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Lagrange and Appell-Gibbs approaches in problems of non-holonomic dynamic systems

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Hamiltonian functional and relevant Lagrange equation system are popular tools in investigation of dynamic systems. Various generalizations enable to extend the class of problems concerned slightly beyond conventional limits of a Hamiltonian system. This strategy is very effective particularly concerning 2D and simpler 3D systems. However, the governing differential systems of most non-holonomic 3D systems suffer from inadequate complexity, when deduced using this way. Any analytical investigation of such a governing system is rather impossible and its physical interpretation can be multivalent. For easier analysis particularly of systems with non-holonomic constraints the Appell-Gibbs approach seems to be more effective providing more transparent governing systems.

Both Lagrangian as well as Appell-Gibbs procedures originate from a balance of internal energy change in time on one hand and external energy supply (e.g. excitation) and dissipation on the other hand. Formulating an adequate functional equivalence a suitable variational principle is applied. It claims that among all admissible shapes of the system reaction, the selected one is that which represents the minimal energy requirement.

This manipulation being based on the third form of the basic equation of dynamics can bring us to the system of Lagrangian equations of the 2nd kind. It reads for dynamic system with n Degrees of Freedom (DOF) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\Big) - \frac{\partial \mathcal{T}}{\partial q_s} + \frac{\partial \mathcal{V}}{\partial q_s} = Q_s + \frac{\partial \mathcal{R}}{\partial \dot{q}_s} + \sum_{r=1}^{l} \lambda_r C_{rs}, \quad s = 1,..,n, \tag{1}$$

where $q_s, s = 1,..,n$ are Lagrange coordinates, $\mathcal{T}, \mathcal{V}$ is kinetic and potential energy, respectively, Q_s are generalized forces, $\mathcal{R}$ is the Rayleigh function representing a potential of damping forces, if it exists. Otherwise individual expressions characterizing more complex laws of the damping should be included. For more information, see either monographs [2, 5] or hundreds of papers, e.g., [3, 4].

The system of equations (1) should be completed by $l < n$ equations of links, which can be expressed in a form inherent to both holonomic/non-holonomic links

$$\sum_{s=1}^{n} C_{rs}\dot{q}_s + D_r = 0,\ r = 1,..,l,\ \mathbf{D} = |D_r(\mathbf{q},t)|\} \in \mathbb{R}^l\ ,\ \mathbf{C} = |C_{rs}(\mathbf{q},t)| \in \mathbb{R}^{ln}. \tag{2}$$

Eqs. (1) and (2) represent the system of $n + l$ differential equations with $n + l$ unknowns $\mathbf{q}, \boldsymbol{\lambda}$. After rewriting into the normal form, we obtain the system

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u},t), \qquad \mathbf{u},\ \mathbf{f} \in \mathbb{R}^{2n}, \tag{3}$$

which includes $2n + l$ equations with $2n + l$ unknowns $\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\lambda}$. Functions $\mathbf{f}(\mathbf{u}, t)$ are nonlinear functions (smooth enough) of displacements, time t and system parameters representing geometry and physical properties of the system. If the dynamic system is holonomic and constraints (2) are integrable, then it is (sometimes) possible to reduce the number of unknowns eliminating relevant "fixed" degrees of freedom and multipliers $\boldsymbol{\lambda}$. Then we have $2k$ unknowns, where $k = n - l$.

Let us pay attention to Appell-Gibbs approach. The method follows from the 5th Gaussian form of basic equation of Dynamics, for details see, e.g., [1]. The core consists in the Gibbs function $\mathcal{G}$ and function $\mathcal{H}$

$$\mathcal{G} \;=\; \frac{1}{2}\sum_{r=1}^{k} m_r \ddot{q}_r^2, \quad \mathcal{H} = \mathcal{G} - \sum_{r=1}^{k} Q_r \ddot{q}_r. \tag{4}$$

Take a note that summations (4) reflect only "live" degrees of freedom $k = n - l$. It can be shown, that the governing differential system follows from minimization of the function $\mathcal{H}$ with respect to acceleration coordinates. In other words it holds

$$\frac{\partial \mathcal{G}}{\partial \ddot{q}_r} \;=\; Q_r, \quad r = 1, .., k. \tag{5}$$

The energy dissipation terms on the right side of Eq. (5) are to be added similarly like generalized forces using the virtual works principle. The system (5) should be completed by geometric links (2). The differential system of equations (2) and (5) includes $k+l$ unknowns. It represents the most simple and in the same time the most general form of equations of the dynamic system movement. The form of this system is very simple and it can be used with the same effectiveness to investigation of holonomic as well as non-holonomic systems, as the constraints can represent non-holonomic but also holonomic type of links. Unlike the Lagrangian approach the non-holonomic links do not augment the number of differential equations.

The procedure working with accelerations instead of velocities provides incomparably simpler and more transparent governing differential system. It enables easier and more effective analytical investigation and qualitative analysis unlike the Lagrangian governing system. Also the number of unknowns and equations in general is either the same or lower than that provided using the Lagrangian procedure.

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