Analysis and Synthesis of Realizable Non-equilibrium Dissipative Structures

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Abstract –**Computation complexity of a broad variety of practical design problems is known to be strongly depending on an algebraic complexity of corresponding mathematical system representations. Especially some vector-matrix models are frequently used in numerous interdisciplinary fields. One way to overcome the complexity problems is based on some special algebraic structures of low order model approximations, such as e.g. balanced representations. Another approach based on the concept of sparse matrices has also become very popular. As a very successful special case of sparse matrix based approach a class of tridiagonal system representations [1] has found applications in solution of partial differential equations, digital signal processing, image processing, computational fluid dynamics, spline curve fitting and many others. In this contribution a generalized sparse matrix motivated multi-diagonal method is proposed and some new results, based on state space energy motivated causal system representations are presented, too [2].**

Keywords – state space; energy; interaction; neutral; dissipation; ladder structure; tri-diagonal; multi-diagonal

I. INTRODUCTION

This paper will present a new continuous-time state-space system structure which has some interesting properties that have proven to be useful in design of both adaptive and fixed filters, in analysis a synthesis of nonlinear generators of chaotic signals, etc. Causal systems with such a structure can be called ladder systems. From physical point of view, it is worthwhile to notice that inherent property to the proposed structure is the fact that the corresponding state variables can be considered as scaled versions of physical energy state variables, e.g. capacitor voltages and inductor currents. From linear system theory point of view, another useful property of such structures is the ability to realize any structurally asymptotic stable transfer-function matrix. The most important feature of the proposed structure is its *structural compatibility* with a form of *abstract energy conservation principle.* For illustration of a practically useful multi-diagonal structure an example of tri-diagonal matrix:

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$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \quad (1)$$

with diagonal elements representing the total *energy dissipation* while the off-diagonal elements represent the *internal system interactions* of a n^{th} *order state space representation* (h - step of space discretization).

$$\Re(S):\ \dot{x} = Ax + Bu,\ \ y = Cx,\ \ x \in R^n \quad (2)$$

of the infinite-dimensional *real system* with *physical structure* shown in the Fig.1.

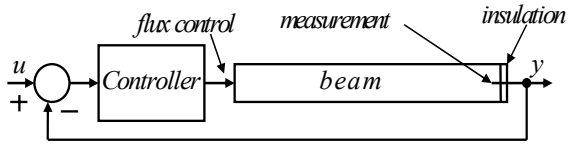

Fig. 1. Example of heated beam with control.

From the strictly theoretical point of view, the adequate model of the real system under consideration takes the form of *parabolic partial differential equation*

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + u \quad (3)$$

The solution of this equation represents the temperature $\theta(x,t)$ in a one-dimensional normalized heat conduction problem, with heat input term $u(x,t)$.

The state space representation given by the eqs.(1), and (2) can be derived using the *space discretization* as a *finite-dimensional approximation* of inherently *infinite-dimensional structure.* This standard technique is known as *finite-elements method.*

It is not very difficult to demonstrate that even in case of low order physical systems the standard ways to derive a proper system representations (using eg. Hamilton`s postulate, energy conservation principle, or Kirchhoff`s laws, etc.), *need not* to lead to symmetric and/or tri-diagonal structures, like eqn. (1). Two relatively simple physical examples will be briefly discussed in order to illustrate this point.

The first one is the *chain structure* of an electrical network consisting of the chain of finite number of controlled interacting (real) *RLC harmonic oscillators* shown in the Fig.2.

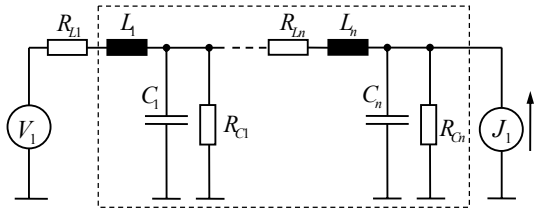

Fig. 2. Ladder electrical network example.

A typical internal structure of the n^{th} order electrical network as shown in the Fig. 2, resulting directly from the given physical structure, can be illustrated by algebraic structure which is *tri-diagonal but not symmetric,* as given by the matrix A (for all R,L,C parameter values normalized to 1):

$$A=\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \quad (4)$$

The second example can be characterized as the *chain structure* of a mechanical system consisting of a finite number of controlled interacting *MKD harmonic oscillators* shown in the Fig.3, with *physical parameters: M* – mass, *K*- stiffness, *D* -damping.

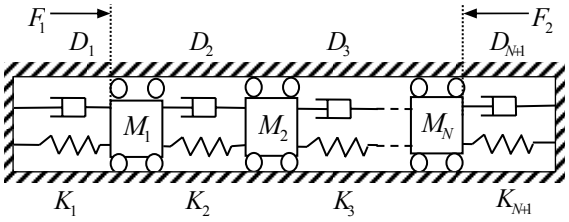

Fig. 3. Mechanical system example

A typical example of the *internal algebraic structure* of the 4th order mechanical system as shown in the Fig. 3, resulting directly from application of Hamilton`s postulate for the *given physical structure* can be illustrated by corresponding *algebraic structure* defined by the matrix A as given by eq.(5). Recall that the class of tri-diagonal structures, such as A in eq. (1) and (4), represents a *sub-class* of the well known *Hessenberg structures.* It is worthwhile to notice that the *structure* (5) is *neither tri-diagonal nor symmetric.*

$$A=\begin{bmatrix} -\frac{D_1+D_2}{M_1} & \frac{D_2}{M_1} & -\frac{K_1+K_2}{M_1} & \frac{K_2}{M_1} \\ \frac{D_2}{M_2} & -\frac{D_2+D_3}{M_2} & \frac{K_2}{M_2} & -\frac{K_2+K_3}{M_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (5)$$

One of the most important features of the Hessenberg matrix (or its transpose) is that all elements below the first sub-diagonal have to be zero. Thus if a Hessenberg matrix is a *composition of symmetric and skew-symmetric* sub-matrices like eqn.(4), it follows that the *structure is tri-diagonal.* Such matrices have the *same eigenvalues* as the original, but *less computation* is needed to reveal them. It is not difficult to prove that there exist *multi-diagonal structures,* (like the 5-diagonal matrix A in eqn. (5) for zero value of the *interaction parameter* $K_2=0$). Such structures are closely related to *composition of symmetric and skew-symmetric parts* leading to the class of *port-Hamiltonian system representations* which can be *optimal* with respect to *energy* motivated *measures of approximation errors.*

The following n^{th} order *state energy equivalent tri-diagonal structure* to the *non-symmetric structure* eq.(5) (for n=4), where ω_2 and ω_4 are the *frequency parameters* of two 2nd order *subsystems in interaction* where σ_3 is the *interaction parameter* and *parameters* Δ_{11} and Δ_{33} represent the *total system dissipation.*

$$\bar{A}=\begin{bmatrix} -\Delta_{11} & \omega_2 & 0 & 0 \\ -\omega_2 & 0 & \sigma_3 & 0 \\ 0 & -\sigma_3 & -\Delta_{33} & \omega_4 \\ 0 & 0 & -\omega_4 & 0 \end{bmatrix},\ \bar{B}=\begin{bmatrix} \sqrt{\Delta_{11}} & 0 \\ 0 & 0 \\ 0 & \sqrt{\Delta_{33}} \\ 0 & 0 \end{bmatrix},\ \bar{C}=\bar{B}^T \quad (6)$$

It is easy to prove that the *existence of the equivalent representation* (6) reduces to *solvability* of equations:

$$\begin{aligned} \Delta_{11}+\Delta_{33} &= \frac{M_1(D_2+D_3)+M_2(D_1+D_2)}{M_1M_2}; \\ \omega_2^2\omega_4^2 &= \frac{K_1K_2+K_1K_3+K_2K_3}{M_1M_2} \\ \Delta_{11}\left(\sigma_3^2+\omega_4^2\right)+\Delta_{33}\omega_2^2 &= \frac{D_1\left(K_2+K_3\right)+D_2\left(K_1+K_3\right)+D_3\left(K_1+K_2\right)}{M_1M_2} \\ \omega_2^2+\sigma_3^2+\omega_4^2+\Delta_{11}\Delta_{33} &= \frac{M_1(K_2+K_3)+M_2(K_1+K_2)}{M_1M_2} \\ &\quad +\frac{D_1D_3+D_2\left(D_1+D_3\right)}{M_1M_2} \end{aligned} \quad (7)$$

II. CONCLUSIONS

The main objective of the contribution was to show how *multi-diagonal* as well as a broad class of generally *non-symmetric system representations* may be transformed into the *tri-diagonal form.* The n^{th} order generalization is straightforward.

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