle$ for a non-empty *finite* set X. $\square$

Medvedev's logic $\mathbf{ML}_{<\omega}$ is the modal logic that corresponds to the Medvedev frames:

$$\mathbf{ML}_n = \bigcap \big\{\Lambda\big(\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq\rangle\big) : |X| = n\big\} \tag{11}$$

$$\mathbf{ML}_{<\omega} = \bigcap \big\{\Lambda\big(\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq\rangle\big) : |X| \text{ non-empty, finite}\big\} \tag{12}$$

A *Skvortsov frame* is defined in the same way except with X is a non-empty set of *any cardinality.* We denote the corresponding Skvortsov logics by $\mathbf{ML}_\alpha$ for sets X of cardinality α. It has been proved (see Theorem 2.2 in [12]) that

$$\mathbf{ML} \stackrel{def}{=} \bigcap_\alpha \mathbf{ML}_\alpha = \mathbf{ML}_\omega \tag{13}$$

As a slight abuse of notation we will use the term *Medvedev frame* for any frame of the form $\mathcal{P}^0(X)$ (thus X need not be finite here). One of the main results in [2] is Theorem 5.2:

Theorem 2.2 (Theorem 5.2 in [2])**.** *Countable Bayes and Medvedev's logics coincide.*

$$\begin{array}{ccccccccc}
\mathbf{ML} & = & \mathbf{ML}_\omega & = & \mathbf{ML}_{\leq\omega} & \subsetneq & \mathbf{ML}_{<\omega} & \subsetneq & \mathbf{ML}_n \\
\cup & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
\mathbf{BL} & \subsetneq & \mathbf{BL}_\omega & = & \mathbf{BL}_{\leq\omega} & \subsetneq & \mathbf{BL}_{<\omega} & \subsetneq & \mathbf{BL}_n
\end{array} \tag{14}$$

To gain not finite axiomatizability results we follow the method presented in Shehtman [11] and we recall the most important lemmas that we make use of.

Lemma 2.3 (cf. Proposition 4 in [11])**.** Let $\mathcal{F}$ be a generated finite **S4**-frame. Then there is a modal formula $\chi(\mathcal{F})$ with the following properties:

(A) For any **S4**-frame $\mathcal{G}$ we have $\mathcal{G} \nVdash \chi(\mathcal{F})$ if and only if $\exists u\, \mathcal{G}^u \twoheadrightarrow \mathcal{F}$.

(B) For any logic $\mathbf{L} \supseteq \mathbf{S4}$ we have $\mathbf{L} \subseteq \Lambda(\mathcal{F})$ if and only if $\chi(\mathcal{F}) \notin \mathbf{L}$.

Definition 2.4 ([11]). For $m > 0$ and $k > 2$ the Chinese lantern is the **S4**-frame $\mathcal{C}(k, m)$ formed by the set

$$\{(i, j) : (1 \leq i \leq k - 2, 0 \leq j \leq 1) \text{ OR } (i = k - 1, 1 \leq j \leq m) \text{ OR } (i = k, j = 0)\}$$

with the accessibility relation being an ordering:

$$(i, j) \leq (i', j') \text{ iff } (i, j) = (i', j') \text{ OR } i > i'$$

□

$\mathcal{C}(m, k)$ is illustrated on page 373 in [11], however, we will not need any particular information about $\mathcal{C}$ apart from two lemmas that we recall below.

Lemma 2.5 (Lemma 22 in [11]). Let ϕ be a modal formula using l variables and let $m > 2^l$. Then $\mathcal{C}(k, m) \nVdash \phi$ implies $\mathcal{C}(k, 2^l) \nVdash \phi$.

Lemma 2.6 (Lemma 24 in [11]). For any $n > 1$ we have $\mathcal{C}(2^n, 2^n) \Vdash \mathbf{ML}_{<\omega}$.

For a natural number l a logic $\mathbf{L}$ is l-axiomatizable if it has an axiomatization using only formulas whose propositional variables are among $p_1, \ldots, p_l$. Every finitely axiomatizable logic is l-axiomatizable for a suitable l: take l to be the maximal number of variables the finitely many axioms in question use.

Theorem 2.7. *Let $\mathbf{L}$ be a normal modal logic with $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. Suppose that for every $l \geq 1$ and $k > l$ there is $n \geq k$ such that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. Then $\mathbf{L}$ is not l-axiomatizable for any number l.*

Proof. By way of contradiction suppose $\mathbf{L}$ is l-axiomatizable, that is, $\mathbf{L} = \mathbf{S4} + \Sigma$ where Σ is a set of formulas that can use only the first l propositional variables. Let $k = 2^l$. By assumption there is $n \geq k$ so that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. That Σ axiomatizes $\mathbf{L}$ means that every formula in $\mathbf{L}$ can be derived (in the normal modal calculus) from a finite set of axioms from Σ. Therefore there is an l-formula $\phi \in \mathbf{L}$ such that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{S4} + \phi$. This implies, by Lemma 2.3(B), that $\mathcal{C}(k, 2^n) \nVdash \phi$. As $n \geq k = 2^l > l$, Lemma 2.5 ensures $\mathcal{C}(k, 2^l) \nVdash \phi$. In particular, $\mathcal{C}(k, 2^l) \nVdash \mathbf{L}$.

On the other hand Lemma 2.5 implies (as $k = 2^l$) that $\mathcal{C}(k, 2^l) \Vdash \mathbf{ML}_{<\omega}$. By assumption $\mathbf{L} \subseteq \mathbf{ML}_{<\omega}$ so it follows that $\mathcal{C}(k, 2^l) \Vdash \mathbf{L}$ which is a contradiction. ∎

Corollary 2.8. Let $\mathcal{F}$ be a frame, $\mathbf{L} = \Lambda(\mathcal{F})$ and assume $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. Suppose for any $k \geq 1$ there is $n \geq k$ such that for all $u \in \mathcal{F}$ we have $\mathcal{F}^u \not\twoheadrightarrow \mathcal{C}(k, 2^n)$. Then $\mathbf{L}$ is not l-axiomatizable for any finite number l.

Proof. Under the given assumptions Lemma 2.3(A) implies that for all $k \geq 1$ there is $n \geq k$ so that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. Then Theorem 2.7 applies. ∎

2.1 Not finite axiomatizability of the logic of finitary Boolean frames

Let $\mathcal{F} = \langle W, \leq \rangle$ be a finite ordering (partially ordered set) and pick $x \in W$. y is an immediate successor of x if $x < y$ and there is no $x < z < y$. (As usual $<$ means $\leq \cap \neq$). The branch index $b_{\mathcal{F}}(x)$ is the cardinality of the set of immediate successors of x, and the depth $d_{\mathcal{F}}(x)$ is the least upper bound of cardinalities of chains in $\mathcal{F}$ whose least element is x. Thus, $d_{\mathcal{F}}(x) = 1$ means that x has no immediate successors. We adopt Lemma 17 in [11] in a slightly more general setting:

Definition 2.9. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a Boolean algebra of subsets of X such that all finite subsets of X are contained in $\mathcal{B}$. A frame isomorphic to $\mathcal{B}^0 = \langle \mathcal{B} \setminus \{\emptyset\}, \supseteq \rangle$ is called a *finitary Boolean frame.* $\square$

The connection with Medvedev frames is transparent: for any set X the frames $\mathcal{P}^0(X) = \langle \wp(X) \setminus \{\emptyset\}, \supseteq \rangle$ are finitary Boolean frames. Note that any point-generated subframe of $\mathcal{B}^0$ is also a finitary Boolean frame.

Lemma 2.10. Let $\mathcal{B}^0$ be a finitary Boolean frame and let $\mathcal{F}$ be a finite, point-generated ordering (frame) such that $b_{\mathcal{F}}(x) \neq 1$ for every $x \in \mathcal{F}$. If $\mathcal{B}^0 \twoheadrightarrow \mathcal{F}$, then $b_{\mathcal{F}}(x) < 2^{d_{\mathcal{F}}(x)}$ for all $x \in \mathcal{F}$.

Proof. Suppose $h : \mathcal{B}^0 \twoheadrightarrow \mathcal{F}$ is a surjective bounded morphism. We show first that for all $x \in \mathcal{F}$ there is a set $A_x \in \mathcal{B}^0$ such that $h(A_x) = x$ and $|A_x| < 2^{d_{\mathcal{F}}(x)}$. We proceed by induction. The case $d(x) = 1$ is straightforward: For some $A \in \mathcal{B}^0$ we have $h(A) = x$. As $d(x) = 1$, x has no proper successors, therefore for all $B \subseteq A$, $B \in \mathcal{B}^0$ we have $h(B) = x$. Pick any $A_x = \{a\} \subseteq A$, then $h(A_x) = x$ and $|A_x| = 1 < 2^1$ (by assumption all finite subsets of X belong to $\mathcal{B}^0$, thus $\{a\} \in \mathcal{B}^0$).

Suppose now (inductive hypothesis) that we know the statement for all $x \in \mathcal{F}$ with $d(x) = n$ and consider the case $d(x) = n + 1$. There is a set $A \in \mathcal{B}^0$ such that $h(A) = x$. By assumption $b(x) \neq 1$, therefore x has at least two immediate successors x_1 and x_2. Then $d(x_i) = n$ therefore, by induction, there are sets $A_1, A_2 \subseteq A$ such that $h(A_i) = x_i$ and $|A_i| < 2^{d(x_i)}$. Let $A_x = A_1 \cup A_2$. It is clear that $|A| < 2^{d(x_1)} + 2^{d(x_2)} \leq 2^{n+1}$, so we need to show $h(A_x) = x$. Since $A \subseteq A_x$ and h is a homomorphism, it follows that $x \leq h(A_x)$. Similarly, $A_x \supseteq A_i$, therefore $h(A_x) \leq h(A_i) = x_i$. As x_1 and x_2 are immediate successors, the only element $h(A_x)$ that can satisfy the equations $x \leq h(A_x) \leq x_i$ is x.

To complete the proof pick an arbitrary $x \in \mathcal{F}$ and a set $A \in \mathcal{B}^0$ such that $h(A) = x$ and $|A| < 2^{d(x)}$. As A is finite we can assume that A is as small as possible: there is no $B \subsetneq A$ such that $h(B) = x$. The branching index of A is $b(A) = |A| < 2^{d(x)}$, thus it suffices to show that $b(A) \geq b(h(A))$. Take an immediate successor y of x. Then there is $B \subseteq A$ such that $h(B) = y$. B is contained in an immediate successor C of A, and as $B \subseteq C \subseteq A$ holds, we have $h(C)$ is either x or y. But it cannot be x, because of the minimality of A. Therefore, with any immediate successor of x we can associate an immediate successor of A. This completes the proof. ∎

Corollary 2.11. Let $\mathcal{B}^0$ be a finitary Boolean frame. Then there is no surjective bounded morphism $\mathcal{B}^0 \twoheadrightarrow \mathcal{C}(k, 2^k)$.

Proof. The point $x = (k, 0)$ in $\mathcal{C}(k, 2^k)$ has depth $d(x) = k$ and branch index $b(x) = 2^k$. If there is $\mathcal{B}^0 \twoheadrightarrow \mathcal{F}$, then Lemma 2.10 applies and we would have $b(x) < 2^{d(x)}$ which is impossible. ∎

Corollary 2.12. Let $\mathcal{B}^0$ be an infinite finitary Boolean frame. Then $\mathbf{L} = \Lambda(\mathcal{B}^0)$ is not finitely axiomatizable.

Proof. We intend to apply Corollary 2.8. It is straightforward that the reverse-Boolean ordering $\supseteq$ is transitive and reflexive, therefore $\mathcal{B}^0$ is an $\mathbf{S4}$-frame (i.e. $\mathbf{S4} \subseteq \mathbf{L}$).

Next, let us verify that $\mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. To this end it is enough to prove $\mathcal{B}^0 \twoheadrightarrow \mathcal{P}^0(\mathbb{N})$ as in this case $\mathbf{L} \subseteq \Lambda(\mathcal{P}^0(\mathbb{N})) = \mathbf{ML}_\omega \subseteq \mathbf{ML}_{<\omega}$. As $\mathcal{B}^0$ is infinite, there are countably infinite many pairwise disjoint elements $a_i \in \mathcal{B}^0$ $(i \in \mathbb{N})$ such that $\bigcup i \in \mathbb{N} a_i = X$. The idea is to extend the mapping $a_i \mapsto i$ to a bounded morphism. Define $f : \mathcal{B}^0 \to \mathcal{P}^0(\mathbb{N})$ as

$$f(a) = \{i \in \mathbb{N} : \ a \cap a_i \neq \emptyset\} \neq \emptyset$$

Then $f(X) = \mathbb{N}$ and $f(a_i) = \{i\}$. We claim that f is a surjective bounded morphism.
Surjectivity: For a non-empty $A \subseteq \mathbb{N}$ we have $f(\bigcup i \in A a_i) = A$.
Homomorphism: Suppose for $a, b \in \mathcal{B}^0$ we have $a \supseteq b$. Then whenever $b \cap a_i \neq \emptyset$ we also have $a \cap a_i \neq \emptyset$, therefore

$$\{i \in \mathbb{N} : \ a \cap a_i \neq \emptyset\} \supseteq \{i \in \mathbb{N} : \ b \cap a_i \neq \emptyset\}$$

meaning that $f(a) \supseteq f(b)$.
Zig-zag property: Suppose $f(a) = A$ and $A \supseteq B \neq \emptyset$. We need to find a b with $a \supseteq b$ and $f(b) = B$. Take $b = a \setminus \bigcup_{i \in A \setminus B} a_i$. Then $a \supseteq b$ and $f(B) = \{i : \ b \cap a_i \neq \emptyset\} = B \setminus A = B$.

Finally, to fulfill all requirements of Corollary 2.8 we show that for all $u \in \mathcal{B}^0$ we have $(\mathcal{B}^0)^u \not\twoheadrightarrow \mathcal{C}(k, 2^k)$. As point-generated subframes of $\mathcal{B}^0$ are also finitary Boolean frames, the result follows from Corollary 2.11. ∎

2.2 Not finite axiomatizability of the logic of certain measure algebra frames

Suppose $\langle X, \mathcal{B}, w\rangle$ is a probability space. Two measurable sets A and B are said to be w-equivalent, $A \sim_w B$ in symbols (when it is clear we omit w from the subscript), if the w-measure of their symmetric difference is 0. $\sim$ is a congruence on the Boolean σ-algebra $\mathcal{B}$ and thus we can consider the quotient structure $\mathcal{B}/_\sim$ which also is a σ-complete Boolean algebra with the quotient operations. E.g. $A/_\sim \leq B/_\sim$ if and only if $A \subseteq B$ modulo w-measure zero for any representatives A and B respectively of $A/_\sim$ and $B/_\sim$. The measure w can be pushed down to $\mathcal{B}/_\sim$ by the definition $\bar{w}(A/_\sim) = w(A)$ for any $A \in \mathcal{B}$. For every $A \in \mathcal{B}$ with $w(A) = 0$ we have $A/_\sim = \emptyset/_\sim$, therefore $\emptyset/_\sim$ is the only element in $\mathcal{B}/_\sim$ which has $\bar{w}$-measure zero. A similar argument shows that only $X/_\sim$ has $\bar{w}$-measure 1 in $\mathcal{B}/_\sim$. $\langle \mathcal{B}/_\sim, \bar{w}\rangle$ is called a measure algebra.

Definition 2.13. A frame isomorphic to $\mathcal{M}(X, \mathcal{B}, w) = \langle \mathcal{B}/_{\sim_w} \smallsetminus \{\emptyset/_{\sim_w}\}, \geq \rangle$ is called a *measure algebra frame*. Here $\langle X, \mathcal{B}, w \rangle$ is a probability space and $\geq$ is the converse of the Boolean ordering of $\mathcal{B}/_{\sim_w}$. □

There is a straightforward connection with Medvedev frames: the set of possible worlds of a Medvedev frame is $\wp(X) \smallsetminus \{\emptyset\}$ for some (finite) set X. $\wp(X)$ is a Boolean algebra, thus we can think of the possible worlds of a Medvedev frame as the structure that results when we cut out the least element of a Boolean algebra. We did the same thing in case of finitary Boolean frames (Definition 2.9) and we do something similar here: we take the Boolean algebra $\mathcal{B}/_{\sim}$ and cut out its least element. If X is a finite set, then every subset of X is measurable, thus $\mathcal{B} = \wp(X)$. Take now a faithful probability measure w over $\mathcal{B}$. Then no proper subsets of X are $\sim_w$ equivalent, therefore $\mathcal{B}/_{\sim}$ is (isomorphic to) $\wp(X)$. Consequently, the Medvedev frame $\mathcal{P}^0(X)$ and the measure algebra frame $\mathcal{M}(X, \mathcal{B}, w)$ are isomorphic when X is finite. For infinite X the connection is more subtle: the Lebesgue measure algebra (where $X = [0,1]$, $\mathcal{B}$ = all Borel sets of $[0,1]$, w = Lebesgue measure) has no σ-complete set representation (it is not realized as a σ-complete Boolean algebra of subsets of some set), therefore it is not isomorphic to any (finitary) Boolean frame.

From now on, in this subsection, $\mathcal{B}$ denotes the Borel σ-algebra over the unit interval $[0,1]$, $\mathcal{L}$ the Lebesgue σ-algebra over $[0,1]$, and λ the Lebesgue measure. $\sim$ means $\sim_\lambda$. As $\mathcal{L}$ is the completion of $\mathcal{B}$, it follows that the measure algebras $\mathcal{L}/_{\sim}$ and $\mathcal{B}/_{\sim}$ are isomorphic. Therefore $\mathcal{M}([0,1], \mathcal{L}, \lambda)$ and $\mathcal{M}([0,1], \mathcal{B}, \lambda)$ are also isomorphic (as frames). Note that both $\mathcal{B} \smallsetminus \{\emptyset\}$ and $\mathcal{L} \smallsetminus \{\emptyset\}$ are infinite finitary Boolean frames (in the sense of Definition 2.9), thus their modal logics are not finitely axiomatizable (Corollary 2.12).

Lemma 2.14. If $\mathcal{M}$ is a measure algebra frame and $a \in \mathcal{M}$, then the generated subframe $\mathcal{M}^a$ is also a measure algebra frame. If $\mathcal{M} = \mathcal{M}([0,1], \mathcal{L}, \lambda)$, then $\mathcal{M}^a$ is isomorphic to $\mathcal{M}$.

Proof. Pick any $\mathcal{M} = \mathcal{M}(X, \mathcal{B}, w)$ and an element $a \in \mathcal{M}$. There is $A \in \mathcal{B}$ such that $A/_{\sim_w} = a$. Measurable subsets of A are exactly the sets in $\mathcal{B} \restriction A = \{A \cap B : B \in \mathcal{B}\}$ and $v(\cdot) = w(\cdot \mid A)$ is a probability measure on $\mathcal{B} \restriction A$. It is not hard to see that the generated subframe $\mathcal{M}^a$ is isomorphic to $\mathcal{M}(A, \mathcal{B} \restriction A, v)$.

For the second statement we only have to note that for a Lebesgue measurable $A \subseteq [0,1]$ we have $\mathcal{M}(A, \mathcal{L} \restriction A, \lambda) \cong \mathcal{M}([0,1], \mathcal{L}, \lambda)$. ∎

Theorem 2.15. *The modal logic of $\mathcal{M} = \mathcal{M}([0,1], \mathcal{L}, \lambda)$ is not finitely axiomatizable.*

Proof. Write $\mathbf{L} = \Lambda(\mathcal{M})$. We intend to apply Corollary 2.8; the first part of the proof is almost identical to that of Corollary 2.12. It is straightforward that the reverse-Boolean ordering $\geq$ is transitive and reflexive, therefore $\mathcal{M}$ is an **S4**-frame (i.e. $\mathbf{S4} \subseteq \mathbf{L}$).

Next, we verify $\mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. It is enough to prove $\mathcal{M} \twoheadrightarrow \mathcal{P}^0(\mathbb{N})$ as in this case $\mathbf{L} \subseteq \Lambda(\mathcal{P}^0(\mathbb{N})) = \mathbf{ML}_\omega \subseteq \mathbf{ML}_{<\omega}$. As $\mathcal{M}$ is infinite, there are countably infinite many pairwise disjoint elements $a_i \in \mathcal{M}$ $(i \in \mathbb{N})$ such that $\bigvee_{i \in \mathbb{N}} a_i = [0,1]/_{\sim}$. (Note that each a_i has a representative $A_i \in \mathcal{B}$ so that $a_i = A_i/_{\sim}$. Then $A_i \cap A_j$ is λ-measure zero for all $i \neq j$ and $\bigcup_{i \in \mathbb{N}} A_i = [0,1]$ (modulo

λ-measure zero)). Define $f : \mathcal{M} \to \mathcal{P}^0(\mathbb{N})$ as

$$f(a) = \{i \in \mathbb{N} : \ a \wedge a_i \neq 0\} \neq \emptyset$$

Then $f([0,1]/_{\sim}) = \mathbb{N}$ and $f(a_i) = \{i\}$. We claim that f is a surjective bounded morphism.
Surjectivity: For a non-empty $A \subseteq \mathbb{N}$ we have $f(\bigvee_{i \in A} a_i) = A$.
Homomorphism: Suppose for $a, b \in \mathcal{M}$ we have $a \geq b$. Then whenever $b \wedge a_i \neq 0$ we also have $a \wedge a_i \neq 0$, therefore

$$\{i \in \mathbb{N} : \ a \wedge a_i \neq 0\} \supseteq \{i \in \mathbb{N} : \ b \wedge a_i \neq 0\}$$

meaning that $f(a) \supseteq f(b)$.
Zig-zag property: Suppose $f(a) = A$ and $A \supseteq B \neq \emptyset$. We need to find a b with $a \geq b$ and $f(b) = B$. Take $b = a - \bigvee_{i \in A \smallsetminus B} a_i$. Then $a \geq b$ and $f(B) = \{i : \ b \wedge a_i \neq 0\} = B \smallsetminus A = B$.

Finally, to fulfill all requirements of Corollary 2.8 we prove that for all $u \in \mathcal{M}$ we have $\mathcal{M}^u \not\twoheadrightarrow \mathcal{C}(k, 2^k)$. As generated subframes of measure algebra frames are measure algebra frames (Lemma 2.14) it is enough to prove that $\mathcal{M} \not\twoheadrightarrow \mathcal{C}(k, 2^k)$.

Suppose, seeking a contradiction, that there is $f : \mathcal{M} \twoheadrightarrow \mathcal{C}(k, 2^k)$. Write $\mathcal{L}^+ = \{B \in \mathcal{L} : \lambda(B) > 0\}$ and let $\mathcal{N}$ be the frame $\mathcal{N} = \langle \mathcal{L}^+, \supseteq \rangle$. The mapping f can be lifted up to a surjective bounded morphism $f^+ : \mathcal{N} \twoheadrightarrow \mathcal{C}(k, 2^k)$ by letting $f^+(B) = f(B/_{\sim})$. Clearly, if $A \sim B$, then $f^+(A) = f^+(B)$. To complete the proof we would like to apply Corollary 2.11, but the problem is that $\mathcal{N}$ is not a finitary Boolean frame as finite subsets of $[0,1]$ has λ-measure zero. Therefore, we need to further extend f^+ as follows. Let A be a non-empty Lebesgue measurable set and suppose $\{A_i\}_{i \in \mathbb{N}}$ is a sequence of positive measure sets ($A_i \in \mathcal{L}^+$) such that $A_i \supseteq A_{i+1}$ modulo measure zero and $A = \bigcap_{i \in \mathbb{N}} A_i$. Then define $F(A) = \sup_{\{A_i\}_{i \in \mathbb{N}}} \lim_i f^+(A_i)$. For every such sequence $\{A_i\}$ the limit exists as $\mathcal{C}(k, 2^k)$ is finite and $f^+(A_i) \leq f^+(A_{i+1})$. If two sequences $\{A_i\}$ and $\{B_i\}$ give a different limit, then $\{A_i \cap B_i\}$ yields a greater (or equal) limit as $f^+(A_i \cap B_i) \geq f^+(A_i), f^+(B_i)$. It follows that the supremum always exists. Denote the frame extended with elements of such form by $\mathcal{N}^\star = \langle \mathcal{L}^\star, \supseteq \rangle$, where

$$\mathcal{L}^\star = \big\{A : \ A \in \mathcal{L}, \ \exists \{A_i\} \subseteq \mathcal{L}^+ : \ A = \cap_i A_i, \ A_i \supseteq A_{i+1}\big\}$$

It is straightforward to verify that $F : \mathcal{N}^\star \twoheadrightarrow \mathcal{C}(k, 2^k)$ is a surjective bounded morphism.

To complete the proof observe that each finite subset can be obtained in the form $\bigcap_{i \in \mathbb{N}} A_i$: if Z is a finite subset of $[0,1]$, then $Z = \bigcap_{i \in \mathbb{N}} (Z \cup (0, \frac{1}{i}))$. Therefore the frame $\mathcal{N}^\star$ is a finitary Boolean frame, thus Corollary 2.11 implies that there is no bounded morphism $\mathcal{N}^\star \twoheadrightarrow \mathcal{C}(k, 2^k)$. This contradicts to F being such a bounded morphism. ∎

Suppose X is a finite or countably infinite set and consider the probability space $\langle X, \wp(X), w \rangle$ for some probability measure w. We can assume that w is faithful, otherwise we would switch to a smaller X. As noted above $\mathcal{M}(X, \wp(X), w)$ is isomorphic to the Medvedev frame $\mathcal{P}^0(X)$. Consider now the probability space whose underlying set is $[0,1] \cup X$, its σ-algebra is the algebra generated by $\mathcal{L} \cup \wp(X)$ and the probability measure is $\nu = r \cdot \lambda + (1 - r) \cdot w$ for some $r \in (0,1)$.

Denote the corresponding measure algebra frame by $\mathcal{M}([0,1] \cup X, \mathcal{L} + \wp(X), \nu)$. (By replacing $\mathcal{L}$ with $\mathcal{B}$ we obtain the same measure algebra frame).

Theorem 2.16. *The modal logic of $\mathcal{M} = \mathcal{M}([0,1] \cup X, \mathcal{L} + \wp(X), \nu)$ is not finitely axiomatizable.*

Proof. The same proof as that of Theorem 2.15 works with the obvious modifications. The proof for $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$ is identical. To show that for all $u \in \mathcal{M}$ we have $\mathcal{M}^u \not\twoheadrightarrow \mathcal{C}(k, 2^k)$ we need two cases: if $u \leq X/_\sim$, then $\mathcal{M}^u$ is isomorphic to a subframe of $\mathcal{P}^0(X)$ which is a finitary Boolean frame, thus the proof completes by applying Corollary 2.11; if u intersects $[0,1]/_\sim$, then the proof continues identically. ∎

3 Standard Bayes logic is not finitely axiomatizable

Recall (from the Introduction) that every standard probability space is isomorphic (modulo zero) to $\langle \mathbb{N} + [0,1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda \rangle$, where $\langle \mathbb{N}, \wp(\mathbb{N}), p \rangle$ is an arbitrary probability space.

Lemma 3.1. There is no surjective bounded morphism $\mathcal{M}(\mathbb{N}+[0,1], \wp(\mathbb{N})+\mathcal{L}, p+\lambda) \twoheadrightarrow \mathcal{C}(k, 2^k)$.

Proof. The proof of this statement is part of the proof of Theorem 2.16 (cf. the proof of Theorem 2.15). ∎

Lemma 3.2. Let $\mathcal{F} = \mathcal{F}(X, \mathcal{B})$ be a Bayes frame. For any possible world (probability measure) $w \in \mathcal{F}$ the generated subframe $\mathcal{F}^w$ is isomorphic to $\mathcal{M}(X, \mathcal{B}, w)$.

Proof. As $\mathcal{F}^w$ is generated by w, for any $u \in \mathcal{F}^w$ there is a non w-measure zero set A_u such that $u(\cdot) = w(\cdot \mid A_u)$. This A_u is unique up to w-measure zero equivalence: if $A_u \sim_w A'_u$, then $w(\cdot \mid A_u) = w(\cdot \mid A'_u)$. Therefore u can be identified with the element $A_u/_\sim \in \mathcal{B}/_\sim$. Indeed, let f be the mapping $f : \mathcal{F}^w \to \mathcal{B}/_\sim \setminus \{\emptyset/_\sim\}$ defined by $f(u) = A_u/_\sim$. (In particular $f(w) = X/_\sim$ which is the top element of the Boolean algebra $\mathcal{B}/_\sim$, i.e. the element which generates $\mathcal{M}$). It is fairly easy to check that f is an isomorphism between the frames $\mathcal{F}$ and $\mathcal{M}$. ∎

Theorem 3.3. *Let $\langle X, \mathcal{B} \rangle$ be a standard Borel space and $\mathcal{F}(X, \mathcal{B})$ be the corresponding Bayes frame. The logic $\Lambda(\mathcal{F}(X, \mathcal{B}))$ is not finitely axiomatizable.*

Proof. Let $\mathbf{L} = \Lambda(\mathcal{F}(X, \mathcal{B}))$. We intend to apply Corollary 2.8. We already know that $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$ (cf. Theorem 1.3) thus we only need to see that for any k and every $u \in \mathcal{F}$ we have $\mathcal{F}^u \not\twoheadrightarrow \mathcal{C}(k, 2^k)$. By Lemma 3.2 we know that $\mathcal{F}^u$ is isomorphic to $\mathcal{M}(X, \mathcal{B}, u)$. But as $\langle X, \mathcal{B} \rangle$ is a standard Borel space, the probability space $\langle X, \mathcal{B}, u \rangle$ is also standard (more precisely, the completion of it is standard), therefore it is isomorphic modulo zero to $\langle \mathbb{N} + [0,1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda \rangle$ for some p. Lemma 3.1 states there is no bounded morphism $\mathcal{M}(\mathbb{N}+[0,1], \wp(\mathbb{N})+\mathcal{L}, p+\lambda) \twoheadrightarrow \mathcal{C}(k, 2^k)$, which completes the proof. ∎

Theorem 3.4. *The standard Bayes logic*

$$\mathbf{BL}_{st} = \{\phi : \ (\forall \textit{Standard Borel Bayes frames } \mathcal{F})\ \mathcal{F} \Vdash \phi\}$$

is not finitely axiomatizable.

Proof. Take $\mathcal{F} = \mathcal{F}(\mathbb{N} + [0,1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda)$, where p is any faithful measure on $\wp(\mathbb{N})$. Then every standard Borel Bayes frame is a generated subframe of $\mathcal{F}$, thus $\Lambda(\mathcal{F}) = \mathbf{BL}_{st}$. But then Theorem 3.3 applies and completes the proof. ∎

Open problems. We have seen that none of the logics $\mathbf{BL}_{<\omega}$, $\mathbf{BL}_{\omega}$ and $\mathbf{BL}_{st}$ are finitely axiomatizable (moreover, none of them can be axiomatized by formulas using finitely many propositional variables only). We know that the following containments hold:

$$\mathbf{S4} \ \subseteq \ \mathbf{BL} \ \subseteq \ \mathbf{BL}_{st} \ \subsetneq \ \mathbf{S4.1} \ \subsetneq \ \mathbf{BL}_{\omega} = \mathbf{BL}_{\leq\omega} \ \subsetneq \ \mathbf{S4.1} + \mathbf{Grz} \ \subsetneq \ \mathbf{BL}_{<\omega}$$

Problem 3.5. Is $\mathbf{BL} = \mathbf{BL}_{st}$?

Problem 3.6. Is $\mathbf{BL} = \mathbf{S4}$?

Problem 3.7. If $\mathbf{S4} \neq \mathbf{BL} \neq \mathbf{BL}_{st}$, then is $\mathbf{BL}$ finitely axiomatizable?

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