THE RELATION BETWEEN *RATS*-SPLINES AND THE CATMULL AND CLARK B-SPLINES

Andreas Savva
School of Computing, Intercollege
46 Makedonitissas Ave, P.O.Box 24005, 1700 Nicosia, Cyprus
Tel: +357-2-357735
Fax: +357-2-357481
e-mail: savva@intercol.edu

Gordon J. Clapworthy
Department of Computer & Information Sciences,
De Montfort University
Hammerwood Gate, Kents Hill, Milton Keynes, MK7 6HP, UK
Tel: +44-1908-834931
Fax: +44-1908-834948
e-mail: gc@dmu.ac.uk

ABSTRACT

This paper presents the relationship between the Recursive Arbitrary Topology Splines (RATS) method, derived by the authors, and the Catmull and Clark recursive B-Spline method. Both methods are capable of defining surfaces of any arbitrary topology of control points. They "fill-in" n-sided regions with four-sided patches. The Catmull & Clark method is derived from the midpoint subdivision of B-splines whereas the RATS method is derived from the midpoint subdivision of Bézier splines. RATS generates an additional set of patches defining the border of the surface but the RATS inner surface is identical to the Catmull and Clark surface. This paper illustrates this relationship between the two methods.

Keywords: surface modelling, recursive splines, geometric design, arbitrary topology figures.

1. INTRODUCTION

The generation of a curve from a polygon by successively refining the polygon with the addition of new vertices and edges was introduced by Chaikin [Chaik74] in 1974. A few years later, in 1978, Catmull & Clark [Catmu78] and Doo and Sabin [Doo78] generalized the idea to surfaces. Both methods were extended in defining surfaces with arbitrary topology of control points. They refine, or subdivide, an irregular mesh by creating a new mesh, with more faces and vertices, that approximates the old. By repeating the process by a number of subdivisions a smooth surface is formed.

The Doo and Sabin method generates biquadratic B-spines, and the Catmull and Clark method generates bicubic B-splines. A more recent method that generalizes quartic triangular B-splines was developed by Loop and De Rose [Loop90]. However, due to the popularity of bicubic patches more researchers and modellers have given a lot of emphasis on the Catmull and Clark method. Even until today it could be considered as the most popular method for describing surfaces among those methods that are based on recursive algorithms.

A large amount of research carried out is based on the Catmull and Clark splines, including, Ball and Story [Ball88], and Doo and Sabin [Doo78] who studied the behaviour of the surface at the extraordinary points. Also, Halstead, Kass and DeRose [Halst93] derived a Catmull and Clark surface that interpolates the control points, and Nasri [Nasri87] treated the problem of shrinking of boundaries.

Like the Catmull and Clark method, the *RATS* (Recursive Arbitrary Topology Splines) method, which is derived by Savva [Savva98, Savva00], describes bicubic surfaces that are defined by an arbitrary topology of control points. The RATS method is derived from a standard bicubic Bézier patch and is based on a recursive patch midpoint subdivision algorithm on a rectangular framework of control points, which is then generalized to arbitrary nets of control points.

The Catmull and Clark method is an approximation to the control points defining the surface. The RATS surface is also an approximation to the control points but interpolates the corner control points. In fact, it generates an additional set of patches at the border of the surface. This paper demonstrates the relationship of the two methods. The Catmull and Clark surface is identical to the RATS inner surface. The *inner* surface is defined as the RATS surface without the patches at its border.

In section 2 the two methods are introduced. Section 3 derives the relationship between them and section 4 summarizes and concludes.

2. THE METHODS

Both methods are based on recursion and at each step they construct a new set of points with more vertices and smaller faces than the original set of points. After a number of iterations the result is a smooth surface. As the methods are based on recursion, the points that are generated are called "*new points*", while the points defining the control polygon at each iteration are called "*old points*".

In Fig.2 the RATS new points, which are generated from the old points given in Fig.1, are illustrated. Fig.3 shows the Catmull and Clark new-points that are also generated from the old points given in Fig.1. Fig.4 and Fig.5 illustrate the difference between a RATS surface and a Catmull and Clark surface. Both surfaces were generated from the same control points. The additional patches generated by RATS are clearly noted in these figures.

The Catmull and Clark new points are divided into *three* types: (1) new vertex points – new points corresponding to the old vertices, (2) new edge points – new points corresponding to old edges, and (3) new face points – new points lying in the centre of the squares of the original mesh. On the other hand, the RATS new points are divided into *ten* types: These consists of 3 subtypes of new vertex points, 4 subtypes of new edge points, and 3 subtypes of new face points. The RATS new points are listed below and illustrated in Fig.6. The new points are represented by Os whereas the old points are represented by Xs.

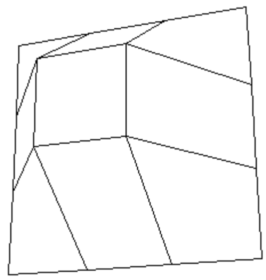

Fig.1 The old points

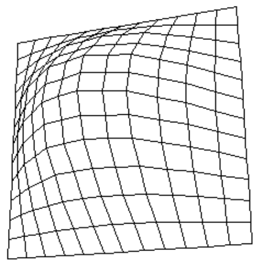

Fig.2 The RATS new points

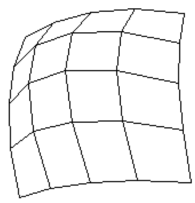

Fig.3 The Catmull and Clark new points

(1) *new vertex points* V_α – new points corresponding to the old vertices which are at the corners of the surface,

(2) *new vertex points* V_β – new points corresponding to old vertices which are on the border but not on the corners of the surface,

(3) *new vertex points* V_γ – new points corresponding to old vertices which are not on the border of the surface,

(4) *new edge points* E_α – new points corresponding to edges on the border of the surface where at least one of the vertices sharing the edge is at a corner of the surface,

(5) *new edge points* E_β – new points corresponding to edges on the border of the surface where none of the vertices sharing the edge is at any corner of the surface,

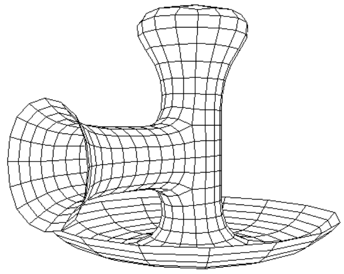

Fig.4 A RATS surface

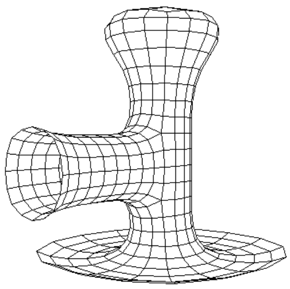

Fig.5 A Catmull & Clark surface

(6) *new edge points* E_γ – new points corresponding to edges which are not on the border of the surface, but at least one of the vertices sharing the edge is on the border,

(7) *new edge points* E_δ – new points corresponding to edges where none of the vertices sharing the edge is on the border of the surface,

(8) *new face points* F_α – new points lying in the centre of the faces of the original mesh where at least one of the vertices defining the face is at a corner of the surface,

(9) *new face points* F_β – new points lying in the centre of the faces of the original mesh where at least one of its edges is on the border of the surface and none of the vertices defining the face is at a corner of the surface, and

(10) *new face points* F_γ – new points lying in the centre of the faces of the original mesh where none of its edges is on the border of the surface.

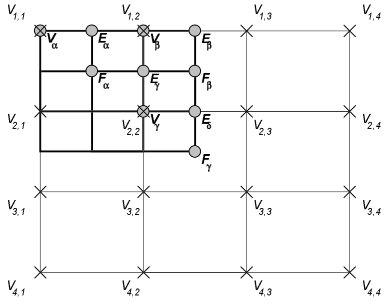

Fig.6 The new points in respect to the old points

Given an $n \times m$ mesh of old points, $V_{i,j}$, for $1 \le i \le n$ and $1 \le j \le m$, the new points that will be generated by the two methods are given in section 2.1 and 2.2. It must be noted that the Catmull and Clark method generates $(2n\text{-}3)\times(2m\text{-}3)$ new points, whereas the RATS method generates $(2n\text{-}1)\times(2m\text{-}1)$ new points. Thus, the RATS method produces $4(n+m\text{-}2)$ more new points than the Catmull and Clark method at every iteration. These additional points describe the patches that are generated by RATS at the border of the surface.

2.1. THE CATMULL AND CLARK SURFACE

The $(2n\text{-}3)\times(2m\text{-}3)$ new points, $P_{i,j}$, for $1 \le i \le 2n\text{-}3$ and $1 \le j \le 2m\text{-}3$, that are generated by the Catmull and Clark method [Catmu78], [Savva96] are given by Eq.1 – Eq4.

<u>*New Face Points*</u>

$$P_{2i\text{-}3,2j\text{-}3} = \frac{(V_{i\text{-}1,j\text{-}1} + V_{i\text{-}1,j} + V_{i,j\text{-}1} + V_{i,j})}{4} \qquad (1)$$

$$for \begin{cases} 2 \le i \le n \\ 2 \le j \le m \end{cases}$$

<u>*New Edge Points*</u>

$$P_{2i\text{-}3,2j\text{-}2} = \frac{1}{2}\left(\frac{(A+B)}{2} + \frac{(V_{i\text{-}1,j} + V_{i,j})}{2}\right) \qquad (2a)$$

$$for \begin{cases} 2 \le i \le n \\ 2 \le j \le m-1 \end{cases}$$

and

$$P_{2i\text{-}2,2j\text{-}3} = \frac{1}{2}\left(\frac{(B+C)}{2} + \frac{(V_{i,j\text{-}1} + V_{i,j})}{2}\right) \qquad (2b)$$

$$for \begin{cases} 2 \le i \le n-1 \\ 2 \le j \le m \end{cases}$$

where

$$A = P_{2i\text{-}3,2j\text{-}3} = \frac{(V_{i\text{-}1,j\text{-}1} + V_{i\text{-}1,j} + V_{i,j\text{-}1} + V_{i,j})}{4}$$

$$B = P_{2i\text{-}3,2j\text{-}2} = \frac{(V_{i\text{-}1,j} + V_{i\text{-}1,j+1} + V_{i,j} + V_{i,j+1})}{4}$$

$$C = P_{2i\text{-}2,2j\text{-}3} = \frac{(V_{i,j\text{-}1} + V_{i,j} + V_{i+1,j\text{-}1} + V_{i+1,j})}{4}$$

<u>*New Vertex Points*</u>

$$P_{2i\text{-}2,2j\text{-}2} = \frac{F}{4} + \frac{E}{2} + \frac{V_{i,j}}{4} \quad for \begin{cases} 2 \le i \le n-1 \\ 2 \le j \le m-1 \end{cases} \qquad (3)$$

where

$$E=\frac{1}{4}\left(\frac{V_{i,j}+V_{i,j\text{-}1}}{2}+\frac{V_{i,j}+V_{i,j+1}}{2}+\frac{V_{i,j}+V_{i\text{-}1,j}}{2}+\frac{V_{i,j}+V_{i+1,j}}{2}\right)$$

$$F=\frac{(A+B+C+D)}{4}$$

A, B, C are as defined above, and

$$D=P_{2i\text{-}2,2j\text{-}2}=\frac{(V_{i,j}+V_{i,j+1}+V_{i+1,j}+V_{i+1,j+1})}{4}$$

2.2. THE RATS SURFACE

The (2n-1)×(2m-1) new points, $P_{i,j}$, for $1\le i\le 2n$-1 and $1\le j\le 2m$-1, that are generated by the RATS method [Savva98], [Savva00] are given by Eq.4 – Eq19. Fig.7 illustrates two RATS surfaces.

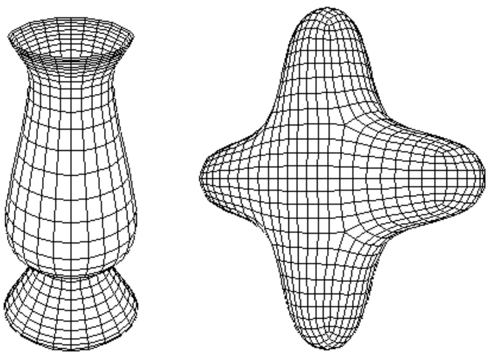

Fig.7 Two RATS surfaces

It must be noted that:

$$F_{i,j}=\frac{(V_{i,j}+V_{i,j+1}+V_{i+1,j}+V_{i+1,j+1})}{4}$$

New Vertex Points V_α

$$P_{2i-1,2j-1}=V_{i,j} \qquad \text{for} \begin{cases} i=1,n \\ j=1,m \end{cases} \tag{4}$$

New Vertex Points V_β

$$P_{2i-1,2j-1}=\frac{1}{2}\left(\frac{(V_{i-1,j}+V_{i,j})}{2}+\frac{(V_{i,j}+V_{i+1,j})}{2}\right) \tag{5}$$

$$\text{for} \begin{cases} 2\le i\le n-1 \\ j=1,m \end{cases}$$

and

$$P_{2i-1,2j-1}=\frac{1}{2}\left(\frac{(V_{i,j-1}+V_{i,j})}{2}+\frac{(V_{i,j}+V_{i,j+1})}{2}\right) \tag{6}$$

$$\text{for} \begin{cases} i=1,n \\ 2\le j\le m-1 \end{cases}$$

New Vertex Points V_γ

$$P_{2i-1,2j-1}=\frac{1}{4}\left(F_{i-1,j-1}+F_{i,j-1}+F_{i-1,j}+F_{i,j}\right) \tag{7}$$

$$\text{for} \begin{cases} 2\le i\le n-1 \\ 2\le j\le m-1 \end{cases}$$

New Edge Points E_α

$$P_{2i,2j-1}=\frac{1}{2}\left(P_{i,j}+P_{i+1,j}\right) \quad \text{for} \begin{cases} i=1,n-1 \\ j=1,m \end{cases} \tag{8}$$

and

$$P_{2i-1,2j}=\frac{1}{2}\left(P_{i,j}+P_{i,j+1}\right) \quad \text{for} \begin{cases} i=1,n \\ j=1,m-1 \end{cases} \tag{9}$$

New Edge Points E_β

$$P_{2i-2,2j-1}=\frac{1}{4}\left(\frac{(P_{i-2,j}+P_{i-1,j})}{2}+\frac{2(P_{i-1,j}+P_{i,j})}{2}+\frac{(P_{i,j}+P_{i+1,j})}{2}\right)$$

$$\text{for} \begin{cases} 3\le i\le n-1 \\ j=1,m \end{cases} \tag{10}$$

and

$$P_{2i-1,2j-2}=\frac{1}{4}\left(\frac{(P_{i,j-2}+P_{i,j-1})}{2}+\frac{2(P_{i,j-1}+P_{i,j})}{2}+\frac{(P_{i,j}+P_{i,j+1})}{2}\right)$$

$$\text{for} \begin{cases} i=1,n \\ 3\le j\le m-1 \end{cases} \tag{11}$$

New Edge Points E_γ

$$P_{2i-1,2j}=\frac{1}{2}\left(F_{i-1,j}+F_{i,j}\right) \quad \text{for} \begin{cases} 2\le i\le n-1 \\ j=1,m-1 \end{cases} \tag{12}$$

and

$$P_{2i,2j-1}=\frac{1}{2}\left(F_{i,j-1}+F_{i,j}\right) \quad \text{for} \begin{cases} i=1,n-1 \\ 2\le j\le m-1 \end{cases} \tag{13}$$

New Edge Points E_δ

$$P_{2i-2,2j-1}=\frac{1}{2}\left(\frac{(F_{i-2,j-1}+F_{i-2,j}+F_{i-1,j-1}+F_{i-1,j})}{4}+\frac{(F_{i-1,j-1}+F_{i-1,j,}+F_{i,j-1}+F_{i,j})}{4}\right)$$

$$\text{for} \begin{cases} 3\le i\le n-1 \\ 2\le j\le m-1 \end{cases} \tag{14}$$

and

$$P_{2i-1,2j-2}=\frac{1}{2}\left(\frac{(F_{i-1,j-2}+F_{i,j-2}+F_{i-1,j-1}+F_{i,j-1})}{4}+\frac{(F_{i-1,j-1}+F_{i,j-1,}+F_{i-1,j}+F_{i,j})}{4}\right) \tag{15}$$

$$\text{for} \begin{cases} 2\le i\le n-1 \\ 3\le j\le m-1 \end{cases}$$

New Face Points F_α

$$P_{2i,2j} = F_{i,j} \qquad \text{for } \begin{cases} i = 1, n-1 \\ j = 1, m-1 \end{cases} \tag{16}$$

New Face Points F_β

$$P_{2i-2,2j} = \frac{1}{2}\left(\frac{(F_{i-2,j} + F_{i-1,j})}{2} + \frac{(F_{i-1,j} + F_{i,j})}{2} \right) \tag{17}$$

$$\text{for } \begin{cases} 3 \le i \le n-1 \\ j = 1, m-1 \end{cases}$$

and

$$P_{2i,2j-2} = \frac{1}{2}\left(\frac{(F_{i,j-2} + F_{i,j-1})}{2} + \frac{(F_{i,j-1} + F_{i,j})}{2} \right) \tag{18}$$

$$\text{for } \begin{cases} i = 1, n-1 \\ 3 \le j \le m-1 \end{cases}$$

New Face Points F_γ

$$P_{2i-2,2j-2} = \frac{1}{4}\left(\begin{array}{l} \frac{(F_{i-2,j-2} + F_{i-2,j-1} + F_{i-1,j-2} + F_{i-1,j-1})}{4} + \\ \frac{(F_{i-1,j-2} + F_{i-1,j-1} + F_{i,j-2} + F_{i,j-1})}{4} + \\ \frac{(F_{i-2,j-1} + F_{i-2,j} + F_{i-1,j-1} + F_{i-1,j})}{4} + \\ \frac{(F_{i-1,j-1} + F_{i-1,j} + F_{i,j-1} + F_{i,j})}{4} \end{array} \right)$$

$$\text{for } \begin{cases} 3 \le i \le n-1 \\ 3 \le j \le m-1 \end{cases} \tag{19}$$

3. RELATION OF *RATS* TO B-SPLINES

Let a surface be defined by $n \times m$ control points $V_{i,j}$, for $1 \le i \le n$ and $1 \le j \le m$, and let $E_{i,j}$, for $1 \le i \le n$-1 and $1 \le j \le m$, be the midpoint of the edge $V_{i,j}\ V_{i+1,j}$, $E'_{i,j}$, for $1 \le i \le n$ and $1 \le j \le m$-1, be the midpoint of the edge $V_{i,j}\ V_{i,j+1}$, and $F_{i,j}$, for $1 \le i \le n$-1 and $1 \le j \le m$-1, be the average of the face defined by $V_{i,j}$, $V_{i,j+1}$, $V_{i+1,j}$ $V_{i+1,j+1}$. This is shown in Fig.8 where n=m=4.

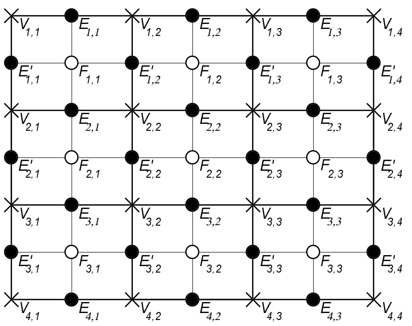

Fig.8 Representation of points defining a RATS surface

In Savva and Clapworthy [Savva98] and Savva [Savva00] it is shown that in the RATS method, the *new face points* F_γ are the average of all the *new vertex points* V_γ corresponding to the old points defining the face, and the *new edge points* E_δ are the midpoints of the two *new vertex points* V_γ corresponding to the old vertices defining the edge, as can be seen in Fig.2. Wireframe models are therefore better visualised if the F_γ and E_δ points are not displayed, as shown in the two surfaces displayed in Fig.7. Comparing Eq.4 – Eq.19 with the example given in this section and shown in Fig.8, it can be seen that the points defined in this section represent a RATS set of new points.

Since $E_{i,j}$ and $E'_{i,j}$ are the midpoints of the edges, the RATS new vertex points corresponding to them will represent the Catmull and Clark *new edge points*. Similarly, the RATS new vertex points for $F_{i,j}$ correspond to the Catmull and Clark *new face points*. This is illustrated in Fig.8. Calculating these points for $E_{2,2}$, $F_{2,2}$ and $V_{2,2}$ that are of type F_γ, E_δ, and V_γ points respectively, yields the equations that follow. Note that the E and E' points are treated identically since they are the same type of points.

New Vertex Points $V\gamma$ for $E_{2,2}$

$$V_\gamma \text{ for } E_{2,2} = \frac{1}{4}\left(\begin{array}{l} \frac{(V_{2,2} + E'_{1,2} + E_{2,2} + F_{1,2})}{4} + \\ \frac{(V_{2,3} + E'_{1,3} + E_{2,2} + F_{1,2})}{4} + \\ \frac{(V_{2,2} + E'_{2,2} + E_{2,2} + F_{2,2})}{4} + \\ \frac{(V_{2,3} + E'_{2,3} + E_{2,2} + F_{2,2})}{4} \end{array} \right) \tag{20}$$

New Vertex Points $V\gamma$ for $F_{2,2}$

$$V_\gamma \text{ for } F_{2,2} = \frac{1}{4}\left(\begin{array}{l} \frac{(V_{2,2} + E'_{2,2} + E_{2,2} + F_{2,2})}{4} + \\ \frac{(V_{2,3} + E'_{2,3} + E_{2,2} + F_{2,2})}{4} + \\ \frac{(V_{3,2} + E'_{2,2} + E_{3,2} + F_{2,2})}{4} + \\ \frac{(V_{3,3} + E'_{2,3} + E_{3,2} + F_{2,2})}{4} \end{array} \right) \tag{21}$$

New Vertex Points $V\gamma$ for $V_{2,2}$

$$V_\gamma \text{ for } V_{2,2} = \frac{1}{4}\left(\begin{array}{l} \frac{(V_{2,2} + E'_{1,2} + E_{2,1} + F_{1,1})}{4} + \\ \frac{(V_{2,2} + E'_{1,2} + E_{2,2} + F_{1,2})}{4} + \\ \frac{(V_{2,2} + E'_{2,2} + E_{2,1} + F_{2,1})}{4} + \\ \frac{(V_{2,2} + E'_{2,2} + E_{2,2} + F_{2,2})}{4} \end{array} \right) \tag{22}$$

From the definition given in the beginning of this section, it is known that

$$E_{i,j} = \frac{(V_{i,j} + V_{i+1,j})}{2}$$

$$E'_{i,j} = \frac{(V_{i,j} + V_{i,j+1})}{2}$$

$$F_{i,j} = \frac{(V_{i,j} + V_{i,j+1} + V_{i+1,j} + V_{i+1,j+1})}{4}$$

Substituting the above equations in Eq.20 – Eq.22 results in Eq.23 – Eq.25.

New Vertex Points $V\gamma$ for $F_{2,2}$ or ***New Face points***

$$V_\gamma \text{ for } F_{2,2} = \frac{(V_{2,2} + V_{2,3} + V_{3,2} + V_{3,3})}{4} \quad (23)$$

New Edge Points $V\gamma$ for $E_{2,2}$ or ***New Edge points***

$$V_\gamma \text{ for } E_{2,2} = \frac{1}{2}\left(\frac{(F_{1,2} + F_{2,2})}{2} + \frac{(V_{2,2} + V_{2,3})}{2} \right) \quad (24)$$

New Vertex Points $V\gamma$ for $V_{2,2}$ or ***New Vertex points***

$$V_\gamma \text{ for } V_{2,2} = \frac{F}{4} + \frac{2E}{4} + \frac{V_{2,2}}{4} \quad (25)$$

where

$$F = \frac{(F_{1,1} + F_{1,2} + F_{2,1} + F_{2,2})}{4}$$

and

$$E = \frac{1}{4}\left(\frac{V_{2,2}+V_{2,1}}{2} + \frac{V_{2,2}+V_{2,3}}{2} + \frac{V_{2,2}+V_{1,2}}{2} + \frac{V_{2,2}+V_{3,2}}{2} \right)$$

Comparing Eq.24 – Eq.26 to Eq.1 – Eq.3 it can be seen that the resulting *new face*, *new edge* and *new vertex points* are the same as in the Catmull & Clark B-spline recursive method. Therefore, the inner surface produced by RATS is identical to a B-spline surface. However, RATS produces an additional set of patches at the border of the surface, comparing to the Catmull and Clark method.

4. SUMMARY AND CONCLUSIONS

The RATS (Recursive Arbitrary Topology Splines) method defines surfaces with an arbitrary topology of control points. It is a recursive spline method and is derived from the midpoint subdivision of Bézier splines.

The Catmull and Clark method is the most popular recursive spline method for defining surfaces based on an arbitrary topology of control points. The two methods are in fact related. This paper derives this relationship: The Catmull and Clark surface is *identical* to the inner RATS surface.

However, the RATS method generates an additional set of patches describing the border of the surface. Thus, the RATS surface is fitted nearer to the control mesh border than the Catmull & Clark surface, giving a better definition of the control mesh.

REFERENCES

[Ball88] Ball A.A. and Storry J.T. Conditions for tangent plane continuity over recursively defined B-spline surfaces. *ACM Transactions on Computer Graphics*, Vol.7, No.2, pp.83-102, 1988

[Catmu78] Catmull E. and Clark J. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Design*, Vol.10, No.6, pp.350-355, 1978

[Chaik74] Chaikin G. An algorithm for high speed curve generation. *Computer Graphics and Image Processing*, Vol.3, pp.346-349, 1974

[Doo78] Doo D. and Sabin M. Behaviour of recursive division surfaces near extraordinary points. *Computer Aided Design*, Vol.10, No.6, pp.356-360, 1978

[Halst93] Halstead M., Kass M. and DeRose T. Efficient, fair interpolation using Catmull-Clark surfaces, *SIGRAPH*, *Proc Computer Graphics*, 1993, 36-44

[Loop90] Loop C.T. and De Rose T. Generalized B-spline surfaces of arbitrary topology. *Proc SIGGRAPH'90*, *Computer Graphics*, Vol.24, No.4, pp.347-355, 1990

[Nasri87] Nasri A.H. Polyhedral subdivision methods for free-form surfaces, *ACM Transactions on Graphics*, Vol.6, pp.29-73, 1987

[Savva96] Savva A. Modelling articulated figures on arbitrary meshes of control points, *PhD Dissertation*, Univ. of North London, 1996.

[Savva98] Savva A. and Clapworthy G.J. A Recursive Approach to Parametric Surfaces Containing Non-rectangular Patches, *International Conference on Information Visualisation*, IEEE Press, pp.300-306, London, July 1998

[Savva00] Savva A. and Clapworthy G.J. A Spline Method for Figure Modelling, *IASTED Inter. Conference in Modelling and Simulation*, pp.179-183, Pittsburgh, USA, May 2000