Resistance of thin disks and rings

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***Abstract*—In this paper the following, classic problem [1] is taken into consideration: calculate the resistance between two, perfectly conducting contacts on the rim of a thin disk ($t \ll a$, where a indicates disk's radius and t is its thickness) of conductivity σ. An example of a simple, approximate solution to above problem can be found in [2], however, it requires contacts to be sufficiently small (i.e. $d \gg \delta$, where δ is the contacts' size and d indicates the distance between them). Our purpose was to find more general solution, valid for contacts of finite size. Moreover, we propose the extension of above problem to thin rings.**

The exact analytical method of solving the Laplace equation to find the scalar potential was used. Afterwards, the Ohm law to obtain the formulas for disks and rings resistance was utilized. Using the dilogarithm function we obtained simplified, approximate formulas for the disks and rings resistance.

I. Introduction

Analytical solution to the problem of calculation of thin discs and rings resistance for different electrodes configurations, requires in each case solving the Laplace equation with appropriate boundary conditions. Such a solution can be represented generally by some infinite series. However, we will show that considered series can be approximated and written using the closed-form expressions which retain good agreement with the exact result, moreover do not require the use of numerical procedures during calculations.

In this paper we calculate the resistance for the following setups: the thin disks with equal (Figure 1) and different (Figure 2) electrodes and the thin rings with equal (Figure 3) and different (Figure 4) electrodes.

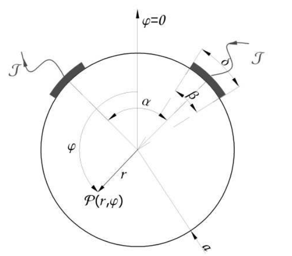

Fig. 1. A thin disk with two electrodes of equal sizes.

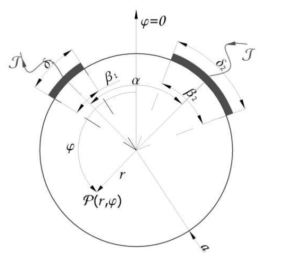

Fig. 2. A thin disk with two electrodes of different sizes.

II. Analytical solution of the Laplace equation for thin discs and rings

The method of separation of variables [3] was employed to obtain the general solution for scalar potential:

$$V(r,\varphi) = \sum_{n=1}^{\infty} A_n r^n \cos n\varphi \tag{1}$$

Using proper boundary conditions for current density and applying the Ohm law to obtain resistivity, the final formulas for resistance can be written as follows:

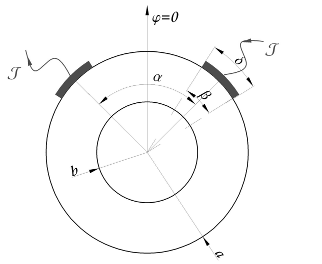

Fig. 3. A thin ring with two electrodes of equal sizes.

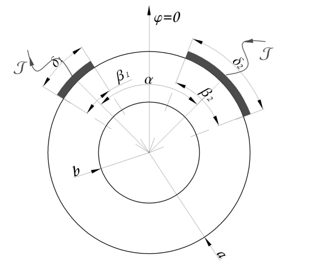

Fig. 4. A thin ring with two electrodes of different sizes.

- for disk with equal electrodes (Figure 1):

$$R = \frac{8}{\pi\sigma\beta t}\sum_{n=1}^{\infty}\frac{\sin\frac{n\beta}{2}}{n^2}\sin^2\frac{n\alpha}{2} \tag{2}$$

- for disk with non-equal electrodes'(Figure 2):

$$R = \frac{4}{\pi t\sigma}\sum_{n=1}^{\infty}\frac{\sin^2\frac{n\alpha}{2}}{n^2}\left(\frac{1}{\beta_1}\sin\frac{n\beta_1}{2}+\frac{1}{\beta_2}\sin\frac{n\beta_2}{2}\right) \tag{3}$$

- for thin ring with equal electrodes (Figure 3):

$$R = \frac{8}{\pi\sigma\beta t}\sum_{n=1}^{\infty}\frac{1+\left(\frac{b}{a}\right)^{2n}}{1-\left(\frac{b}{a}\right)^{2n}}\cdot\frac{\sin\frac{n\beta}{2}}{n^2}\sin^2\frac{n\alpha}{2} \tag{4}$$

- for thin ring with non-equal electrodes (Figure 4):

$$R = \frac{4}{\pi t\sigma}\sum_{n=1}^{\infty}\frac{1+\left(\frac{b}{a}\right)^{2n}}{1-\left(\frac{b}{a}\right)^{2n}}\cdot\frac{\sin^2\frac{n\alpha}{2}}{n^2}\left(\frac{1}{\beta_1}\sin\frac{n\beta_1}{2}+\right.$$
$$\left.+\ \frac{1}{\beta_2}\sin\frac{n\beta_2}{2}\right) \tag{5}$$

III. Approximation of the formulas for disks and rings resistances

In this section we will show only the final results of resistance formulas approximations. The detailed proofs for each obtained expression will be shown in a full paper.

Previously obtained formulas for disks and rings resistance can be approximated using dilogarithm function, which can be defined as follows [4]

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2} \tag{6}$$

Moreover [4]

$$\mathrm{Li}_2(e^{-z}) = \frac{\pi^2}{6} + (z\ln z - z) - \frac{z^2}{4} + \frac{\mathrm{B}_1 z^3}{2\cdot 3\cdot 2!} - \frac{\mathrm{B}_2 z^5}{4\cdot 5\cdot 4!} + \ldots \tag{7}$$

where B_i indicates the i-th Bernoulli number. Using identities (6) and (7), applying Taylor and binomial series expansion and taking into account only terms that contain α we get the approximation of the formulas for disks' resistance:

- for disk with equal electrodes (Figure 1):

$$R \approx \frac{2}{\sigma t \pi}\left(\ln\frac{4\sin\frac{\alpha}{2}}{\beta}+1\right) \tag{8}$$

- for disk with non-equal electrodes(Figure 2):

$$R \approx \frac{2}{\sigma \pi t}\left(\ln\frac{4\sin\frac{\alpha}{2}}{\sqrt{\beta_1\beta_2}}+1\right) \tag{9}$$

To approximate the formulas for rings resistance one needs to consider following identity [5]:

$$\mathrm{Im}\left[\frac{1}{e^{ix}-c}\right]=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\mathrm{E}_{2n+1}(c)x^{2n+1}}{(2n+1)!(c-1)^{2n+2}}, \tag{10}$$

where $\mathrm{E}_n(c)$ indicates Eulerian polynomials. Integrating twice both sides of equation (10) we get the following equation

$$\begin{aligned}\mathrm{Im}\left[\mathrm{Li}_2\left(\frac{e^{ix}}{c}\right)\right] &= c\cdot\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\mathrm{E}_{2n+1}(c)x^{2n+3}}{(2n+3)!(c-1)^{2n+2}}+\\ &- x\ln\left(1-\frac{e^{ix}}{c}\right)+x\ln(e^{ix}-c)\end{aligned} \tag{11}$$

Moreover it can be also easily proved that [5]

$$\mathrm{Im}\left[\mathrm{Li}_2\left(\frac{e^{ix}}{c}\right)\right]=\sum_{n=1}^{\infty}\frac{\sin kx}{c^k k^2} \tag{12}$$

Using above identities we get the final formulas written in terms of sum of Eulerian polynomials. The binomial series expansion was used to rearrange the terms of obtained formula. The terms that contain only α were taken to get the final approximation:

- for thin ring with equal electrodes (Figure 3):

$$R \approx \frac{2}{\sigma t \pi}\left(\ln\frac{4\sin\frac{\alpha}{2}}{\beta}+1-\frac{2\cos\alpha-2}{c^2-1}\right) \tag{13}$$

- for thin ring with non-equal electrodes (Figure 4):

$$R \approx \frac{2}{\sigma t \pi}\left(\ln\frac{4\sin\frac{\alpha}{2}}{\sqrt{\beta_1\beta_2}}+1-\frac{2\cos\alpha-2}{c^2-1}\right) \tag{14}$$

where $c=\frac{a}{b}$ and a and b indicate the external and internal radius of the ring, respectively.

Figures 5 and 6 show the comparison between exact - (2) and (3) and approximate - (8) and (9) formulas for disks resistance for different sizes of electrodes and different distances between them. Analogically Figures 7 and 8 show this comparison in case of rings.

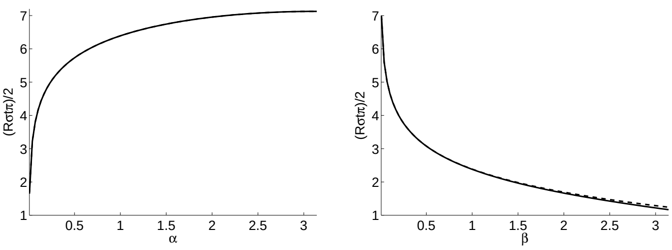

Fig. 5. Comparison between exact (solid line) and approximate (dashed line) formulas for the resistance of thin disk with equal electrodes. Left: for $\beta=\frac{\pi}{360}$, $\beta<\alpha<\pi$. Right: for $\alpha=\pi$ and $0<\beta<\alpha$

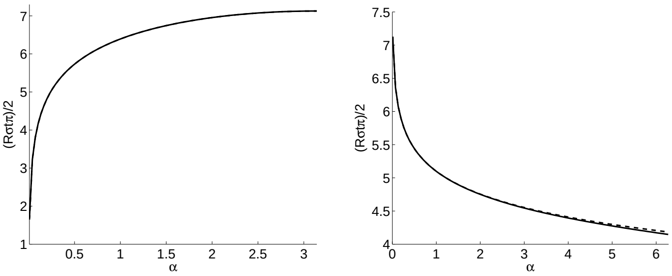

Fig. 6. Comparison between exact (solid line) and approximate (dashed line) formulas for the resistance of thin disk with different electrodes. Left: for $\beta=\frac{\pi}{360}$, $\beta<\alpha<\pi$. Right: for $\alpha=\pi$, $\beta_2=\pi/360$ and $0<\beta_1<2\pi-\beta_2$

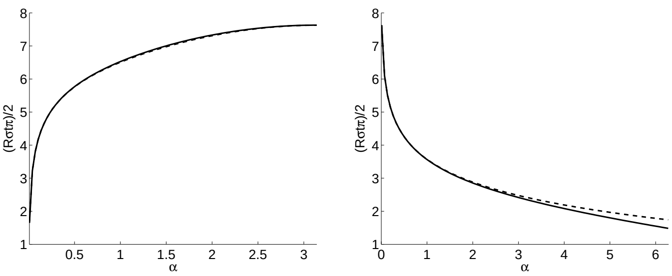

Fig. 7. Comparison between exact (solid line) and approximate (dashed line) formulas for the resistance of thin ring with equal electrodes. Left: for $c=3$, $\beta=\frac{\pi}{360}$, $\beta<\alpha<\pi$. Right: for $c=3$, $\alpha=\pi$ and $0<\beta<\alpha$

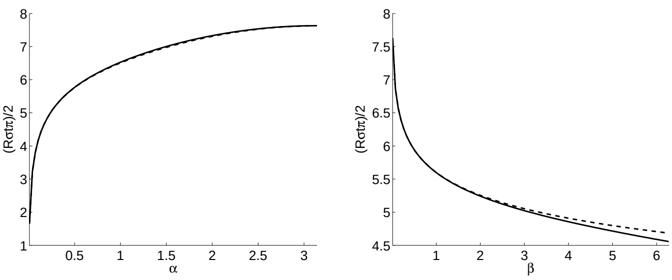

Fig. 8. Comparison between exact (solid line) and approximate (dashed line) formulas for the resistance of thin ring with different electrodes. Left: for $c=3$, $\beta=\frac{\pi}{360}$, $\beta<\alpha<\pi$. Right: for $c=3$, $\alpha=\pi$, $\beta_2=\pi/360$ and $0<\beta<2\pi-\beta_2$

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