Local Analysis of Dynamical Systems — Concepts and Interpretation

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Abstract — We present several terms and definitions related to the local analysis of dynamical systems. Multiple terms for one and the same thing that were found in literature are put together to provide a "dictionary" of terms and to avoid potential confusion due to misleading definitions. Additionally, some important concepts which are necessary to analyze a dynamical system are briefly discussed and a new procedure to locally analyze a dynamical system's behavior near trajectory points is proposed. The paper should give computer graphics specialists working on the visualization of analytically defined dynamical systems a set of mathematically tools for a thorough investigation of the local behavior of such system.

Keywords: dynamical systems, local analysis, visualization, flow field analysis.

1. Introduction

Dynamical systems are found in various fields of research (e.g., flow field analysis), economy (e.g., stock market models), physics, medicine, and others [ArPl90]. They are given by an analytical specification or as sampled data. There are many possible ways to analyze such a system, e.g., analyzing its long term behavior. An important branch of the analysis of dynamical systems is *local analysis*. For certain applications, e.g., the prediction of a system's behavior, it is crucial to know, how initially close states will evolve with respect to each other. Flow field analysts, for example, are often interested in vortices, that may be detected by local analysis of the underlying dynamical system. We therefore concentrate on the local analysis of dynamical systems throughout this paper.

Scientists that are interested in dynamical systems (and the local analysis of these systems) are confronted with a lot of terms, formulas, and definitions. Non-mathematicians get easily confused by studying some of the relevant literature in the beginning. Differing terms for the same object do not help to clear up the situation as well as subtle differences in the interpretation of mathematical symbols do not simplify the understanding. This was one of the reasons to compile relevant terms that occur often in literature and to assemble the different definitions. For example, the curvature of a 3D curve can either be calculated from the Frenét formulas (see section 3) or by analysing the Jacobian matrix of the dynamical system (see section 5).

On the other hand it is interesting to see how some (local) attributes of a dynamical system can be retrieved by rather different approaches. This seems to be especially useful when some of the straight-forward techniques are not possible due to incomplete or insufficient

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specifications. One example is the analysis of dynamical systems that are given as sampled data which do not allow the use of straight-forward analytical approaches in most cases.

Before we start with terms and definitions relevant for local analysis of dynamical systems we list some high-level classifications of dynamical systems (see section 2). Thereafter we span an arc from differential geometry aspects when analysing trajectories of dynamical systems to the analysis of linear dynamical systems and its interpretation. In this part of the paper (see sections 3, 4, and 5) we present well known concepts but concentrate to give a unifying view of various terms and definitions, which are sometimes used ambiguously and interchangeable in literature. Then we discuss dynamical system analysis near special subsets of the topology of behavior to end up with a new approach to locally analyze points on trajectories.

2. Classifications of Dynamical Systems

Dynamical systems are mainly represented by a state that evolves in time. Input as well as the current state of a dynamical system determine the evolution of the system. Typically an output is generated from the state of the system [Rina95]. See figure 1 for an illustration of this principle.

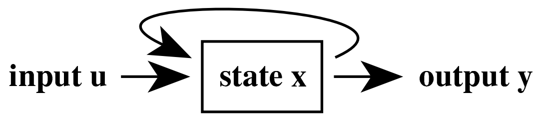

Figure 1: Specification of a dynamical system.

This is the general definition of a dynamical system, where many different systems fit into the scheme as illustrated in figure 1. For investigating dynamical systems it is necessary to specify some characteristics that provide a subdivision with special classes of dynamical systems. Specific methods are available for some of these classes, thus such a classification can help to simplify the analysis.

An important characteristic of a dynamical system is whether it is *continuous* or *discrete*. Continuous systems (often called *flow*s) are given by differential equations (e.g., $\dot{x} = A \cdot x$) whereas discrete dynamical systems (often called *maps*) are specified by difference equations (e.g., $x_{n+1} = A \cdot x_n \quad \Leftrightarrow \quad x_{n+1} - x_n = (A - I) \cdot x_n$) [Tson92].

Autonomous systems are characterized by the fact that input and output are omitted from the definition [Rina95]. Both examples mentioned above present autonomous systems.

An important criterion for the analysis of a dynamical system is whether it is *time-dependent* or not [Lane93] [Lane94]. For time-dependent dynamical systems the function that specifies $\dot{x}$ (continuous case) or x_{n+1} (discrete case) depends on the time itself whereas for time-independent systems this function does not change over time. Both examples above specify a time-independent system, if A is assumed to be constant over time.

When a dynamical system is to be analysed the fact whether it is *linear* or not is very important. Linear dynamical systems are simple to analyse as opposed to non-linear systems, which typically do have intricate dynamical behavior [Tson92]. Often *linearization* is used to get insights into these complex non-linear dynamical systems.

Using linearization, another classification of dynamical systems is crucial for separating simple cases from more complex ones. *Hyperbolic* dynamical systems can be analysed by linearization efficiently, whereas *non-hyperbolic* systems may cause major troubles in combination with linearization [AbSh92] [GlLe91]. Hyperbolic systems are structurally stable, i.e., they are the general case. Non-hyperbolic systems are difficult to investigate, occur rarely and can be considered the transitional phase between two hyperbolic systems of different nature [Rina95].

3. Differential Geometry and Terms

The solution of a continuous dynamical system is a *trajectory* $T_x(t)$ as defined by equation (1) [KeMa92] [PoWi94]. Any point on the trajectory is given by its parameter t and an initial state x of the system. Parameter t can be interpreted as the time passed since the system evolved from x. Note, that (1) is a recursive definition that cannot be expressed explicitly in most cases.

$$\mathrm{T}_x(t) = x + \int_0^t v(\mathrm{T}_x(u))\, du \tag{1}$$

Differential geometry includes the analysis of curves and surfaces in higher dimensions. The construction of a local coordinate system (Frenét-Frame) helps to get insight into local characteristics of a space curve, e.g., curvature and torsion [Beac91] [HaMa94]. Local analysis of trajectories requires a good working knowledge of various terms of differential geometry. They are shortly discussed in the following.

Given a parameterized curve $C(t)$ in three-space a reparameterization is possible such that the curve's new parameter s is equal to the arc length of curve C in the parameter interval $[0, s)$. In respect to these distinct parameters derivations of curve C are written differently:

$$\dot{C} = dC/dt,\ \ddot{C} = d^2C/dt^2, \text{ and } \dddot{C} = d^3C/dt^3 \tag{2}$$

$$C' = dC/ds,\ C'' = d^2C/ds^2, \text{ and } C''' = d^3C/ds^3 \tag{3}$$

By the use of these derivations a local coordinate system (Frenét-Frame) can be built at a curve point by the curve's *tangent vector* $t_C = C'$, its *principal normal* $n_C = C''/|C''|$, and its *binormal* $b_C = t_C \times n_C$. These three vectors span an orthonormal basis at a curve point. Note, that n_C and b_C are ambiguous when the curve is locally equal to a straight line.

By building the Frenét-Frame at a curve point the curvature κ and the torsion τ of curve C at this point can be derived in a straight-forward way from the orthonormal basis [Beac91]:

$$\frac{d}{ds}\begin{pmatrix} t_C \\ n_C \\ b_C \end{pmatrix} = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \cdot \begin{pmatrix} t_C \\ n_C \\ b_C \end{pmatrix} \quad \Rightarrow \quad \kappa = \left|\frac{dt_C}{ds}\right|,\ \tau = \left|\frac{db_C}{ds}\right| \tag{4}$$

Curvature κ and torsion τ of curve C can be described in other terms as well. For example, the curvature of a curve can be written as $1/r$, when r is the radius of the osculating circle [BrSe80]. As a third possibility κ can be derived by the following procedure: Assuming α to be the angle enclosed by the curve's tangent and the line running through $C(s)$ and some slightly ahead point on the curve $C(s+\Delta s)$, the curvature κ can be calculated as $\kappa = \lim_{\Delta s \to 0} \alpha/\Delta s$.

Torsion can be similarly derived by a differential quotient. Assuming β to be the angle enclosed by a line through $C(s)$ and $C(s+\Delta s)$ and the rectifying plane (spanned by t_C and b_C), the torsion τ can be calculated as $\tau = \lim_{\Delta s \to 0} \beta/\Delta s$ [BrSe80].

4. Dynamical Systems as a Babylon of Terms

This section discusses some of the often used terms in combination with dynamical system analysis. Most of terms will be well-known to the reader, but often several differing terms are used in literature to denote the same concept or object. To avoid possible confusion about these many sometimes interchangeable terms a clarifying survey is appropriate.

We start with operator ∇, which is often used to define other important terms for the analysis of dynamical systems. It builds up a vector of the partial derivatives of its operand and is defined as shown in equation (5) [BrSe80]. If ∇'s operand $f(x)$ is a scalar function, then $\nabla f(x)$ is called the *gradient* of f [BrSe80]. If ∇'s operand $v(x)$ is a vector function, then ∇v is the *Jacobian* matrix $J = \partial v/\partial x$ of $v(x)$ [LeWi93].

$$\nabla = \begin{pmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \cdots \end{pmatrix}^{\mathrm{T}},\ \mathrm{grad}\, f(x) = \nabla f(x),\ J = \nabla v(x) = \partial v/\partial x \tag{5}$$

An often used (scalar) term is the *divergence* of a flow $\mathrm{div}\, v(x)$. It can be written as $\nabla \cdot v(x)$ or as the trace Tr of v's Jacobian ∇v [BrSe80]:

$$\mathrm{div}\, v(x) = \nabla \cdot v(x) = Tr(\nabla v) = \sum_i (\partial v/\partial x)_{i,i} \tag{6}$$

The divergence basically describes the local amount of outgoing or incoming flow at a specific location of the dynamical system. It is 0, if the amount of incoming flow is equal to the amount of the outgoing flow.

Another important term for the local analysis of dynamical systems is the *rotation* vector of a flow: $\operatorname{rot} v(x)$ [ScVo91] [PaWa94]. This attribute of a flow is often named *vorticity* instead of rotation and abbreviated by ω [Hans93]. As a third term sometimes *curl* is used instead of rotation [Hans93]. The vorticity/rotation/curl of a flow is defined as follows:

$$\omega = \operatorname{rot} v(x) = \operatorname{curl} v(x) = \nabla \times v(x) \tag{7}$$

Vector $\operatorname{rot} v(x)$ describes the rotation axis and its length the rotation velocity, which is given at state x. Note, that some references define the vorticity slightly different as $\omega = (1/2) \cdot \operatorname{rot} v(x)$.

A scalar term related to the vorticity as defined above is the *stream vorticity* Ω [Hans93] [ScVo91]. It is the cosinus of the angle enclosed by the vorticity vector and the flow vector $v(x)$. This term characterizes the type of rotation in the system. If Ω is 1, the flow rotates around the flow vector, whereas a value of 0 implies, that either there is no vorticity or the flow rotates in a plane which also contains the direction of the flow.

$$\Omega = \frac{v \cdot \omega}{|v| \cdot |\omega|} = \frac{v \cdot (\nabla \times v)}{|v| \cdot |\nabla \times v|} \tag{8}$$

Just slightly different from the above definition is the specification of *helicity* [LeWi93]. Furthermore the *helicity density* H_d as given in the literature is just the same as helicity [PoWa93]. A value of 0 means exactly the same as no stream vorticity, but helicity increases proportional to the length of ω and v. It is defined by:

$$H_d = \Omega \cdot |v| \cdot |\omega| = v \cdot \omega = v \cdot (\nabla \times v) \tag{9}$$

Another term in correlation with the rotation of a flow is its *circulation* Γ_C [Lajo94]. The circulation of a flow can be used to determine if it is possible to use a potential function instead of the vector function v for analysis purposes: If the circulation Γ_C of a flow is 0 for any closed curve C, then a potential function f exists such that $\operatorname{grad} f(x) = v(x)$. In such a case it is often easier to use f instead of v. Additionally $(\forall C\colon \Gamma_C = 0)$ implies that there is no rotation at all in the vector field. By using Stoke's equations, Γ_C can be expressed as follows:

$$\Gamma_C = \oint_C v(x)\, ds = \int_A \operatorname{rot} v(x)\, dA \tag{10}$$

A .... the surface (of an arbitrary volume) containing C.

5. Interpreting the Matrix of an Autonomous and Linear System

As we already stated in section 2, linear dynamical systems are especially simple to analyze. Since we need this procedure for the rest of our paper, we briefly discuss some different approaches of analyzing the matrix of a linear and autonomous dynamical system's A [Tson92].

5.1. Eigenvalues and Eigenvectors

Continuous dynamical systems ($\dot{x} = A \cdot x$) as well as discrete systems ($x_{n+1} = A \cdot x_n$) that are autonomous and linear can be entirely analyzed by investigating the matrix A and its characteristics. One possibility is to compute A's eigenvalues λ_i and its eigenvectors e_i from equations (11) and (12), respectively [Rina95] [Tson92].

$$\det(A - \lambda_i \cdot I) = 0 \tag{11}$$
$$A \cdot e_i = \lambda_i \cdot e_i \tag{12}$$

The interpretation of the eigenvalues λ_i — they can be either real or complex — is different for continuous and discrete dynamical systems, because a continuous system is specified by the change of the current state, whereas a discrete dynamical system is specified by giving the next state of the system.

	Continuous Case	Discrete Case	
Convergence	$\operatorname{Re}\lambda_i < 0$	$\lvert\lambda_i\rvert < 1$	(13)
Divergence	$\operatorname{Re}\lambda_i > 0$	$\lvert\lambda_i\rvert > 1$	(14)
Rotation	$\operatorname{Im}\lambda_{j,k} \neq 0$	$\operatorname{Im}\lambda_{j,k} \neq 0$	(15)

Convergence, divergence, and rotation are to be interpreted relatively to the origin of the coordinate system. Note, that a fix-point of a continuous dynamical system is called hyperbolic, if its eigenvalues do not lay on the imaginary axis ($\operatorname{Re}\lambda_i \neq 0$). Fix-points of discrete dynamical systems are hyperbolic, if $\lvert\lambda_i\rvert \neq 1$ for all eigenvalues.

5.2. Decomposing Matrix A

Another possibility of analyzing matrix A of a linear and autonomous system is by decomposing it into a symmetric matrix A^+ and an asymmetric matrix A^- as follows [LeWi93]:

$$A^+ = \left(A + A^{\mathrm{T}}\right)/2,\ A^- = \left(A - A^{\mathrm{T}}\right)/2 \tag{16}$$

The elements of A^+ and A^- can be interpreted rather straight-forward [ScVo91]:

$$A^+ = \begin{pmatrix} d_x & \bullet & \bullet \\ \bullet & d_y & \bullet \\ \bullet & \bullet & d_z \end{pmatrix}, \text{ and } \left(d_x + d_y + d_z\right) = \operatorname{div} v(x) \tag{17}$$

The elements of A^+ marked with '$\bullet$' built up the shear strain portion of this linear system.

$$A^- = \frac{1}{2}\begin{pmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \operatorname{rot} v(x) \tag{18}$$

5.3. Analyzing the Matrix in a Local Coordinate System (Frenét-Frame)

A third possibility of linear system analysis is especially useful while investigating a flow's Jacobian J. It can be transformed into the local Frenét-frame at some point of a trajectory ($J \to J^{local}$). Then the elements of J^{local} as given in the following equation allow a detailed characterization of the underlying flow [LaWi93]:

$$J^{local} = \begin{pmatrix} \hat{a} & \hat{s} & \hat{s} \\ \hat{c} & \hat{d} & \hat{d}, \hat{t} \\ \hat{c} & \hat{d}, \hat{t} & \hat{d} \end{pmatrix} \tag{19}$$

Elements of matrix J that are marked with '$\hat{a}$', '$\hat{s}$', or '$\hat{c}$' specify changes of the flow that are parallel to $v(x)$. The element marked with '$\hat{a}$' gives the acceleration of the flow, whereas the elements marked with '$\hat{s}$' give the shear strain at this state of the system. Elements that are marked with '$\hat{c}$' give the curvature of the flow.

Remaining elements of matrix J, that are marked with either '$\hat{d}$' alone or '$\hat{d}$' and '$\hat{t}$', specify the changes of the flow that are perpendicular to $v(x)$. Splitting the bottom-right 2×2-matrix into a symmetric and an asymmetric one gives the divergence (by the elements marked with '$\hat{d}$') and the torsion (by the elements marked with '$\hat{t}$') of the flow.

6. Dynamical System Analysis near Fix-Points or Cycles

Linear systems by themselves have a rather simple dynamical behavior. The reason, why linear system analysis is so important, is that non-linear systems are often analyzed by local linearization [Tson92]. This is especially easy near fix-points, since the long-term behavior trivially coincides with the local behavior at these points.

6.1. Dynamical System Analysis near a Fix-Point

Analysing the system's behavior near its fix-points can help to understand the evolution of any state of the system. Assuming the system is non-linear and hyperbolic, linearization can be used to determine the behavior near fix-points completely. Continuous and discrete systems can be treated rather similar [Rina95]:

	The Continuous Case	The Discrete Case	
Vector Field Definition	$\dot{x} = dx/dt = v(x)$	$x_{n+1} = v(x_n)$	(20)
Fix-Point Definition	$\dot{\bar{x}} = v(\bar{x}) = 0$	$\bar{x} = v(\bar{x})$	(21)
Rewriting x (x_n)	$x = \bar{x} + \Delta$	$x_n = \bar{x} + \Delta_n$	(22)
Using Taylor Expansion	$\dot{x} \approx v'(\bar{x}) \cdot \Delta$	$x_{n+1} \approx \bar{x} + v'(\bar{x}) \cdot \Delta_n$	(23)
Linearized System	$\dot{\Delta} = v'(\bar{x}) \cdot \Delta$	$\Delta_{n+1} = v'(\bar{x}) \cdot \Delta_n$	(24)

To keep the analysis simple, we assume the system to be autonomous and time-independent (see (20) for the definitions). Assuming the existence of at least one fix-point (see (21) for the definitions) any state of the dynamical system near fix-point $\bar{x}$ can be rewritten with respect to $\bar{x}$ (see (22)). With this reformulation the dynamical system con be approximated by a Taylor expansion as shown in (23). $v'(\bar{x})$ denotes the Jacobian matrix of $v(x)$ evaluated at $\bar{x}$. Using (22) again, the left side of the Taylor expansion in (23) can be rewritten. This operation yields the linearized systems for small perturbations around fix-point $\bar{x}$ (see (24)). These linear systems can now be analyzed as discussed in section 5.

6.2. Dynamical System Analysis near a Cycle

Cycles are another important class of characteristic subsets within continuous dynamical systems. A cycle is given, when the system returns to a previous state. The system behavior near such a cycle can be analyzed by using a *Poincaré Map*. Such a map is a discrete dynamical system, that is produced from a continuous dynamical system and that is of a lower dimension than the original system. A Poincaré Map is specified by the cross-section of a surface perpendicular to the cycle (usually a plane) and a trajectory near the cycle. The Poincaré Map is a discrete dynamical system with at least one fix-point $\bar{x}$, i.e., $\bar{x}$ is the cross-section of the cycle and the surface. Thus the Poincaré Map can be analysed as shown in the section before and the results are then used for interpreting the system's behavior nearby the cycle [Rina95].

7. Dynamical System Analysis near a Trajectory

In the following we propose another approach to analyze a dynamical system's behavior. It is somewhat similar to the method presented in section 5.3 [LeWi93], as the dynamical system is also transformed into the Frenét-Frame Φ of a point on the trajectory. Contrary to their approach we use the analysis by eigenvalues and eigenvectors to interpret this transformed Jacobian matrix. Expressing a dynamical system $\dot{x} = v(x)$ in terms of Φ one gets

$$\dot{u} = (g2l \circ v \circ l2g)(u) = \tilde{v}(u) \tag{25}$$

u .... a state of the system in terms of Φ.
$g2l$ .... transformation from the global coordinate system into Φ.
$l2g$ .... transformation from Φ into the global coordinate system.

Near the point of interest p (represented in the global coordinate system) a state of the system can be written as $u = 0 + \Delta$ in terms of the local coordinate system. Note, that p represented in terms of Φ is 0. Using a Taylor expansion of $\tilde{v}(u)$ up to first-order terms, we get

$$\dot{u} = \tilde{v}(u) = \tilde{v}(0+\Delta) \approx \tilde{v}(0) + \left.\frac{\partial \tilde{v}}{\partial u}\right|_{u=0} \cdot \Delta = \lambda \cdot \phi_1 + \tilde{v}'(0) \cdot \Delta \tag{26}$$

ϕ_1 .... unit-vector in terms of Φ collinear to the axis corresponding to the trajectory's tangent.
λ .... length of $v(p)$.

Transforming the very left side of (26) by using $u = 0 + \Delta$ we get a linearized system for small perturbations of p (in terms of Φ), because $d0/dt = \tilde{v}(0) = \lambda \cdot \phi_1$.

$$\dot{\Delta} = \tilde{v}'(0) \cdot \Delta \tag{27}$$

The elements of Δ can be separated into a scalar ${}^{(1)}\Delta$ and a vector ${}^{(2\ldots)}\Delta$ that is of one dimension less than Δ. ${}^{(1)}\Delta$ is assumed to be 0, since perturbations of p that are not perpendicular to the trajectory's tangent make no sense at all — a state of the system that is represented as a perturbation of p with a component ${}^{(1)}\Delta \neq 0$ can be more accurately expressed as a (perpendicular) perturbation of another point on exactly the same trajectory. Thus $\dot{\Delta}$ does not depend on the first row of matrix $\tilde{v}'(0)$. The remaining elements of $\tilde{v}$'s Jacobian $J|_0 = \tilde{v}'(0)$ can be decomposed into the first line ${}^{(1,2\ldots)}J|_0$ and the lower-right sub-matrix ${}^{(2\ldots,2\ldots)}J|_0$.

Decomposing $\dot{\Delta}$ similar to Δ yields a part parallel to the trajectory's tangent (scalar ${}^{(1)}\dot{\Delta}$) and a part perpendicular to ϕ_1 (sub-vector ${}^{(2\ldots)}\dot{\Delta}$):

$$\begin{aligned} {}^{(1)}\dot{\Delta} &= {}^{(1,2\ldots)}J\big|_0 \cdot {}^{(2\ldots)}\Delta \\ {}^{(2\ldots)}\dot{\Delta} &= {}^{(2\ldots,2\ldots)}J\big|_0 \cdot {}^{(2\ldots)}\Delta = A \cdot {}^{(2\ldots)}\Delta \end{aligned} \qquad (28)$$

A can be analyzed as already shown for continuous systems at the neighbourhood of fix-points. But we must be careful with the interpretation of this analysis, because all the results hold for the investigated point p only. For example, if the analysis of matrix A reveals that the system's evolution is convergent (fix-point is an attractor) the only thing that can be said is that nearby trajectories are locally attracted by the trajectory at the specific location chosen. To detect convergent, divergent, or saddle regions of a trajectory it must be shown that the structural characteristics of matrix A are persistent for a certain region of the trajectory. This might be not simple analytically, but can be done approximately by numerical simulation..

8. Conclusion

This paper compiles important terms and definitions that are useful for analyzing analytically defined dynamical systems. Widely varying terms and denotations are sometimes used in literature to describe important concepts of dynamical systems. Thus a clarifying survey of these sometimes interchangeable terms and definitions is given.

After presenting a classification of dynamical systems, tools of differential geometry are discussed with respect to the analysis of trajectories of dynamical systems. The description of terms defining flow characteristics of dynamical systems (e.g., divergence, rotation) is followed by discussing linearization techniques for dynamical systems.

Together with an investigation of flow behavior close to a fix-point and cycles a concept for the local analysis of a dynamical system close to an arbitrary trajectory is presented. This approach basically investigates perturbations orthogonal to the chosen trajectory by determining eigenvalues and eigenvectors of a matrix which is closely related to the Jacobian matrix of the dynamical system but with lower dimension.

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