Faster ASV Decomposition for Orthogonal Polyhedra, Using the Extreme Vertices Model (EVM).

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Abstract.

The alternating sum of volumes (ASV) decomposition is a widely used technique for converting a b-rep into a CSG model, with all its implicit uses and advantages -like form feature recognition, among others. The obtained CSG tree has convex primitives at its leaf nodes, while the contents of its internal nodes alternate between the set-union and set-difference operators.

This paper first shows that the obtained CSG tree T can also be expressed as the regularized Exclusive-OR operation among all the convex primitives at the leaf nodes of T, regardless the structure and internal nodes of T. The importance of this result becomes apparent, for example, with those solid modeling schemes, for which the Exclusive-OR operation can be performed much faster than both the set union and set difference operators. This is the case for the Extreme Vertices Model (EVM) for orthogonal polyhedra. Therefore, this paper is then devoted for applying this result to orthogonal polyhedra, using the Extreme Vertices Model. It also includes a comparision of using this result vs. not-using it when finding the ASV decomposition of orthogonal polyhedra, as well as some practical uses for the ASV decomposition of orthogonal polyhedra.

Keywords: Solid Modeling, Boundary representations (b-rep), Constructive Solid Geometry (CSG), Boolean operations, ASV decomposition, Form feature recognition.

1 Introduction.

A complex shape is often divided into simpler shapes to facilitate representing, reasoning and performing various operations on the shape [Sakurai95]. Convexity is a fundamental characteristic that allows easier solutions to most geometric problems [Preparata85, Edelsbrunner87]. To exploit convexity, the given non convex object can be represented as a Boolean combination of convex components, a process called *convex decomposition* [Kim92].

For example, the earliest presentations of form feature recognition [Kyprianou80, Woo82] propose a convex decomposition method which uses convex hulls and set-difference operations. Reflecting the nature of alternating volume contribution, this decomposition was called *Alternating Sum of Volumes* (ASV) [Woo82]. Then, form features can be automatically obtained by a suitable manipulation of the resulting expression [Tang91a, Waco93].

In ASV decomposition, a non convex object is represented by a hierarchical structure of convex components. ASV decomposition, however, may not converge, which severely limits the domain of geometric objects that the method can handle. Kim and Wilde formalized the ASV decomposition, identified the cause of non convergence, and presented a remedy for non convergence [Kim89]. Tang and Woo discussed algorithmic aspects of non convergence detection and its remedy [Tang91a, b]. By combining ASV decomposition and remedial partitioning using splitting operations, Kim and Wilde proposed the *Alternating Sum of Volumes with Partitioning* (ASVP) and proved its convergence [Kim89]. Different feature-recognition approaches are also proposed elsewhere [Henderson84, Falcidieno89, Gavankar90, Joshi88]. A good survey in feature recognition can be found in [Pratt95].

On the other hand, the Extreme Vertices Model (EVM), first published in [Aguilera96], was introduced as a restricted model for *two-manifold Orthogonal Polyhedra* (OP). Also in that paper, a Boolean operations algorithm that works for that model was presented and analyzed. In more recent works [Aguilera97, 98b], a natural domain extension for the EVM basics that now handles *Orthogonal Pseudo-Polyhedra* (OPP) (see a definition in section 3.2), is presented. The EVM is an unambiguous model for representing and handling OPP by means of a single subset of the polyhedron's vertices, -a very simple and concise, yet powerful and versatile way. It also allows simple and robust algorithms for performing the most usual and demanding tasks, as Set Membership Classification and Boolean Operations.

A very important feature of EVM algorithms is that, even though their input data (i.e., vertices' coordinates) can be floating-point values, no time-consuming floating-point arithmetic is ever performed, so there are absolutely no propagation errors due to partial results (which do not exist). All results are obtained by just classifying and selecting vertices' coordinates of the initial data. All the theoretical foundations for the EVM in its full domain can be found in [Aguilera98c].

The paper is organized as follows: sections 2 and 3 give the needed background respectively on the ASV decomposition, and the Extreme Vertices Model. Section 4 develops the presented approach. Section 5 presents some experimental results, and the last section discusses the conclusions and shows the proposed future work.

2 Background on ASV.

2.1 Alternating Sum of Volumes Decomposition.

The ASV decomposition of a general polyhedron is a method that provides a CSG tree that represents the original object by means of convex primitives components.

Let $CH(P)$ be the convex hull of a polyhedron P and $CHD^*(P)$ the regularized convex-hull difference, also called deficiency of P, $CHD^*(P) = CH(P) -^* P$. The ASV decomposition, $ASV(P)$, of a polyhedron P is defined as the following recursive expression [Kim89, 92]:

$$ASV(D_k) = \begin{cases} D_k & \text{if } D_k \text{ is convex} \\ H_{k+1} -^* ASV(D_{k+1}) & \text{if } D_k \text{ is not convex} \end{cases}$$

where $D_0 = P$, $H_{k+1} = CH(D_k)$, and $D_{k+1} = H_{k+1} -^* D_k$.
(D stands for deficiency). Then P is expressed as:

$$P = H_1 -^* (H_2 -^* (H_3 -^* (H_4 -^* ...))) \qquad (1)$$
$$\text{or} \quad P = H_1 -^* H_2 +^* H_3 -^* H_4 +^* ... \qquad (2)$$
$$\text{or} \quad P = (H_1 -^* H_2) +^* (H_3 -^* H_4) +^* ... \qquad (3)$$
$$\text{or} \quad P = H_1 -^* (H_2 -^* H_3) -^* (H_4 -^* H_5) -^* ... \qquad (4)$$

Where Eqn.2 to Eqn.4 are obtained thorough algebraic manipulation of Eqn.1, and these series are known as alternating sum of volumes; conjunctive; and disjunctive forms, respectively. Here, $+^*$ denotes a quasi-disjoint union operator [Kim92], or the *gluing* operator [Mäntylä88].

From the construction of the ASV series, each resulting convex hull H_k will completely enclose all subsequent convex hulls, i.e., $H_j \supseteq H_k$ for any $j > k$. Furthermore, the ASV decomposition will succeed when a deficiency D_n is found to be convex for some n. In that case $H_{n+1} = D_n$ and $D_{n+1} = \emptyset$, then the relation $H_{k+1} \subseteq H_k$ holds for all $k \leq n$.

2.2 Non Convergence of ASV and its Remedy.

It has been shown [Woo82] that an ASV series is non-terminating if and only if two consecutive convex hulls coincide, i.e., $H_k = H_{k+1}$. In such a case, the deficiency D_{k-1} is ASV-irreducible [Kim92], or nonconvergent [Tang91a]. Moreover, the identification of $H_k = H_{k+1}$ implies that, for any $j \geq k$, $H_j = H_k$ and $D_{j+1} = D_{j-1}$ [Tang91b], therefore this process becomes cyclic.

When a deficiency becomes ASV-irreducible, one solution is to split the ASV-irreducible deficiency by an appropriate plane, in such a way that the split parts are themselves convergent, and find the ASV series of each subset. Kim provides some methods for finding such a plane. The process is called Alternating Sum of Volumes with Partitioning (ASVP) [Kim89, 92].

2.3 Non-Extremal Faces Reduction.

A face of a polyhedron P is said to be an extremal face if the plane containing the face is a supporting plane of P, i.e., if P is on one side of the closed planar halfspace determined by the plane; otherwise it is a non-extremal face. So, the boundary of P can be partitioned into the sets of extremal and non-extremal faces [Kim89].

In the ASV-decomposition step, the boundary of H_k is composed by the set of extremal faces of D_{k-1} plus a new set of fictitious hull faces, while the boundary of D_k is composed by the set of non-extremal faces of D_{k-1} plus the same set of fictitious hull faces. Therefore, the set of non-extremal faces of D_k is a subset of the non-extremal faces of D_{k-1}. The recursion finishes when D_k does not contain any non-extremal face, i.e., when it is convex. The non-convergence of ASV-decomposition corresponds to the irreducibility of the set of non-extremal faces. The remedy

for non-convergence is to partition the non-extremal faces of an ASV-irreducible deficiency, so that the resulting sets of non-extremal faces can be reduced.

2.4 Recursive definitions of the ASVP.

The decomposition process to compute the ASVP tree, T, of a generic non-convex polyhedron, P, can be described with the following simple terms:

a) If the non-extremal faces of P can be reduced, then P must be decomposed as $P = H -^* D$, where H and D are, respectively, the convex hull and deficiency of P, see Fig. 1.a. Moreover, H is a leaf node and D a generic node in T, so only D should be recursively processed.

b) Otherwise P is ASV-irreducible, then it must be split as $P = Q +^* R$ according to section 2.2, where both Q and R are generic nodes in T, so both of them should be recursively processed. See Fig. 1.b.
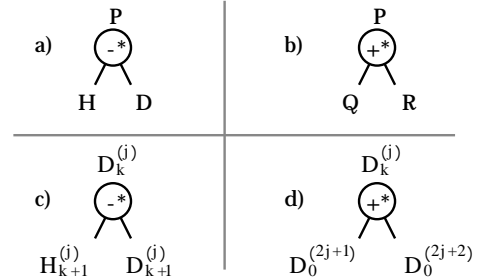

Figure 1: Generic nodes in the ASVP tree with and without indexes.

The expression of a complex tree, however, cannot be accurately represented with the above definition. This can be solved by using indexes. Let $D_k^{(j)}$ denote a generic polyhedron, then, in a similar way as above, the recursive definition of the ASVP process is shown in Fig. 2.

The value of k is incremented every time a convex hull and a deficiency are computed, see Fig. 1.c. This value is reset to zero whenever a split operation must be made, then value j is updated in a way that provides uniqueness within subtrees. See Fig. 1.d.

2.5 Presence of the XOR Operation in the ASVP Decomposition.

The regularized Exclusive-Or (XOR) operation, denoted by $\oplus^*$, between two solids P and Q, is defined as:

$$P \oplus^* Q = (P \cup^* Q) -^* (P \cap^* Q) \qquad (5)$$
$$\text{or} \quad P \oplus^* Q = (P -^* Q) \cup^* (Q -^* P) \qquad (6)$$

These definitions lead to the following properties:

Property 1: *Let P and Q be two quasi-disjoint solids, i.e., $P \cap^* Q = \emptyset$. Then $P +^* Q = P \oplus^* Q$.*

Property 2: *Let P and Q be two solids with $P \supseteq Q$, i.e., $Q -^* P = \emptyset$. Then $P -^* Q = P \oplus^* Q$.*

$$ASVP(D_k^{(j)}) = \begin{cases} D_k^{(j)} & \text{if } D_k^{(j)} \text{ is convex} \\ H_{k+1}^{(j)} -^* ASVP(D_{k+1}^{(j)}) & \text{if the non-extremal faces of } D_k^{(j)} \text{ can be reduced} \\ ASVP(D_0^{(2j+1)}) +^* ASVP(D_0^{(2j+2)}) & \text{if } D_k^{(j)} \text{ is ASV-irreducible} \end{cases}$$

where
- $H_{k+1}^{(j)} = CH(D_k^{(j)})$, and $D_{k+1}^{(j)} = CHD^*(D_k^{(j)})$. See Fig. 1.c.
- $D_0^{(2j+1)}$ and $D_0^{(2j+2)}$ are the split parts of $D_k^{(j)}$, i.e. $D_k^{(j)} = D_0^{(2j+1)} +^* D_0^{(2j+2)}$. See Fig. 1.d.
- $D_0^{(0)}$ represents the original polyhedron, P.

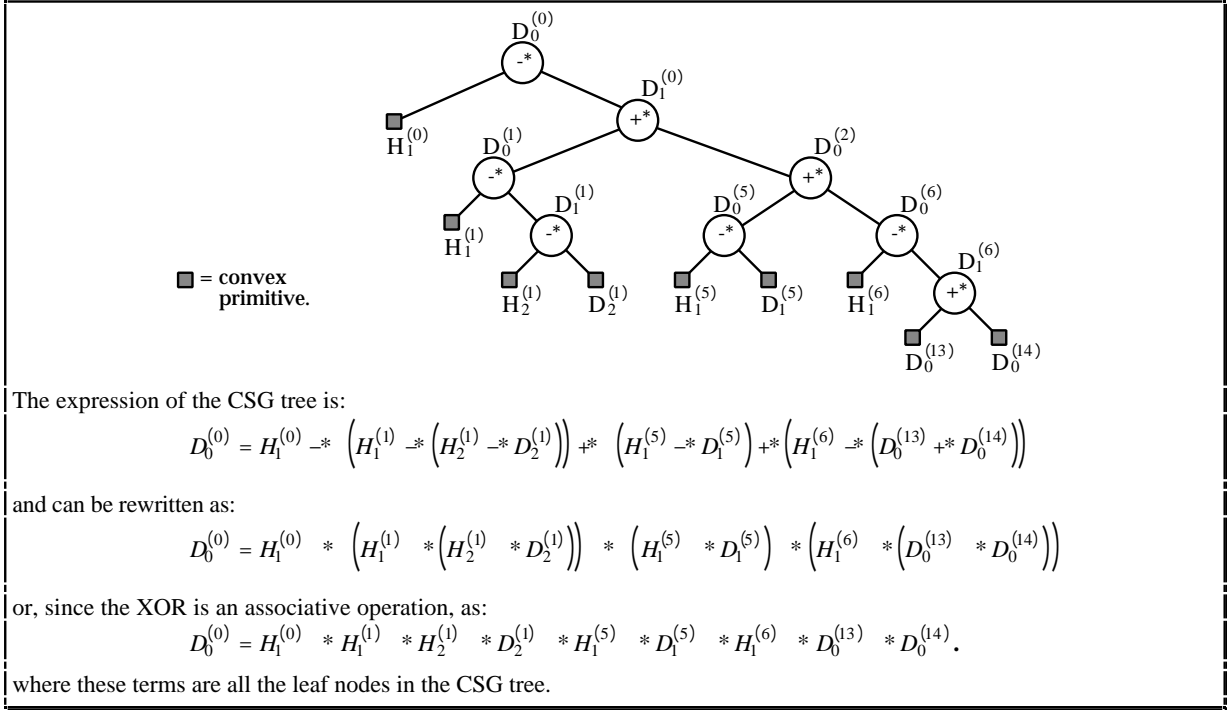**Figure 2:** Recursive definition of the ASVP process.

The expression of the CSG tree is:

$$D_0^{(0)} = H_1^{(0)} -^* \left(H_1^{(1)} -^* \left(H_2^{(1)} -^* D_2^{(1)} \right) \right) +^* \left(H_1^{(5)} -^* D_1^{(5)} \right) +^* \left(H_1^{(6)} -^* \left(D_0^{(13)} +^* D_0^{(14)} \right) \right)$$

and can be rewritten as:

$$D_0^{(0)} = H_1^{(0)} *^* \left(H_1^{(1)} *^* \left(H_2^{(1)} *^* D_2^{(1)} \right) \right) *^* \left(H_1^{(5)} *^* D_1^{(5)} \right) *^* \left(H_1^{(6)} *^* \left(D_0^{(13)} *^* D_0^{(14)} \right) \right)$$

or, since the XOR is an associative operation, as:

$$D_0^{(0)} = H_1^{(0)} *^* H_1^{(1)} *^* H_2^{(1)} *^* D_2^{(1)} *^* H_1^{(5)} *^* D_1^{(5)} *^* H_1^{(6)} *^* D_0^{(13)} *^* D_0^{(14)}.$$

where these terms are all the leaf nodes in the CSG tree.

Figure 3: Example of a complex tree.

Now, the first reuslt of this paper is theorem 1, regarding the presence of the XOR operation in the ASVP process.

Theorem 1: *Let P be a polyhedron and T the CSG tree obtained by the ASVP decomposition of P. Then P can be expressed as the regularized Exclusive-OR operation among all the convex polyhedra found at the leaf nodes of T, regardless the structure and internal nodes of T.* (See an example in Fig. 3)

Proof. The ASVP-process of an object results in a CSG tree T, whose leaf nodes are convex polyhedra. Let us use the recursive definition of ASVP, where P is a generic internal node in T. Then, P is represented by means of either a Boolean difference or by a quasi-disjoint union:

a) If the non-extremal faces of P can be reduced, then $P = H -^* D$, where H and D are, respectively, the convex hull and deficiency of P, thus $D \subseteq H$. Then, by Property 2, $P = H *^* D$.

b) If P is ASV-irreducible, then $P = Q +^* R$, where Q and R are the split parts of P. Then, by Property 1, $P = Q *^* R$.

Therefore, whether or not P is ASV-irreducible, the Boolean operation in the corresponding internal node of T can be replaced by the regularized XOR operation between the same two subtrees of P. Thus, the obtained expression for P will contain only XOR operators, with all the needed pairs of parenthesis to reflect the tree structure. The XOR, however, is an associative and commutative operation and thus, its final result does not depend on the order of applying them. So all the parenthesis can be removed from the expression, which proves this theorem. ∎

3 Background on the Extreme Vertices Model (EVM).

In this section the theoretical basis for EVM is set up. We begin with several definitions and then we show the properties of EVM. Not all proofs have been included here due to the obvious space limitations. The reader, however, can find them in [Aguilera98c].

3.1 Terminology.

A pseudo-polyhedron is the space enclosed by a finite collection of planar faces such that (a) every edge has at least two adjacent faces, and (b) if any two faces meet, they meet at a common edge [Tang91a]. A two-manifold edge is adjacent to exactly two faces, and a two-manifold vertex is the apex of only one cone of faces. Conversely, a non-manifold edge is adjacent to more than two faces, and a non-manifold vertex is the apex of more than one cone of faces [Rossignac91].

Polyhedra are two-manifold r-sets. Pseudo-polyhedra are r-sets with non-manifold boundary. A non-regular polyhedron is a non homogeneously three-dimensional object, i.e., it has "dangling" faces or edges. [Tang91a, Rossignac91].

3.2 Orthogonal Polyhedra.

Orthogonal polyhedra (OP) are polyhedra with all their edges and faces oriented in three orthogonal directions [Preparata85, Juan89] Orthogonal PseudoPolyhedra (OPP) are defined as regular and orthogonal polyhedra with non-manifold boundary [Aguilera98a, 98c]. In an OPP, a non-manifold edge is adjacent to exactly four faces and a non-manifold vertex is the apex of exactly two cones of faces, see Fig. 4.
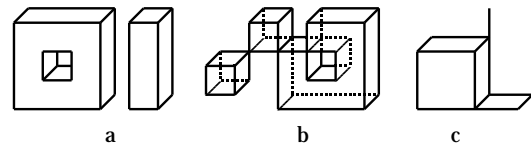

Figure 4: a) An OP. b) An OPP. c) A non-regular orthogonal polyhedron.

3.3 The Extreme Vertices Model for OPP.

A brink (orextended-edge) is an uninterrupted segment built out of the maximal sequence of collinear and contiguous two-manifold edges of an OPP, P.
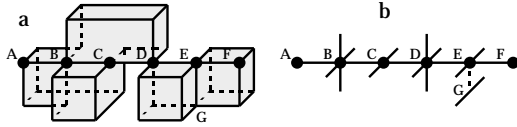
Figure 5: An OPP and its brinks.

Fig. 5.a shows an OPP with a brink having five edges and six vertices (*A* to *F*). There are also other seven brinks of two edges each, shown in Fig. 8.b,, and the remaining brinks are composed of one edge. The non-manifold elements in this OPP are vertex *D* and edge $\overline{EG}$ (shown as a dashed line in Fig. 8.b).

According to the brink's definition, non-manifold edges <u>do not</u> belong to brinks. Every two-manifold edge belongs to a brink, whereas every brink consists of one or more edges and contains as many vertices as the number of edges plus one.

The <u>Extreme Vertices (EV)</u> of an OPP are defined as the ending (or extreme) vertices of all the OPP brinks. EV always have three incident two-manigold edges, regardless of the number of non-manifold edges (if any). The <u>Extreme Vertices Model (EVM)</u> for OPP is defined as a model that only stores all their EV. Finally, an <u>ABC-sorted EVM</u> is an EVM where its EV are sorted first by coordinate *A*, then by *B*, and then by *C*. An EVM can be sorted on six different ways: *XYZ*, *XZY*, *YXZ*, *YZX*, *ZXY*, and *ZYX*. From now on, <u>EVM(P)</u> will denote the ABC-sorted EVM of an OPP *P*. Although the EVM has been defined for 3D-OPP, it is also defined for 2D-OPP and 1D-OPP [Aguilera97, 98c].

A <u>plane of vertices</u> of an OPP *P*, denoted as *plv(P)*, is the set of faces of *P* lying on a plane perpendicular to a main axis. Similarly, a <u>line of vertices</u> is the set of brinks lying on a line parallel to a main axis.

Theorem 2: *The Extreme Vertices Model is a complete B-Rep model for OPP.*

This theorem is proven in [Aguilera98c], and states that EVM can unambiguously represent OPP, because all the information which is not explicitly represented can be obtained from the extreme vertices set. Moreover, it has been developed an algorithm that computes a complete hierarchical B-Rep from the EVM [Aguilera98c].

EVM is a very simple and concise model easy to validate [Aguilera98b]. It allows simple and robust algorithms for performing the most usual and demanding tasks such as Boolean operations [Aguilera96] and set membership classification [Aguilera98b]. In this paper we only explain the exclusive OR and splitting operations in the next subsections, because they are needed to understand the proposed approach showed in the next section.

3.4 The Exclusive-OR Operation on the EVM.
The regularized exclusive-OR operation can be easily carried out on EVM, as stated by the following theorem:

Theorem 3: *Let P and Q be two d-dimensional OPP having EVM(P) and EVM(Q) as their respective models, then EVM(P * Q) = EVM(P) EVM(Q).*

This theorem is proven in [Aguilera98c] for 1D-OPP, and then, by induction, for 2D and 3D-OPP. Note that while *P*, *Q*, and *P* *Q* are <u>infinite</u> sets of points, *EVM(P)*, *EVM(Q)*, and *EVM(P* *Q)* are <u>finite</u> sets of points. Therefore, *P* *Q* can be easily carried out on EVM, by means of simple point-wise XOR operations. This process takes linear time [Aguilera98c].

Finally, the following two corollaries can be stated, which correspond to specific situations of the XOR operands. They allow to compute the union and difference of two OPPs when those specific situations are met. For general Boolean operations see [Aguilera96, 98c].

Corollary 1: *Let P and Q be two d-dimensional disjoint or quasi-disjoint OPPs having EVM(P) and EVM(Q) as their respective models, then EVM(P Q) = EVM(P) EVM(Q).*

Corollary 2: *Let P and Q be two d-dimensional Orthogonal Pseudo-Polyhedra such that P Q, with EVM(P) and EVM(Q) as their respective models, then EVM(P −*Q) = EVM(P) EVM(Q). (that is, the complement of Q with respect P).*

Proofs: These Corollaries are proved by combining Theorem 3 with Properties 1 and 2, respectively. ∎

3.5 The Splitting Operation on the EVM.
Splitting an OPP *P* with an orthogonal splitting plane *SP* produces two polyhedra *Q* and *R*, each at the IN and OUT halfspaces of *SP*, respectively. This process is based on corollary 1 and the fact that the Extreme Vertices of each of the resulting objects will be a subset of *EVM(P)*, except for some new Extreme Vertices that could be created and they will lie on *SP*. The idea is that if a brink perpendicular to *SP* is not split by *SP*, then both of its Extreme Vertices (and the brink itself) will be assigned to either *Q* or *R*, accordingly. However if a brink is split by *SP*, then each part will become a new brink for the corresponding resulting objects. Each new brink must be defined by its two Extreme Vertices, one comes from *P* and the other must be created at the intersection of the brink with *SP*. It is shown that only those brinks perpendicular to *SP* need to be considered. This process also takes linear time. See a full discussion in [Aguilera98a, c].

4 Extracting Form Features from OPP.

4.1 The Approach.
This section proposes the use of ASVP to extract OPP form features using EVM. For this purpose, the proposed approach uses orthogonal hulls instead of convex hulls; replaces both the set-difference and quasi-disjoint union operations by the faster XOR operation; and proves that these changes do not nullify convergence. Then with these changes all intermediate and resulting objects are OPP, so EVM can be used. Let us call this process as *Alternating Sum of <u>Orthogonal</u> Volumes* (ASOV).

4.2 Use of Orthogonal Hulls.
"Any convex volume whose boundary includes the extremal faces of the given object can be used as convex component throughout the decomposition instead of convex hulls" [Kim92]. Therefore, the orthogonal hull, or minimum bounding box, whose boundary is a superset of the set of extremal faces of an OPP, can be used just as well.

Theorem 4: *Let P be an OPP, CH(P) its convex hull, and OH(P) its orthogonal hull, or minimum bounding box. Let A be the set of faces of P lying on the boundary of OH(P), and let B be the set of faces of P lying on the boundary of CH(P). Then A = B.*

This theorem is proved by Juan-Arinyo in [Juan95], and states that all the extremal faces of an OPP, P, lie on the boundaries of both $CH(P)$ and $OH(P)$. Therefore, when applying the ASV-decomposition method to P, computing deficiency sets with respect to $CH(P)$ is equivalent to computing deficiency sets with respect to $OH(P)$. This is true because the set of faces of P is partitioned into the same set of extremal and non-extremal faces, whether by using $CH(P)$, or by using $OH(P)$. The only and unimportant difference is that the resulting set of fictitious hull faces when using $OH(P)$ will obviously contain only orthogonal faces.

4.3   Use of EVM in ASOV and its Behavior.

If P is an OPP and orthogonal hulls are to be used throughout its ASOV-decomposition, then all intermediate and resulting objects are OPP, so EVM can be used to handle all necessary OPP operations.

Let P be an OPP, and $D_0 = P$, $H_{k+1} = CH(D_k)$, and $D_{k+1} = H_{k+1} -^* D_k$, then the same recursive expression holds:

$$ASOV(D_k) = \begin{cases} D_k & \text{if } D_k \text{ is a box} \\ H_{k+1} -^* ASOV(D_{k+1}) & \text{otherwise} \end{cases}$$

The use of EVM greatly simplifies handling all these OPP operations. Given $EVM(D_0)$, then:

a)   computing $EVM(H_{k+1})$ is as simple as searching for the minimum and maximum coordinate values stored in $EVM(D_k)$, and ABC-ordering the resulting eight points (vertices of H_{k+1}).

b)   computing $EVM(D_{k+1}) = EVM(H_{k+1} -^* D_k)$ is also simple. Since $H_{k+1} \supseteq D_k$ then, by corollary 2, $EVM(H_{k+1} -^* D_k) = EVM(H_{k+1}) \otimes EVM(D_k)$. Thus, $EVM(D_{k+1}) = EVM(H_{k+1}) \otimes EVM(D_k)$.

Furthermore, the following theorem regarding the use of EVM in ASOV, holds:

Theorem 5: *Let P be an ASOV-decomposable OPP, and H_1, H_2, …, H_n, the orthogonal hulls obtained by the ASOV decomposition of P, then*

$$EVM(P) = \overset{n}{\underset{k=1}{\otimes}} EVM(H_k).$$

Proof: The proof comes directly from theorems 1 and 3. ∎

For clarity purposes, the use of EVM in the 2D case will be presented first, then the 3D case will be a simple extension of the 2D case.

4.4   The Use of EVM in the 2D Case.

Similarly to extremal faces in 3D (see section 2.3), an edge of a polygon is an <u>extremal edge</u> if the line containing the edge is a supporting line of the polygon, i.e., if the polygon is on one side of the closed linear halfplane determined by the line; otherwise it is a <u>non-extremal edge</u>. Also, the boundary of the polygon can be partitioned into the sets of extremal and non-extremal edges by means of its 2D orthogonal hull.

Fig. 6 shows the extremal edges of P, and D_1 to D_4 as solid lines. Note that a 2D-OPP has at least four extremal edges, while a 3D-OPP has at least six extremal faces (i.e., twice as many as the number of dimensions).
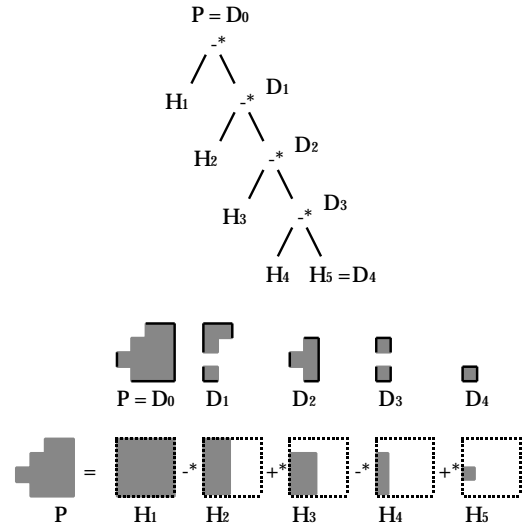

Figure 6: A 2D example of ASOV-decomposition (solid lines are extremal edges).

Fig. 7.a shows the EVM of each shape in Fig. 6 using solid dots. Each pair of consecutive extreme vertices in the vertical direction is joined by the corresponding brink. When computing the EVM of the next shape, by means of an XOR operation with the orthogonal hull, some new vertices will be added (shown as empty dots) and will appear as part of the EVM of the next shape, and some others will be removed (shown as solid and crossed dots). Note that each corner of the orthogonal hull has either an empty dot or a crossed dot, according to the absence or presence, respectively, of a vertex of the shape on that corner. The addition and removal of those vertices are handled by the XOR operation.

Definition 1: A <u>fully extremal edge (FEE)</u> of an OPP, P, is an edge of P that coincides completely with an edge of $OH(P)$.

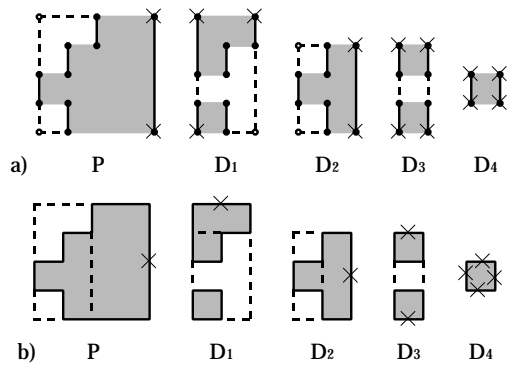We now will prove that fully extremal edges detect ASOV-convergence.

Figure 7: **a)** P and its deficiencies in the EVM.
             b) Fully extremal edges.

Lemma 1: $H_k \supset H_{k+1}$ *if and only if D_{k-1} has at least one fully extremal edge (FEE).*

Proof: We know that $H_j \supseteq H_k$ for any $j > k$. Now, let E be a FEE in D_{k-1}. Since $H_k = OH(D_{k-1})$ then E is also a FEE for H_k. Thus, neither D_k $(= H_k -^* D_{k-1})$ nor H_{k+1} contain edge E, which is a supporting edge for H_k.
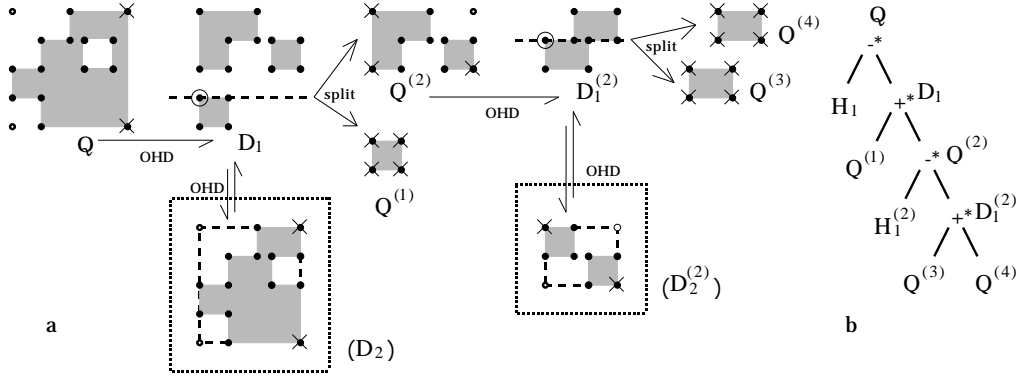
Figure 8: a) EVM process of a ASOVP decomposition. **b)** Resulting CSG tree.

So $H_k \supseteq H_{k+1}$, and therefore $H_k \supset H_{k+1}$. Conversely and for the same reason, if $H_k \supset H_{k+1}$ and D_{k-1} had no FEE, then H_k would have no way to shrink into a smaller H_{k+1} with the ASOV process, therefore $H_k = H_{k+1}$. This implies that, if $H_k \supset H_{k+1}$ then D_{k-1} has at least one FEE. ∎

Fig. 7.b shows that any FEE (shown with a cross in its middle) will be removed. Since no other edge lies on the same supporting line of such extremal edge, the corresponding edge of the orthogonal hull for the next shape will be displaced to the next line of edges towards the interior of the shape. Dashed lines show the orthogonal hull with the displaced edge(s), thus it coincides with the orthogonal hull of the following shape, showing that it becomes smaller at each step.

Theorem 6: *The following statements are equivalent:*
a) *An ASOV series is non terminating.*
b) *There is a D_{k-1} with no FEE.*
c) *Two consecutive orthogonal hulls coincide: $H_k = H_{k+1}$.*
d) *For any $j \le k$, $H_j = H_k$ and $D_{j+1} = D_{j-1}$, therefore this process becomes cyclic.*

Proof: The proof comes directly from lemma 1, from [Woo82, and Tang91b], and from the construction of the ASOV series. ∎

A similar remedial procedure for non convergence is to split the ASOV-irreducible deficiency into subsets that are themselves convergent, and find the ASOV series of each subset. This process will be called *Alternating Sum of Orthogonal Volumes with Partitioning* (ASOVP).

The splitting point is chosen as the first Extreme Vertex of D_{k-1} (according to the ABC-sorting) that does not coincide with a corner of its orthogonal hull H_k, this vertex will be called the <u>Splitting Vertex (*SV*)</u>. Note that *SV* belongs to an extremal edge of D_{k-1}, and will coincide with a corner of the orthogonal hull of at least one of the split parts. This approach leads towards making this extremal edge be a fully extremal edge, and thus enabling convergence.

Fig. 8.a shows the ASOVP-process of object Q, having the same shape as object P of figures 6 and 7, but with a hole. Q has one FEE, thus $D_1 = OHD^*(Q)$ is computed. D_1 has no FEE, thus D_1 is split at *SV* (shown as a circled dot) producing $Q^{(1)}$ and $Q^{(2)}$, and the ASOVP-process is recursively applied to them. Continuing in this way, the resulting CSG tree, shown in Fig. 8.b, can be obtained. Note that D_1 and $D_1^{(2)}$ have no

FEE, thus their ASOV become cyclic with their respective complements D_2 and $D_2^{(2)}$, shown in the dotted boxes. However, by applying the remedial-partition method through *SV*, their ASOVP will converge. See section 3.5 for a basic EVM splitting method, or see [Aguilera98a, c] for a full EVM splitting discussion.

4.5   The Use of EVM in the 3D Case.

The 3D case is a simple extension of the 2D case. Any extremal face that coincides completely with a face of its orthogonal hull will be called <u>fully extremal face (FEF)</u>. In a similar way to fully extremal edges (FEE) of the 2D case, any FEF will be removed by the ASOV process. Since no other face lies on the same supporting plane of such extremal face, the corresponding face of the orthogonal hull for the next shape will be displaced to the next plane of vertices towards the center of the object. So the orthogonal hull becomes smaller at each step.

Fig. 9 shows an object P processed by the proposed orthogonal method. In this case, all deficiencies D_0 to D_3 have at least one FEF, therefore, $H_1 \supset H_2 \supset H_3 \supset H_4$, and thus the ASOV converges (fully extremal faces are shown only in the first row, as shaded polygons).

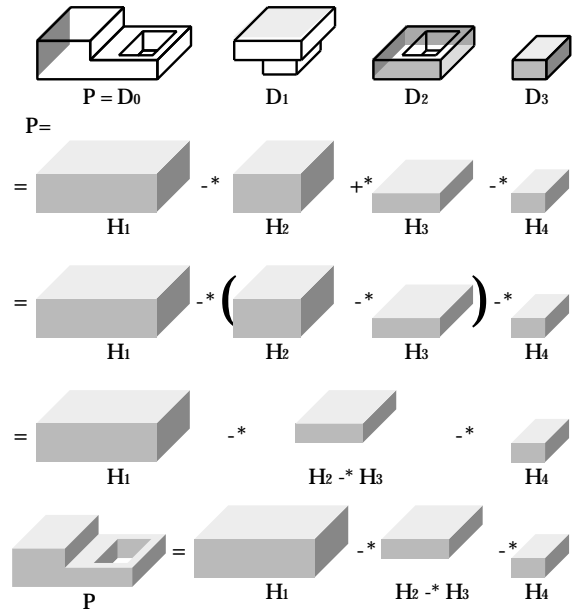**Figure 9:** ASOV decomposition of an OPP, and its algebraic manipulation.

Fig. 9 also presents the algebraic manipulation of Eqn. (4) as $P = H_1 -{}^*(H_2 -{}^* H_3) -{}^* H_4$. Thus, if H_1 represents a raw block of material, then $(H_2 -{}^* H_3)$ and H_4 can be thought as the respective blocks to be removed to create the notch step and the hole.

Lemma 1 and Theorem 6 also apply for the 3D case, we only have to substitute FEE by FEF. That is, $H_k \quad H_{k+1}$ if and only if D_{k-1} has at least one FEF. Therefore, $H_k = H_{k+1}$ and thus the ASOV series is non terminating, if and only if D_{k-1} has no FEF. In this case D_{k-1} is ASOV-irreducible, and the same ASOVP process can be applied for splitting D_{k-1} into subsets that are themselves convergent, and continuing in the same way. The splitting vertex, SV, is also chosen as the first Extreme Vertex of D_{k-1} that does not coincide with a corner of its orthogonal hull H_k. D_{k-1} should be split by all those orthogonal planes passing through SV. Any supporting plane for D_{k-1} passing through SV can be ignored as splitting plane, since it will not split D_{k-1}.

Note that SV belongs to an extremal face of D_{k-1}, and will coincide with a corner of the orthogonal hull of at least one of the split parts. This approach leads towards making this extremal face be a fully extremal face, and thus enabling convergence.

Theorem 7: *At least one of the three orthogonal planes passing through the splitting vertex SV is a supporting plane for D_{k-1}.*

Proof: If D_{k-1} has no FEF, then SV is defined as the first Extreme Vertex of D_{k-1} that does not coincide with a corner of its orthogonal hull H_k. Now, SV must belong to $plv_1(D_{k-1})$, the first plane of vertices of D_{k-1}, because of the following reasoning: four of the eight corners of H_k lie on the supporting plane of $plv_1(D_{k-1})$, and if SV did not belong to $plv_1(D_{k-1})$ then it would mean that all the Extreme Vertices of $plv_1(D_{k-1})$ coincide with corners of H_k, that is, $plv_1(D_{k-1})$ would have four Extreme Vertices coinciding with corners of H_k, and thus $plv_1(D_{k-1})$ itself would be a FEF for D_{k-1}, which is a contradiction of the first sentence of this proof, and D_{k-1} would require no splitting. So, SV belongs to $plv_1(D_{k-1})$ whose supporting plane also supports D_{k-1}. ∎
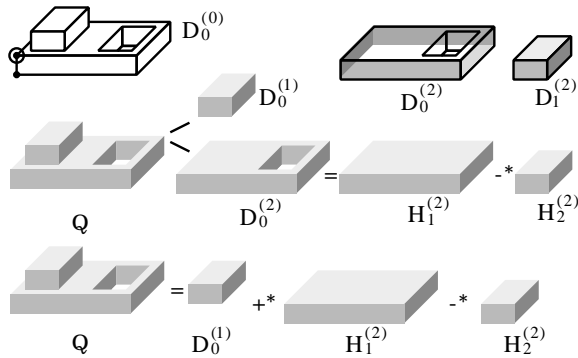

Figure 10: ASOVP decomposition of an OPP.

This theorem states that D_{k-1} may need to be split by at most two planes because SV belongs to $plv_1(D_{k-1})$.

Moreover, experience shows that most of the times SV also belongs to the first line of vertices of $plv_1(D_{k-1})$. In this case SV belongs to two supporting planes of D_{k-1}. Thus, only one splitting plane, perpendicular to this line of vertices, is usually needed.

Fig. 10 shows an ASOV-irreducible object Q processed by the proposed orthogonal method. In this case, $Q = D_0^{(0)}$ has no FEF, thus $D_0^{(0)}$ is split into $D_0^{(1)}$ and $D_0^{(2)}$ by just one plane passing through SV, since SV (shown as a circled dot) is in the first line of vertices of the first plane of vertices of $D_0^{(0)}$. Then, the ASOVP-decomposition process continues for $D_0^{(1)}$ and $D_0^{(2)}$ as usual.

5 Experimental Results.

Standard techniques of ASV decomposition (convex hulls and set difference operators) can be, of course, applied to orthogonal polyhedra. However, by using orthogonal hulls instead of convex hulls, the EVM can be used to handle all the necessary OPP operations in a much faster way than standard algorithms do. EVM algorithms for Boolean operations can be found in [Aguilera96, 98c]. Moreover, by using the result of Theorem 1, i.e., by replacing both the set-difference and disjoint union by the faster XOR operation of the EVM, computation time is further reduced.
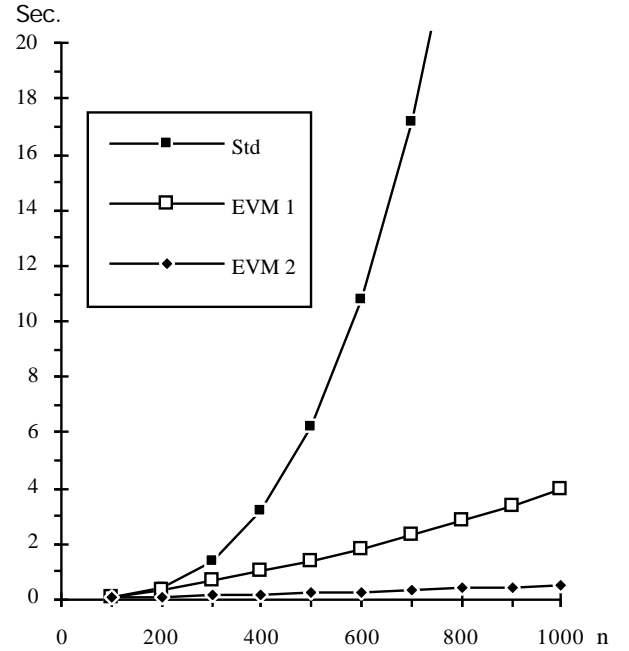

Figure 11: Comparison of methods.

Fig 11. shows a comparison among the above three methods, where n is the number of Extreme Vertices of the polyhedron being ASV-decomposed. Runing times are expressed in seconds. *Std* means the standard method, i.e., the use of convex hulls and set difference operators, in fact, the data structure and algorithms found in [Tang91a, b] were used. *EVM 1* means the use of the Extreme Vertices Model, without using the result of Theorem 1, i.e., by using orthogonal hulls instead of convex hulls, but still using the set-difference and set-union operators. Finally EVM 2 means the use of all results reported in this paper.

Numerical data fitting to all the above experimental results were applied, and showed that the respective complexities are $O(n^3)$, $O(n^{1.45})$ and $O(n)$, approx.

6 Conclusions and Future Work.

In this paper we showed that all the set operations obtained from an ASVP decomposition of a general polyhedron, can be replaced by exclusive-or operations and applied in any order. Then we used this result for presenting the application of the ASV to the particular case of orthogonal polyhedra. It exploits the fact that OPP are represented in the Extreme Vertices Model which have been proved to be a suitable model for OPP.

To sum up the characteristics of the presented method we enumerate the following: it uses orthogonal hulls instead of convex hulls; it replaces both the set-difference and disjoint union by the faster XOR operation; it uses a simple remedial process for non-convergence that is also based on the potential of the EVM [Aguilera98a, c].

As our future work, we are orienting it into two directions. The first one is to continue with the ASOVP decomposition and to study how to recognize and extract *orthogonal* form features from OPP in a fast way. Orthogonal form features means that the recognized type of form features are always orthogonal. For example, Fig. 10 shows that the form features of *Q* are a step to be glued to a base and a hole to be removed, all of them are orthogonal form features (boxes).

Our second focus of interest is to develop more applications for EVM represented OPP. Some applications have already been developed. The use of OPP as geometric bounds in CSG has been discussed in [Aguilera96, 98b]. Now, we are going to study the applicability of using the EVM in the fields of 3D digital images and volume modeling.

7 Acknowledgments.

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