Planar graphs have two-coloring number at most 8*

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Abstract

We prove that the two-coloring number of any planar graph is at most 8. This resolves a question of Kierstead et al. [SIAM J. Discrete Math. 23 (2009), 1548–1560]. The result is optimal.

1 Introduction

We study the two-coloring number of graphs. This parameter was introduced by Chen and Schelp [2] under the name of p-arrangeability; they related it to the Ramsey numbers of graphs and the Burr–Erdős conjecture [1]. It was subsequently found to be related to coloring properties of graphs, such as the game chromatic number, the acyclic chromatic number or the degenerate chromatic number (see [3] and the references therein).

We now recall the definition of the two-coloring number. Let G be a graph and let $\prec$ be a linear ordering of its vertices. (In this paper, graphs are allowed to have parallel edges, but not loops.) For a vertex $v \in V(G)$, let $L_{G,\prec}(v)$ be the set consisting of the vertices $u \in V(G)$ such that $u \prec v$ and either

- $uv \in E(G)$, or

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- u and v have a common neighbor $w \in V(G)$ such that $v \prec w$.

We say that an ordering $\prec$ is *d-two-degenerate* if $|L_{G,\prec}(v)| \leq d$ for every $v \in V(G)$. The *two-coloring number* $\mathrm{col}_2(G)$ of G is defined as $d+1$ for the smallest integer d such that there exists a d-two-degenerate ordering of the vertices of G.

Already in [2], the two-coloring number of planar graphs was bounded by an absolute constant, namely 761. The bound was improved to 10 in [4] and eventually to 9 in [3]. On the other hand, a planar graph with two-coloring number equal to 8 was constructed in [4]. Kierstead et al. [3] found simpler examples yielding the same lower bound (namely, any 5-connected triangulation in which the degree 5 vertices are non-adjacent has this property) and asked whether the two-coloring number of all planar graphs is bounded by 8.

We answer this question in the affirmative:

Theorem 1. *The two-coloring number of any planar graph is at most* 8*.*

It was observed in [3] that the list-degenerate chromatic number of a graph G is bounded by the two-coloring number of G. By this observation, Theorem 1 improves the known upper bound for the list-degenerate chromatic number of planar graphs, as well as for the ordinary degenerate chromatic number of planar graphs, to 8.

The structure of this paper is as follows. In the remainder of this section, we formulate a more general version of Theorem 1 that is better suited for an inductive proof (Theorem 2 below). Section 2 focuses on the basic structural properties of a hypothetical minimal counterexample. These properties are used in Section 3 in a discharging procedure that provides a contradiction, establishing Theorem 2 and hence also Theorem 1.

It will be useful to consider the following relative version of the notion of d-two-degenerate ordering. Let G be a graph, let C be a subset of its vertices and let $\prec$ be a linear ordering of $V(G) \setminus C$. For a vertex $v \in V(G) \setminus C$, let $L_{G,C,\prec}(v)$ be the set consisting of the vertices $u \in V(G) \setminus C$ such that $u \prec v$ and either

- $uv \in E(G)$, or
- u and v have a common neighbor $w \in V(G) \setminus C$ such that $v \prec w$, or
- u and v have a common neighbor $w \in C$.

We say that an ordering $\prec$ is *d-two-degenerate relative to C* if $|L_{G,C,\prec}(v)| \leq d$ for every $v \in V(G) \setminus C$.

Theorem 2. *Let G be a plane graph and let K be a set of at most three vertices incident with the outer face of G. Let C be a subset of $V(G)$ disjoint from K such that every vertex of C has at most 4 neighbors in $V(G) \setminus C$. There exists an ordering $\prec$ of $V(G) \setminus C$ that is 7-two-degenerate relative to C, such that $u \prec v$ for every $u \in K$ and $v \in V(G) \setminus (C \cup K)$.*

Note that Theorem 1 follows from Theorem 2 by setting $C = K = \emptyset$.

2 Basic properties of a minimal counterexample

Before we embark on the study of the properties of a minimal counterexample to Theorem 2, let us define the notion of minimality more precisely.

A *target* is a triple (G, K, C), where G is a plane graph, K is the set of all vertices incident with the outer face of G, $2 \le |K| \le 3$, and C is a subset of $V(G)$ disjoint from K such that every vertex of C has at most 4 neighbors in $V(G) \setminus C$. Note that it suffices to show that Theorem 2 holds for every target, since if $|K| \le 1$, then we can add $2 - |K|$ new isolated vertices into the outer face of G and include them in K, and we can add edges between the vertices of K to ensure that the outer face of G is only incident with the vertices of K. An ordering $\prec$ of $V(G) \setminus C$ is *valid* if $\prec$ is 7-two-degenerate relative to C and $u \prec v$ for every $u \in K$ and $v \in V(G) \setminus (C \cup K)$. We say that a target (G, K, C) is a *counterexample* if there exists no valid ordering $\prec$ of $V(G) \setminus C$. Let $s(G, K, C) = (n, -c, e_C, q, -t, e)$, where $n = |V(G)|$, $c = |C|$, e_C is the number of edges of G with at least one end in C, q is the number of components of G, t is the number of triangular faces of G, and $e = |E(G)|$. A target (G', K', C') is *smaller* than (G, K, C) if $s(G', K', C')$ is lexicographically smaller than $s(G, K, C)$ (observe that this establishes a well-quasiordering on targets). We say that a counterexample is *minimal* if there exists no smaller counterexample.

In a series of lemmas, we now establish the basic properties of minimal counterexamples.

Lemma 3. *If (G, K, C) is a minimal counterexample, then the following hold:*

(i) *C is an independent set,*

(ii) *all vertices of C have degree 4,*

(iii) *G is connected, and*

(iv) *all faces of G except possibly for the outer face have length 3.*

Proof. We prove claim (i). If an edge $e \in E(G)$ has both ends in C, then $(G-e, K, C)$ is a target smaller than (G, K, C), and by the minimality of (G, K, C), there exists a valid ordering $\prec$ for the target $(G-e, K, C)$. Note that $L_{G,C,\prec}(v) = L_{G-e,C,\prec}(v)$ for every $v \in V(G) \setminus C$, and thus $\prec$ is also valid for the target (G, K, C), which is a contradiction. Hence, C is an independent set in G.

We continue with claim (iii). Suppose that G is not connected. Hence, G contains a face f incident with at least two distinct components G_1 and G_2 of G. If G_1 or G_2 consists of only one vertex $v \in C$, then $(G-v, K, C \setminus \{v\})$ is a target smaller than (G, K, C) and its valid ordering is also valid for (G, K, C), which is a contradiction. Otherwise, since C is an independent set, there exist vertices $v_1 \in V(G_1) \setminus C$ and $v_2 \in V(G_2) \setminus C$ incident with f. Then, $(G + v_1v_2, K, C)$ is a target smaller than (G, K, C) (with fewer components) and its valid ordering is also valid for (G, K, C), which is a contradiction. Hence, G is connected.

We next prove claim (iv). Suppose that G has a non-outer face f of length other than three. If f has length 2 and not all its edges belong to the boundary of the outer face, then removing one of its edges results in a target smaller than (G, K, C) whose valid ordering is also valid for (G, K, C), which is a contradiction. If f has length two and all its edges belong to the boundary of the outer face, then $V(G) = K$ and (G, K, C) has a valid ordering, which is a contradiction. Hence, f has length at least 4. Let $f = v_1v_2v_3v_4\ldots$, with the labels chosen so that $v_2 \in C$ if any vertex of C is incident with f. Since C is an independent set, it follows that $v_1, v_3 \notin C$. If $v_1 \neq v_3$, then $(G + v_1v_3, K, C)$ is a target smaller than (G, K, C) (with more triangular faces) and its valid ordering is also valid for (G, K, C), which is a contradiction. Hence, $v_1 = v_3$.

If $v_2 \in C$ and v_2 has degree at least two, then removing one of at least two edges between v_2 and $v_1 = v_3$ results in a target smaller than (G, K, C) whose valid ordering is also valid for (G, K, C). If $v_2 \in C$ and v_2 has degree exactly one, then $(G - v_2, K, C \setminus \{v_2\})$ is a target smaller than (G, K, C) whose valid ordering is also valid for (G, K, C). In both cases, we obtain a contradiction, and thus $v_2 \notin C$.

By the choice of the labels of f, it follows that no vertex of C is incident with f. Furthermore, note that $v_1 = v_3$ is a cut in G, and thus $v_2 \neq v_4$. Consequently, $(G + v_2v_4, K, C)$ is a target smaller than (G, K, C) whose valid ordering is also valid for (G, K, C). This contradiction shows that every non-outer face of G has length three.

It remains to prove claim (ii). Suppose that a vertex $v \in C$ has degree at most three. Since $v \notin K$, the faces incident with v have length three, and thus the neighborhood of v forms a clique in G. The target $(G-v, K, C\setminus\{v\})$ is smaller than (G, K, C), and thus it has a valid ordering $\prec$. Suppose that for some vertices $x, y \in V(G) \setminus C$, we have $x \in L_{G,C,\prec}(y)$. If v is not a common neighbor of x and y, then clearly $x \in L_{G-v,C\setminus\{v\},\prec}(y)$. If v is a common neighbor of x and y, then x and y are adjacent, and thus $x \in L_{G-v,C\setminus\{v\},\prec}(y)$. It follows that $L_{G,C,\prec}(y) = L_{G-v,C\setminus\{v\},\prec}(y)$ for every $y \in V(G) \setminus C$, and thus $\prec$ is a valid ordering for (G, K, C). This is a contradiction, and thus all vertices of C have degree at least 4.

Note that a vertex of C is not incident with parallel edges, as suppressing them would result in a target smaller than (G, K, C) whose valid ordering is also valid for (G, K, C). Since C is an independent set and every vertex of C has at most 4 neighbors not in C, it follows that all vertices in C have degree exactly 4. □

Consider a target (G, K, C). For a vertex $v \in V(G) \setminus C$, let $a(v)$ be the number of neighbors of v in $V(G)\setminus C$, and let $b(v)$ be the number of neighbors of v in C (counted with multiplicity when v is incident with parallel edges). For non-negative integers a, b, we say that v is an *(a, b)-vertex* if $a(v) = a$ and $b(v) = b$. Similarly, we say that v is an *$(a, \leq b')$-vertex* if $a(v) = a$ and $b(v) \leq b'$.

Corollary 4. *If (G, K, C) is a minimal counterexample and $v \in V(G) \setminus C$, then $a(v) \geq b(v)$. Furthermore, if $a(v) = b(v)$ and $u \in V(G)\setminus C$ is a neighbor of v, then $b(u) \geq 2$.*

Proof. If $v \notin K$, then all faces incident with v are triangles. If $v \in K$, then all faces except possibly for the outer one are triangles, and no vertex of the outer face belongs to C. Since C is an independent set, at most half of the neighbors of v belong to C, and thus $b(v) \leq a(v)$. Furthermore, if $b(v) = a(v)$, then every second neighbor of v belongs to C, and thus u and v have two common neighbors belonging to C. □

Lemma 5. *If (G, K, C) is a minimal counterexample and $v \in V(G) \setminus (K \cup C)$, then $a(v) \geq 4$, and if $a(v) = 4$, then $b(v) = 4$.*

Proof. Suppose for a contradiction that $v \in V(G) \setminus (K \cup C)$ satisfies either $a(v) \leq 3$, or $a(v) = 4$ and $b(v) \leq 3$. By Corollary 4, in the former case we have $b(v) \leq a(v)$.

Since v has at most 4 neighbors in $V(G)\setminus C$, it follows that $(G, K, C\cup\{v\})$ is a target. Note that $(G, K, C \cup \{v\})$ is smaller than (G, K, C), and let $\prec$ be its valid ordering. Extend $\prec$ to $V(G) \setminus C$ by letting $u \prec v$ for every $u \in V(G) \setminus (C \cup \{v\})$. Note that $L_{G,C\cup\{v\},\prec}(w) = L_{G,C,\prec}(w)$ for every

$w \in V(G) \setminus (C \cup \{v\})$. Furthermore, $L_{G,C,\prec}(v)$ contains only the neighbors of v that do not belong to C, and the vertices z such that z and v have a common neighbor $w \in C$. However, since all faces of G incident with v are triangles and all vertices in C have degree 4, each neighbor $w \in C$ of v has at most one neighbor z not adjacent to v. Therefore, $|L_{G,C,\prec}(v)| \leq \deg(v) = a(v) + b(v) \leq 7$, and thus $\prec$ is a valid ordering for (G, K, C). This is a contradiction. □

Lemma 6. *Suppose that (G, K, C) is a minimal counterexample. If $|K| = 3$, then G contains no parallel edges and all triangles in G bound a face. If $|K| = 2$, then the edges bounding the outer face of G are the only parallel edges in G, and every non-facial triangle in G contains a vertex of C and both vertices of K.*

Proof. Consider either a pair of parallel edges that do not bound the outer face of G, or a non-facial triangle in G. Since all faces of G except for the outer one have length three, in the former case G contains a non-facial cycle of length two. Hence, let Q be a non-facial cycle of length 2 or 3 in G.

Suppose first that $V(Q) \cap C = \emptyset$. Let G_1 be the subgraph of G drawn in the closure of the outer face of Q, and let G_2 be the subgraph of G drawn in the closure of the inner face of Q. Let $C_1 = C \cap V(G_1)$ and $C_2 = C \cap V(G_2)$. Note that (G_1, K, C_1) and $(G_2, V(Q), C_2)$ are targets, and since Q is a non-facial cycle, they are both smaller than (G, K, C) and they have valid orderings $\prec_1$ and $\prec_2$, respectively. Let $\prec$ be the ordering of $V(G) \setminus C$ such that $u \prec v$ if $u, v \in V(G_1)$ and $u \prec_1 v$, or if $u, v \in V(G_2) \setminus V(Q)$ and $u \prec_2 v$, or if $u \in V(G_1)$ and $v \in V(G_2) \setminus V(Q)$.

Observe that for any $v \in V(G_1) \setminus (V(Q) \cup C_1)$, we have $L_{G,C,\prec}(v) = L_{G_1,C_1,\prec_1}(v)$, since v has no neighbors in $V(G_2)$ other than those belonging to Q (which are also contained in G_1), and since $v \prec w$ for every $w \in V(G_2) \setminus V(Q)$. Similarly, for any $v \in V(G_2) \setminus (V(Q) \cup C_2)$, we have $L_{G,C,\prec}(v) = L_{G_2,C_2,\prec_2}(v)$, since v has no neighbors in $V(G_1)$ other than those belonging to Q, and all the vertices of Q are contained in G_2 and are smaller than v in both orderings $\prec$ and $\prec_2$. Finally, for $v \in V(Q)$ we have $L_{G,C,\prec}(v) = L_{G_1,C_1,\prec_1}(v)$, since all vertices of $V(G_2) \setminus (V(Q) \cup C_2)$ are greater than v in $\prec$ and Q is a clique, so all vertices of Q smaller than v in $\prec$ or $\prec_1$ belong to both $L_{G,C,\prec}(v)$ and $L_{G_1,C_1,\prec_1}(v)$. Furthermore, since $K \subseteq V(G_1)$, the choice of $\prec$ ensures that $u \prec v$ for every $u \in K$ and $v \in V(G) \setminus (C \cup K)$. Hence, $\prec$ is a valid ordering of (G, K, C), which is a contradiction.

Therefore, every non-facial (≤ 3)-cycle in G intersects C. Since C is an independent set, Q contains exactly one vertex of C. If Q has length two, then removing one of the parallel edges of Q results in a target smaller than (G,K,C) whose valid ordering is also valid for (G,K,C). It follows that G contains no parallel edges except possibly for those bounding its outer face, and in particular Q is a triangle.

Let $Q = v_1v_2v_3$, where $v_1 \in C$. Let e and e' be the edges of G incident with v_1 distinct from v_1v_2 and v_1v_3. If exactly one of the edges e and e' is contained in the open disk bounded by Q, then consider the neighbor v_4 of v_1 in the open disk bounded by Q. Since all faces incident with v_1 have length three, v_4 is adjacent to v_2 and v_3. Since the triangle $v_2v_3v_4$ does not intersect C, it bounds a face. However, this implies that v_4 is a $(2,1)$-vertex, which contradicts Lemma 5.

If neither e nor e' is contained in the open disk bounded by Q, then since Q is not a facial triangle and all faces incident with v_1 have length three, it follows that v_2 and v_3 would be joined by a parallel edge drawn inside the open disk bounded by Q; however, this is impossible, since such a parallel edge is not incident with the outer face of G. Finally, consider the case that both e and e' are contained in the open disk bounded by Q. Similarly, v_2 and v_3 are joined by a parallel edge, and thus $K = \{v_2, v_3\}$. We conclude that every non-facial triangle in G contains a vertex of C and two vertices of K. □

Corollary 7. *If (G,K,C) is a minimal counterexample, then every vertex of K has degree at least 4.*

Proof. Suppose first that a vertex $v \in K$ has degree two. Since all faces of G except for the outer one are triangles, if $|K| = 2$, this would imply that G contains a loop, which is a contradiction. If $|K| = 3$, then since all faces of G are triangles and G does not contain parallel edges, we have $V(G) = K$, and any ordering of $V(G)$ is valid, which is a contradiction.

Next, suppose that v has degree three, and let x be the neighbor of v not belonging to K. If $|K| = 2$, then since all faces incident with x are triangles and x is not incident with a parallel edge, it follows that $V(G) = K \cup \{x\}$ and x has degree two. If $|K| = 3$, say $K = \{v, y_1, y_2\}$, then since all faces of G are triangles, it follows that vxy_1 and vxy_2 are triangles. Also, every triangle in G is facial, and thus x has degree three. In both cases, $x \notin C$ and x is a $(2,0)$-vertex or a $(3,0)$-vertex, which contradicts Lemma 5. □

Let $\prec$ be an ordering of $V(G) \setminus C$ in a target (G,K,C). For adjacent vertices $u \in V(G) \setminus C$ and v, a vertex $w \in V(G) \setminus C$ distinct from u is a *friend of u via v* if $w \prec u$ and

- $w = v$, or
- $vw \in E(G)$, $uw \notin E(G)$, and $v \in C$, or
- $vw \in E(G)$, $uw \notin E(G)$, u and w do not have a common neighbor in C, and $u \prec v$.

Note that $L_{G,C,\prec}(u)$ consists exactly of the friends of u via its neighbors. We will frequently use the following observations.

Lemma 8. *Let (G, K, C) be a minimal counterexample and let $u \in V(G) \setminus (C \cup K)$ and $v \in V(G)$ be neighbors. Let $\prec$ be an ordering of $V(G) \setminus C$. Then the following hold:*

(i) if $v \prec u$ or $v \in C$, then u has at most one friend via v,

(ii) if $v \notin C \cup K$ and $u \prec v$, then u has at most $a(v) - 3$ friends via v,

(iii) if $v \notin C \cup K$, $u \prec v$, and v has a neighbor $r \notin C$ non-adjacent to u such that $u \prec r$ and no vertex of C is a common neighbor of u, v and r, then u has at most $a(v) - 4$ friends via v.

Proof. (i) If $v \prec u$, then v is the only friend of u via v. If $v \in C$, then since all faces incident with v are triangles and v has degree 4, the vertex v has at most one neighbor not adjacent to u, and thus u has at most one friend via v.

(ii) Suppose that $v \notin C \cup K$ and $u \prec v$. By Lemma 5, we have $a(v) \geq 4$, and since all faces incident with v have length three, it follows that v has at least two neighbors $z_1, z_2 \notin C$ distinct from u such that for $i \in \{1, 2\}$, either uvz_i is a face, or z_i and u have a common neighbor $z'_i \in C$ such that uvz'_i is a face. In either case, z_i is not a friend of u via v. Since u is not a friend of u via v, it follows that u has at most $a(v) - 3$ friends via v.

(iii) Let us now additionally assume that v has a neighbor r as described in the last case of the lemma. Using the notation from the previous case, we first show that the vertex z_1 is distinct from r. This is clearly the case if z_1 is adjacent to u. Suppose then that z_1 is not adjacent to u, and thus z_1 is a neighbor of a vertex $z'_1 \in C$ such that uvz'_1 is a face. But then u, v and z_1 have a common neighbor in C, and thus $r \neq z_1$. By a symmetric argument, $r \neq z_2$.

Since $u \prec r$, the vertex r is not a friend of u, and thus u has at most $a(v) - 4$ friends via v. □

Lemma 9. *If (G, K, C) is a minimal counterexample, then G contains no path $P = v_1 v_2 \dots v_k$ with $k \geq 2$ disjoint from K, such that v_1 is a $(5, \leq 1)$-vertex, $v_2, \dots, v_{k-1}$ are $(6, 0)$-vertices, and v_k is a $(5, \leq 2)$-vertex.*

Proof. Suppose for a contradiction that G contains such a path P. Without loss of generality, P is an induced path. Furthermore, P is disjoint from C by Lemma 3. Note that each vertex of P has at most 4 neighbors in $V(G) \setminus (C \cup V(P))$, and thus $(G, K, C \cup V(P))$ is a target smaller than (G, K, C). Let $\prec$ be a valid ordering of $(G, K, C \cup V(P))$, and let us extend the ordering to (G, K, C) by setting $u \prec v_1 \prec v_2 \prec \dots \prec v_k$ for every $u \in V(G) \setminus (C \cup V(P))$. Observe that $L_{G,C \cup V(P),\prec}(u) = L_{G,C,\prec}(u)$ for every $u \in V(G) \setminus (C \cup V(P))$. By Lemma 8, v_k has at most one friend via each of its neighbors, and thus $|L_{G,C,\prec}(v_k)| \leq 7$. The vertex v_{k-1} has at most 2 friends via v_k and at most one friend via each of its neighbors distinct from v_k, and thus $|L_{G,C,\prec}(v_{k-1})| \leq 7$. Consider any $i = 1, \dots, k-2$. By Lemma 8, the vertex v_i has at most 2 friends via v_{i+1} (because v_{i+1} is a $(6, 0)$-vertex and we can set $r = v_{i+2}$) and at most one friend via each of its neighbors distinct from v_{i+1}, and thus $|L_{G,C,\prec}(v_i)| \leq 7$. Therefore, $\prec$ is a valid ordering for (G, K, C), which is a contradiction. □

Lemma 10. *If (G, K, C) is a minimal counterexample, then G contains no induced cycle $Q = v_1 v_2 \dots v_k$ with $k \geq 4$ disjoint from K, such that v_k is a $(5, \leq 2)$-vertex and $v_1, \dots, v_{k-1}$ are $(6, 0)$-vertices.*

Proof. Suppose for a contradiction that G contains such an induced cycle Q. Clearly, Q is disjoint from C by Lemma 3. Note that each vertex of Q has at most 4 neighbors in $V(G) \setminus (C \cup V(Q))$, and thus $(G, K, C \cup V(Q))$ is a target smaller than (G, K, C). Let $\prec$ be a valid ordering of $(G, K, C \cup V(Q))$, and let us extend the ordering to (G, K, C) by setting $u \prec v_1 \prec v_2 \prec \dots \prec v_k$ for every $u \in V(G) \setminus (C \cup V(Q))$. Observe that $L_{G,C \cup V(Q),\prec}(u) = L_{G,C,\prec}(u)$ for every $u \in V(G) \setminus (C \cup V(Q))$. By Lemma 8, v_k has at most one friend via each of its neighbors, and thus $|L_{G,C,\prec}(v_k)| \leq 7$. The vertex v_{k-1} has at most 2 friends via v_k and at most one friend via each of its neighbors distinct from v_k, and thus $|L_{G,C,\prec}(v_{k-1})| \leq 7$. Consider any $i = 2, \dots, k-2$. By Lemma 8, the vertex v_i has at most 2 friends via v_{i+1} (because v_{i+1} is a $(6, 0)$-vertex and we can set $r = v_{i+2}$) and at most one friend via each of its neighbors distinct from v_{i+1}, and thus $|L_{G,C,\prec}(v_i)| \leq 7$. Finally, the $(6, 0)$-vertex v_1 has at most two friends via v_2, at most one friend via v_k (since we can set $r = v_{k-1}$), and at most one friend via each of its neighbors distinct from v_2 and v_k, and thus $|L_{G,C,\prec}(v_1)| \leq 7$. Therefore, $\prec$ is a valid ordering for (G, K, C), which is a contradiction. □

Lemma 11. *If (G, K, C) is a minimal counterexample, then G contains no path $P = v_1 v_2 \ldots v_k$ with $k \geq 3$ disjoint from K, such that v_1 is a $(5, \leq 1)$-vertex, $v_2, \ldots, v_{k-2}$ are $(6, 0)$-vertices (if $k \geq 4$), v_{k-1} is a $(6, 1)$-vertex and v_k is a $(5, 0)$-vertex.*

Proof. Suppose for a contradiction that G contains such a path P. Without loss of generality, P is an induced path (v_k has no neighbors in P distinct from v_{k-1} by Lemma 9). Note that P is disjoint from C, and each vertex of P has at most 4 neighbors in $V(G) \setminus (C \cup V(P))$, and thus $(G, K, C \cup V(P))$ is a target smaller than (G, K, C). Let $\prec$ be a valid ordering of $(G, K, C \cup V(P))$, and let us extend the ordering to (G, K, C) by setting $u \prec v_1 \prec v_2 \prec \ldots \prec v_{k-2} \prec v_k \prec v_{k-1}$ for every $u \in V(G) \setminus (C \cup V(P))$. Observe that $L_{G,C\cup V(P),\prec}(u) = L_{G,C,\prec}(u)$ for every $u \in V(G) \setminus (C \cup V(P))$. By Lemma 8, v_{k-1} has at most one friend via each of its neighbors, and thus $|L_{G,C,\prec}(v_{k-1})| \leq 7$. The vertex v_k has at most 3 friends via v_{k-1} and at most one friend via each of its neighbors distinct from v_{k-1}, and thus $|L_{G,C,\prec}(v_k)| \leq 7$. Consider any $i = 1, \ldots, k-2$. By Lemma 8, the vertex v_i has at most 2 friends via v_{i+1} (because we can set $r = v_{i+2}$ and either v_{i+1} is a $(6, 0)$-vertex, or r is a $(5, 0)$-vertex) and at most one friend via each of its neighbors distinct from v_{i+1}, and thus $|L_{G,C,\prec}(v_i)| \leq 7$. Therefore, $\prec$ is a valid ordering for (G, K, C), which is a contradiction. □

3 Discharging

Let us now proceed with the discharging phase of the proof. Let (G, K, C) be a minimal counterexample. Let us assign charge $c'_0(v) = 10\deg(v) - 60$ to each vertex $v \in V(G)$. Since all faces of G except possibly for the outer one have length three, we have $|E(G)| = 3|V(G)| - 3 - |K|$, and thus

$$\begin{aligned}\sum_{v\in V(G)} c'_0(v) &= -60|V(G)| + 10 \sum_{v\in V(G)} \deg(v) \\ &= -60|V(G)| + 20|E(G)| = -60 - 20|K|.\end{aligned}$$

Next, every vertex of $v \in V(G) \setminus C$ sends charge of 5 to every adjacent vertex in C, and let c_0 denote the resulting assignment of charge. Since the total amount of charge does not change, we have $\sum_{v\in V(G)} c_0(v) = -60 - 20|K|$. If $v \in C$, then $\deg(v) = 4$, $c'_0(v) = -20$, and v receives 5 from each of its neighbors, and thus $c_0(v) = 0$. An (a, b)-vertex $v \in V(G) \setminus C$ has $c'_0(v) = 10a + 10b - 60$ and v sends 5 to b of its neighbors, and thus $c_0(v) = 10a + 5b - 60$.

We say that a vertex $v \in V(G) \setminus C$ is *big* if $v \in K$ or $c_0(v) > 0$ (i.e., v is not a $(4,4)$-vertex, a $(5,\leq 2)$-vertex, or a $(6,0)$-vertex). We call vertices not belonging to K *internal*. Next, we redistribute the charge according to the following rules, obtaining the *final charge* c.

R1 Every big vertex sends 2 to each neighboring internal $(5,0)$-vertex.

R2 Every big vertex sends 1 to each neighboring internal $(5,1)$-vertex.

R3 If $v_1v_2\dots v_k$ with $k \geq 3$ is a path in G such that v_1xv_2, v_2xv_3, $\dots$, $v_{k-1}xv_k$ are faces for some vertex x, v_1 is big, x is either big or an internal $(6,0)$-vertex, v_2, $\dots$, v_{k-1} are internal $(6,0)$-vertices, and v_k is an internal $(5,\leq 1)$-vertex, then v_1 sends 1 to v_k.

In the case of rule R3, we say that the charge *arrives to* v_k *through pair* (v_{k-1}, x), and *departs* v_1 *through pair* (v_2, x). Note that it is possible for charge to arrive through (x, v_{k-1}) or depart through (x, v_2) as well, if x is an internal $(6,0)$-vertex. If the charge departs through both (v_2, x) and (x, v_2), we say that the edge v_2x is *heavy for* v_1. The key observations concerning the rule R3 are the following.

Lemma 12. *Let (G,K,C) be a minimal counterexample, let v be an internal $(5,\leq 1)$-vertex, and let vu_1x be a face of G. If u_1 is an internal $(6,0)$-vertex, then charge arrives to v through (u_1, x).*

Proof. By Lemma 9, x is not an internal $(5,\leq 2)$-vertex, and by Corollary 4, x is not a $(4,4)$-vertex. Hence, x is either big or an internal $(6,0)$-vertex.

Let vu_1x, u_1u_2x, u_2u_3x, $\dots$, $u_{k-1}u_kx$ be faces of G incident with x in order, where $k \geq 2$ is chosen minimum such that u_k is not an internal $(6,0)$-vertex (possibly $u_k = v$). If u_k is big, then it sends charge to v by R3 and this charge arrives through (u_1, x). Hence, assume that u_k is not big. Since u_{k-1} is a $(6,0)$-vertex, Corollary 4 implies that u_k is not a $(4,4)$-vertex. Therefore, u_k is an internal $(5,\leq 2)$-vertex. By Lemma 9, it follows that $u_k = v$. Since x does not have a big neighbor, x is an internal vertex. Since x is internal big or $(6,0)$-vertex, its degree is at least 6, and thus $k \geq 6$. However, Lemma 6 implies that $vu_1u_2\dots u_{k-1}$ is an induced cycle, which contradicts Lemma 10. □

Lemma 13. *Let (G,K,C) be a minimal counterexample, let v be a big vertex, and let vu_1u_2, vu_2u_3, and vu_3u_4 be distinct faces of G.*

- *If u_1u_2 is heavy for v, and u_1u_2w is the face of G with $w \neq v$, then w is an internal $(5,\leq 1)$-vertex. Furthermore, no charge departs v through (u_2, u_3), and u_3u_4 is not heavy for v.*

- *If* u_1 *is an internal* $(5, \leq 1)$*-vertex, then charge does not depart* v *through* (u_2, u_3)*.*

- *If* v *is an internal* $(6,1)$*-vertex adjacent to an internal* $(5,0)$*-vertex and charge departs* v *through* (u_1, u_2)*, then* u_3 *is an internal* $(5,0)$*-vertex.*

Proof. Suppose that charge departs v through both (u_1, u_2) and (u_2, u_1). By the assumptions of the rule R3, both u_1 and u_2 are internal $(6,0)$-vertices. For $i = 1, 2$, there exists a path starting in u_i, passing through internal $(6,0)$-vertices adjacent to u_{3-i}, and ending in an internal $(5, \leq 1)$-vertex x_i adjacent to u_{3-i}. By Lemma 9, we have $x_1 = x_2$. Hence, $u_1 u_2 x_1$ is a triangle, and by Lemma 6, we have $w = x_1 = x_2$.

- Suppose that in this situation, charge departs through (u_2, u_3) because of a path in the neighborhood of u_3 ending in an internal $(5, \leq 1)$-vertex x. By Lemma 9, we have $x = w$, and by Lemma 6, $u_2 u_3 w$ bounds a face. However, then u_2 has degree 4, which is a contradiction since u_2 is a $(6,0)$-vertex.

- Suppose that in this situation, $u_3 u_4$ is heavy for v. Then the vertex $w' \neq v$ of the face $u_3 u_4 w'$ is an internal $(5, \leq 1)$-vertex, and by Lemma 9, we have $w = w'$. By Lemma 6, it follows that u_2 and u_3 have degree 4, which is a contradiction, since they are $(6,0)$-vertices.

Suppose now that u_1 is an internal $(5, \leq 1)$-vertex, and that charge departs v through (u_2, u_3) because of a path in the neighborhood of u_3 ending in an internal $(5, \leq 1)$-vertex x. By Lemma 9, we have $x = u_1$. But then u_3 is adjacent to x, and Lemma 6 would imply that $u_1 u_2 u_3$ is a face and u_2 has degree three, which is a contradiction.

Suppose that v is an internal $(6,1)$-vertex adjacent to an internal $(5,0)$-vertex z and that charge departs v through (u_1, u_2) because of a path in the neighborhood of u_2 ending in an internal $(5, \leq 1)$-vertex x. By Lemma 11, we have $x = z$. But then u_2 is adjacent to z, and the triangle $u_2 v z$ bounds a face by Lemma 6. Hence, $z = u_3$. □

Let us now analyze the final charge of the vertices of G.

Lemma 14. *Let* (G, K, C) *be a minimal counterexample. If* v *is an internal* $(5,0)$*-vertex of* G*, then* $c(v) \geq 0$*.*

Proof. We have $c_0(v) = -10$.

By Corollary 4 and Lemma 9, every neighbor of v in G is either big or an internal $(6,0)$-vertex. Suppose that v is adjacent to β big vertices; each of them sends 2 to v by the rule R1. By Lemma 12, charge arrives to v through $10 - 2\beta$ pairs. Hence, $c(v) = c_0(v) + 2\beta + (10 - 2\beta) = 0$. □

Lemma 15. *Let (G, K, C) be a minimal counterexample. If v is an internal $(5, 1)$-vertex of G, then $c(v) \geq 0$.*

Proof. We have $c_0(v) = -5$.

By Corollary 4 and Lemma 9, all neighbors of v except for the one belonging to C are either big or internal $(6, 0)$-vertices. Let $v_1, \ldots, v_6$ be the neighbors of v in order, where $v_2 \in C$. Since $(6, 0)$-vertices have no neighbor in C, both v_1 and v_3 are big. Let $\beta \geq 2$ be the number of big vertices adjacent to v; each of them sends 1 to v by the rule R2. By Lemma 12, charge arrives to v through $10-2\beta$ pairs. Since $\beta \leq 5$, $c(v) = c_0(v)+\beta+(10-2\beta) \geq 0$. □

Lemma 16. *Let (G, K, C) be a minimal counterexample. If v is a big (a, b)-vertex, then $c(v) \geq 8a+7b-60$. In particular, if v is internal and v is neither a $(6, 1)$-vertex nor a $(7, 0)$-vertex, then $c(v) \geq 0$.*

Proof. By Lemma 6, the neighborhood of v in G induces a cycle, which we denote by Q. If v is an internal vertex or $|K| = 3$, then the length of Q is $a + b$. If $v \in K$ and $|K| = 2$ then the length of Q is $a + b - 1$. Note that if $v \in K$, then $a + b \geq 4$ by Corollary 7.

Let us define a weight $w(e)$ for an edge $e = xy$ of Q as follows. If charge departs v through at least one of (x, y) and (y, x), then let $w(e) = 2$. If x or y is an internal $(5, \leq 1)$-vertex and neither x nor y belongs to C, then let $w(e) = 1$. Otherwise, let $w(e) = 0$. Note that no two internal $(5, \leq 1)$-vertices are adjacent by Lemma 9, and that if charge departs v through at least one of (x, y) and (y, x), then neither x nor y is an internal $(5, \leq 1)$-vertex. Furthermore, if xyz is subpath of Q and y is an internal $(5, 0)$-vertex, then $w(xy) = w(yz) = 1$. We conclude that $\sum_{e\in E(Q)} w(e)$ is an upper bound on the amount of charge sent by v.

Note that $w(e) \leq 2$ for every $e \in E(Q)$, and $w(e) = 0$ if e is incident with a vertex of C. Since C is an independent set, exactly $2b$ edges of Q are incident with a vertex of C, and thus $\sum_{e\in E(Q)} w(e) \leq 2(a+b-2b) = 2(a-b)$. Therefore, $c(v) \geq c_0(v) - 2(a-b) = (10a + 5b - 60) - 2(a-b) = 8a + 7b - 60$.

If $a \geq 8$, then $c(v) \geq 8a - 60 \geq 4$. If $a = 7$ and $b \geq 1$, then $c(v) \geq 8 \cdot 7 + 7 - 60 = 3$. If $a = 6$ and $b \geq 2$, then $c(v) \geq 8 \cdot 6 + 7 \cdot 2 - 60 = 2$. Finally, if $a = 5$ and $b \geq 3$, then $c(v) \geq 8 \cdot 5 + 7 \cdot 3 - 60 = 1$. Hence, if v is an internal big vertex, it follows that $c(v) \geq 0$ unless $a = 7$ and $b = 0$, or $a = 6$ and $b = 1$. □

Lemma 17. *Let (G, K, C) be a minimal counterexample. If v is an internal $(7, 0)$-vertex, then $c(v) \geq 0$.*

Proof. Note that $c_0(v) = 10$. Let $v_1 v_2 \dots v_7$ denote the cycle induced by the neighbors of v, and let n_5 denote the number of internal $(5, \leq 1)$-vertices of G adjacent to v.

Since no two internal $(5, \leq 1)$-vertices are adjacent, it follows that $n_5 \leq 3$. By rules R1 and R2, the vertex v sends at most $2n_5$ units of charge. Furthermore, v sends charge over at most $7 - 2n_5$ edges by rule R3. If $n_5 \geq 2$, then $c(v) \geq c_0(v) - 2n_5 - 2(7 - 2n_5) = 2(n_5 - 2) \geq 0$.

If $n_5 = 1$, then suppose that v_1 is the internal $(5, \leq 1)$-vertex. By Lemma 13, no charge departs v through (v_2, v_3) or (v_7, v_6). Also, at most one of the edges $v_3 v_4$, $v_4 v_5$, and $v_5 v_6$ is heavy for v. Therefore, charge departs v through at most 6 pairs, and $c(v) \geq c_0(v) - 2n_5 - 6 > 0$.

Finally, suppose that $n_5 = 0$. By Lemma 13, for $i = 1, \dots, 7$, at most one of the edges $v_i v_{i+1}$, $v_{i+1} v_{i+2}$, $v_{i+2} v_{i+3}$ (with indices taken cyclically) is heavy. Therefore, charge departs v through at most 9 pairs. Hence, $c(v) \geq c_0(v) - 9 > 0$. $\square$

Lemma 18. *Let (G, K, C) be a minimal counterexample. If v is an internal $(6, 1)$-vertex, then $c(v) \geq 0$.*

Proof. Note that $c_0(v) = 5$. Let $Q = v_1 v_2 \dots v_7$ denote the cycle induced by the neighbors of v, where $v_2 \in C$.

Suppose first that v is adjacent to an internal $(5, 0)$-vertex, to which v sends 2 by the rule R1. By Lemma 11, v is adjacent only to one internal $(5, 0)$-vertex and no other internal $(5, \leq 1)$-vertex. Furthermore, by the third part of Lemma 13, charge departs v through at most two pairs. Hence, $c(v) \geq c_0(v) - 2 - 2 > 0$.

Hence, we can assume that v is not adjacent to internal $(5, 0)$-vertices. Let n_5 be the number of internal $(5, 1)$-vertices adjacent to v; v sends 1 to each of them by the rule R2. Note that $n_5 \leq 3$, since no two internal $(5, 1)$-vertices are adjacent by Lemma 9. Since $v_2 \in C$, neither v_1 nor v_3 is a $(6, 0)$-vertex, and thus the edges $v_1 v_7$ and $v_3 v_4$ are not heavy for v.

Suppose first that $n_5 = 0$. If no edge of Q is heavy for v, then charge departs v through at most 5 pairs and $c(v) \geq c_0(v) - 5 = 0$. Hence, by symmetry we can assume that $v_4 v_5$ or $v_5 v_6$ is heavy for v. Lemma 13 implies that no other edge of Q is heavy for v. Let us distinguish the cases.

- If $v_4 v_5$ is heavy, then Lemma 13 implies that charge does not depart through the pair (v_4, v_3), and it does not depart through the pair (v_3, v_4) since v_3 is not a $(6, 0)$-vertex.

- If v_5v_6 is heavy, then Lemma 13 implies that the common neighbor $w \neq v$ of v_5 and v_6 is an internal $(5, \leq 1)$-vertex, and furthermore, that charge may only depart v through pairs (v_4, v_5), (v_7, v_6), (v_4, v_3), and (v_7, v_1) in addition to (v_5, v_6) and (v_6, v_5).

  Suppose that the charge departs v through all these pairs. By Lemma 9, all the charge arrives to w. However, then w is adjacent to v_1, v_3, v_5, v_6, as well as at least two $(6, 0)$-vertices of the paths showing that the charge departing through the pairs (v_4, v_5) and (v_7, v_6) arrives to w. This is a contradiction, since w has at most 5 neighbors not belonging to C.

In both cases, we conclude that charge departs v through at most 5 pairs, and thus $c(v) \geq c_0(v) - 5 = 0$.

Suppose now that $n_5 = 1$. If neither v_1 nor v_3 is an internal $(5, 1)$-vertex, then v sends charge over at most three edges by the rule R3 and at most one of them is heavy for v by Lemma 13, and $c(v) \geq c_0(v) - n_5 - 4 = 0$. Hence, by symmetry, we can assume that v_3 is an internal $(5, 1)$-vertex. By Lemma 13, only one of the edges v_5v_6 and v_6v_7 may be heavy. If v_6v_7 is heavy, then charge does not depart v through (v_7, v_1) or (v_1, v_7), by Lemma 13 and since v_1 is not a $(6, 0)$-vertex. If v_5v_6 is heavy, then charge does not depart v through (v_4, v_5) or (v_5, v_4) by Lemma 13. In either case, charge departs v through at most 4 pairs, and again $c(v) \geq 0$.

Suppose that $n_5 = 2$. Recall that no two internal $(5, \leq 1)$-vertices are adjacent by Lemma 9. If at least one of v_1 and v_3 is not an internal $(5, 1)$-vertex, then v sends charge over at most two edges by rule R3 and neither of them is heavy for v by Lemma 13, hence $c(v) \geq c_0(v) - n_5 - 2 > 0$. If both v_1 and v_3 are internal $(5, 1)$-vertices, then only the edge v_5v_6 may be heavy for v by Lemma 13, and if it is heavy, then no charge departs v through (v_4, v_5), (v_5, v_4), (v_6, v_7) and (v_7, v_6). Hence, charge departs v through at most 3 pairs and $c(v) \geq c_0(v) - n_5 - 3 = 0$.

Finally, suppose that $n_5 = 3$. In this case, Lemma 13 shows that no charge departs v, and thus $c(v) = c_0(v) - n_5 > 0$. □

Proof of Theorem 2. Suppose for a contradiction that Theorem 2 is false. Then, there exists a minimal counterexample (G, K, C). Assign and redistribute charge among its vertices as we described above. Note that the redistribution of the charge does not change its total amount, and thus

$$\sum_{v \in V(G)} c(v) = \sum_{v \in V(G)} c_0(v) = -60 - 20|K|.$$

Recall that $c(v) = c_0(v) = 0$ for every $v \in C$. If v is an internal big vertex, then $c(v) \geq 0$ by Lemmas 16, 17 and 18. If v is an internal vertex with $c_0(v)$ negative, then by Lemma 5, it follows that v is either a $(5,0)$-vertex, or a $(5,1)$-vertex, and $c(v) \geq 0$ by Lemmas 14 and 15. If v is an internal vertex with $c_0(v) = 0$ (i.e., v is a $(4,4)$-vertex, or a $(5,2)$-vertex, or a $(6,0)$-vertex), then $c(v) = c_0(v) = 0$. Therefore,

$$\sum_{v \in V(G)} c(v) \geq \sum_{v \in K} c(v).$$

Consider an (a,b)-vertex $v \in K$. Since v is incident with two edges of the outer face of G, we have $a \geq 2$, and $a + b \geq 4$ by Corollary 7. By Lemma 16, $c(v) \geq 8 \cdot 2 + 7 \cdot 2 - 60 = -30$. Therefore,

$$\sum_{v \in K} c(v) \geq -30|K|.$$

However, since $|K| \leq 3$, we have $-30|K| > -60 - 20|K|$, which is a contradiction. Therefore, no counterexample to Theorem 2 exists. □

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