A METHOD FOR OBTAINING THE TESSERACT BY UNRAVELING THE 4D HYPERCUBE

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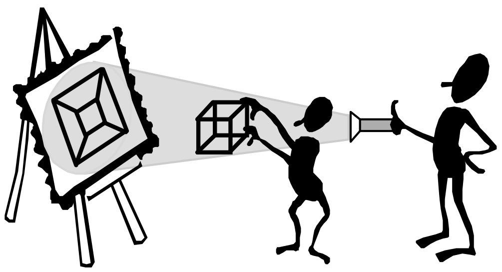
ABSTRACT

This article presents a method for unraveling the hypercube and obtaining the 3D-cross (tesseract) that corresponds to the hyper-flattening of its boundary. The hypercube can be raveled back using the method in an inverse way. Also a method for visualizing the processes is presented. The transformations to apply include rotations around a plane (characteristic of the 4D space). All these processes can be viewed using a computer animation system.

Keywords: 4D-Modeling, 4D-Animation, Computational Geometry.

1. INTRODUCTION

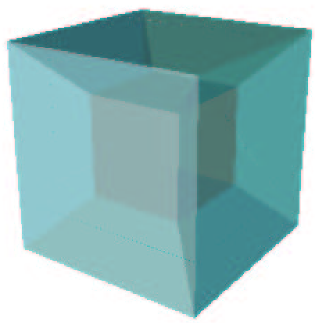
[Coxet84], [Rucke84], [Kaku94], [Robbi92] and [Banch96] start their introductions to the 4D space study presenting three methods for visualizing the hypercube: through their shadows (projections), their cross sections with 3D space and their unravelings.



Projecting a cube on a plane (central projection).
Figure 1

If it is possible to make drawings of 3D solids when they are projected onto a plane, then it is possible to make drawings or 3D models of 4D polytopes when they are projected onto a hyperplane [Coxet84]. The shadows method is based in this principle.

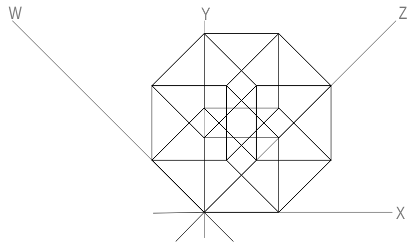
Let us follow the analogy presented in "Flatland" [Abbot84]. If a 3D being wants to show a cube to a 2D being (a flatlander) then the first one must project the cube's shadow onto the plane where the flatlander lives. For this case, the projected shape could be, for example, a square inside another square (Fig. 1).



Hypercube's central projection onto the 3D space.
Figure 2

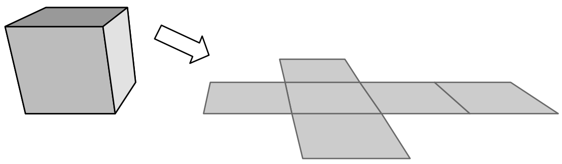
If a 4D being wants to show us a hypercube, he must project the shadow onto the 3D space where we live. The projected body could be a cube inside another cube [Kaku94] called central projection (Fig. 2). We know that a projected cube onto a plane is just an approximation of the real one.

Analogously, the hypercube projected onto our 3D space is also a mimic of the real one. Another useful projection is due to Claude Bragdon (see [Rucke77] for details about this projection). See Fig. 3.



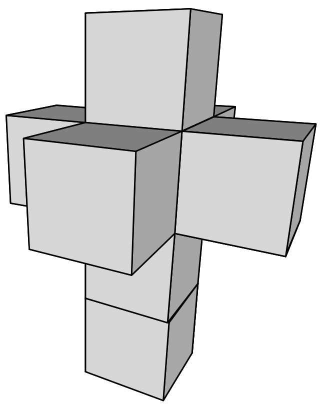
Claude Bragdon's hypercube projection.
Figure 3

A cube can be unraveled as a 2D cross. The six faces on the cube's boundary will compose the 2D cross (Fig. 4). The set of unraveled faces is called the unravelings of the cube.



Unraveling the cube.
Figure 4

In analogous way, a hypercube also can be unraveled as a 3D cross. The 3D cross is composed by the eight cubes that forms the hypercube's boundary [Kaku94]. This 3D cross was named ***tesseract*** by C. H. Hinton (Fig. 5).

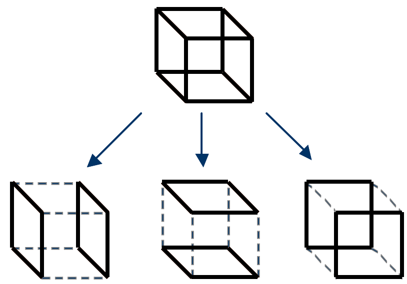


The unraveled hypercube (the tesseract).
Figure 5

A flatlander will visualize the 2D cross, but he will not be able to assembly it back as a cube (even if the specific instructions are provided). This fact is true because of the needed face-rotations in the third dimension around an axis which are physically impossible in the 2D space. However, it is possible for the flatlander to visualize the raveling process through the projection of the faces and their movements onto the 2D space where he lives.

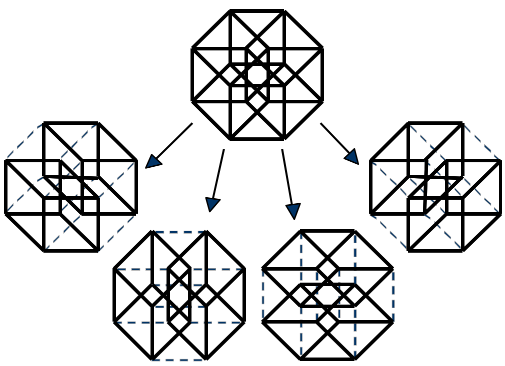
Analogously, we can visualize the tesseract but we won't be able to assembly it back as a hypercube. We know this because of the needed volume-rotations in the fourth dimension around a plane which are physically impossible in our 3D space.

Before going any further, we would like to underline that the cube’s boundary faces can be grouped into three pairs of parallel faces, where their supporting planes define two 2D-spaces parallel to each other. Each pair can be obtained by ignoring all those edges parallel to each main axis (X, Y and Z), see Fig. 6



Viewing the cube’s boundary faces.
Figure 6

It is interesting to analyze the hypercube using its analogy with the cube and the visualization methods above described. [Hilbe52] has determined that a hypercube is composed out of sixteen vertices, thirty-two edges, twenty-four faces and eight bounding cubes (also called cells or volumes). Similarly, and as shown in Fig. 7, all these volumes can be grouped into four pairs of parallel cubes, furthermore, their supporting hyper-planes define two 3D-spaces parallel to each other.

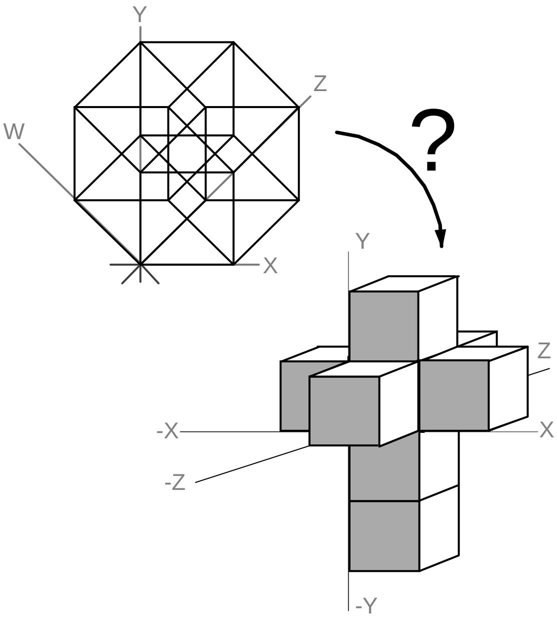


Viewing the hypercube’s boundary volumes.
Figure 7

[Coxet63] points out that each face is shared by two cubes not in the same three-dimensional space, because they form a right angle through a rotation around the shared face's supporting plane. These properties are visible through Bragdon's projection (Fig. 3). The Bragdon's projection as well as the central projection will be used throughout the remainder of this article.

2. PROBLEM

[Kaku94] and [Banch96] describe with detail a representation model for the hypercube through their unravelings. They also mention the physical incapacity of a 3D being to ravel the hypercube back, because the required transformations are not possible in our 3D space (Fig. 8).



The hypercube's unraveling process.
Figure 8

[Kaku94] and [Banch96] also describe that if we witness the raveling process, seven of eight cubes that compose the tesseract will suddenly disappear, because they have moved in the direction of the fourth dimension. However, they don't provide a methodology that indicates the transformations and their parameters to execute the raveling process. In spite of our physical incapacity, we can visualize a projection onto our 3D space of the cubes on the hypercube's boundary through the unraveling and raveling processes.

This article presents a method for unraveling the hypercube and getting the 3D-cross (tesseract) that corresponds to the hyper-flattening of its boundary (Fig. 8). The hypercube can be raveled back using the same method in an inverse way. The transformations to apply include rotations around a plane (characteristic of the 4D space). All these processes can be viewed using a computer animation system.

3. HYPERCUBES'S UNRAVELING METHODOLOGY

The process will be easier if we take the following considerations:

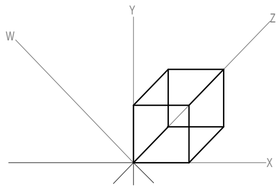
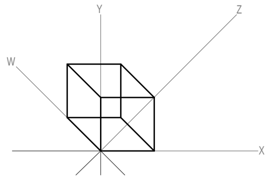
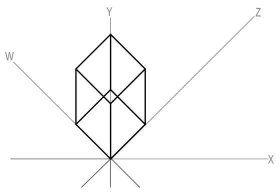
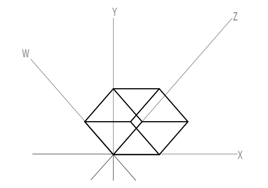
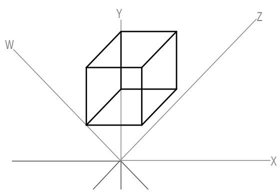
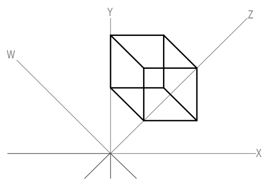
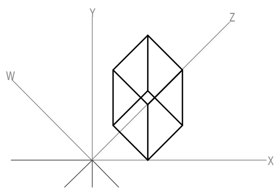
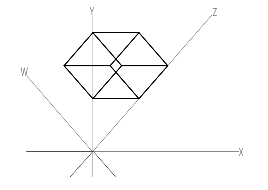
- Select the hypercube's position in the 4D space.
- Select the hyperplane (a 3D subspace embedded in the hyperspace) where the volumes will be rotated to.
- Establish the angles which guarantee that all volumes will be totally embedded in the selected hyperplane.
- All the volumes through their movement into the selected hyperplane must maintain a face adjacent to another volume.

The hypercube's position in the 4D space is essential, because it will define the rotating planes used by the volumes to be positioned onto a hyperplane. For simplicity, one vertex of the hypercube will coincide with the origin, six of its faces will coincide each one with some of XY, YZ, ZX, XW, YW and ZW planes and all the coordinates will be positive (see [Banch96] for the methodology to get the hypercube's coordinates). The coordinates to use are presented in Table 1 (each vertex is arbitrary numbered).

Vertex	***X***	***Y***	***Z***	***W***
0	0	0	0	0
1	1	0	0	0
2	0	1	0	0
3	1	1	0	0
4	0	0	1	0
5	1	0	1	0
6	0	1	1	0
7	1	1	1	0
8	0	0	0	1
9	1	0	0	1
10	0	1	0	1
11	1	1	0	1
12	0	0	1	1
13	1	0	1	1
14	0	1	1	1
15	1	1	1	1

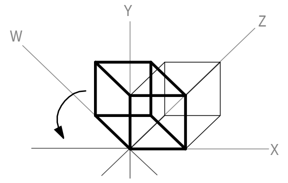
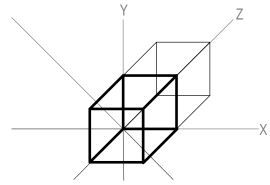
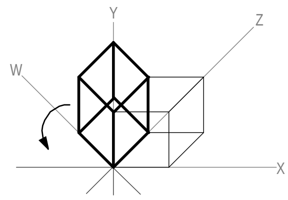
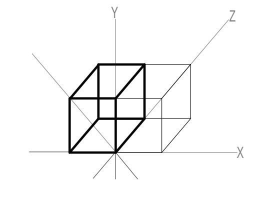
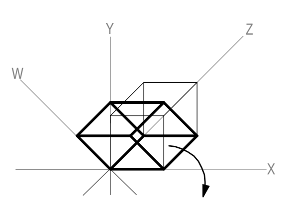
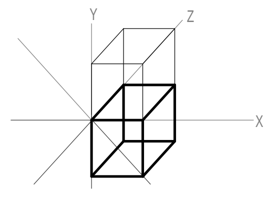
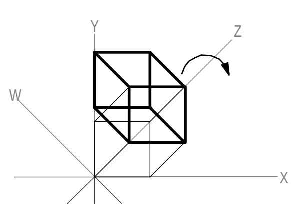
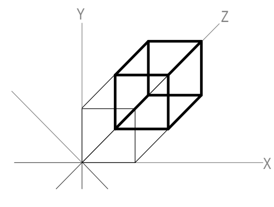
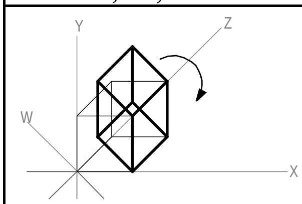
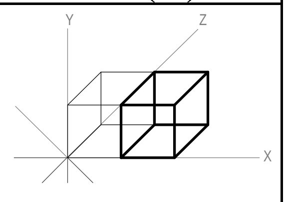
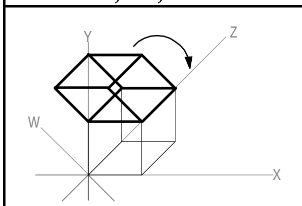
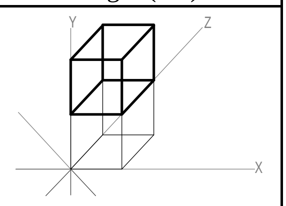
Hypercube's coordinates.
Table 1

We know now why the hypercube's position in the 4D space is important, since it will define the rotating planes to use. The situation is the same for the selected hyperplane, because it is where all the volumes will be finally positioned. Observing the hypercube's coordinates we can see that eight of them present their fourth coordinate value (W) equal to zero. This fact represents that one of the hypercube's volumes (formed by vertexes 0-1-2-3-4-5-6-7) has W=0 as its supporting hyperplane. Selecting the hyperplane W=0 is useful because one of the volumes is "naturally embedded" in the 3D space and it won't require any transformations.

 Volume A (0-1-2-3-4-5-6-7)	 Volume B (0-1-2-3-8-9-10-11)
 Volume C (0-2-4-6-8-10-12-14)	 Volume D (0-1-4-5-8-9-12-13)
 Volume E (8-9-10-11-12-13-14-15)	 Volume F (4-5-6-7-12-13-14-15)
 Volume G (1-3-5-7-9-11-13-15)	 Volume H (2-3-6-7-10-11-14-15)

The hypercube's volumes (the numbers indicate the vertices that compose the volume).
Table 2

Now, it is also useful to identify the hypercube's volumes through their vertices and to label them for future references. Until now we have one identified volume, it is formed by vertexes 0-1-2-3-4-5-6-7, and it will be called volume A. See Table 2.

Adjacent volume (previous to rotation), rotation plane and angle	***Position in the 3D space and in the tesseract after rotation***
 *B, XY, +90°*	 Front (-Z)
 *C, YZ, -90°*	 Left (-X)
 *D, ZX, +90°*	 Down (-Y)
 *F, XY, -90°*	 Back (+Z)
 *G, YZ, -90°*	 Right (+X)
 *H, ZX, -90°*	 Up (+Y)

Applied transformations to the adjacent volumes.
Table 3

We have already described volume A as "naturally embedded" in the 3D space, because it won't require any transformations. Volume A will occupy the central position in the 3D cross and it will called the "central volume".

From the remaining volumes, six of them will have face adjacency with the central volume. Due to this characteristic they can easily be rotated toward our space because their rotating plane is clearly identified. Each of these volumes will rotate around the supporting plane of its shared face with central volume. They will be called "adjacent volumes". Adjacent volumes are B, C, D, F, G and H. The remaining volume E will be called "satellite volume" and it will be discussed later on.

All of the adjacent volumes will rotate right angles. In this way we guarantee that their W coordinate will be equal to zero. It is also important to consider their rotating directions, because the volumes, after the rotations, could otherwise coincide with the central volume. The direction and rotating planes for each adjacent volume are presented in Table 3 (the central volume is also included in each image as a reference for the initial and final position of the volume being analyzed).

At this point, we have seven of the eight hypercube's volumes placed in their final positions (volumes A, B, C, D, F, G and H). Volume E will perform a rather more complex set of transformations. There are two reasons that justify this conclusion:

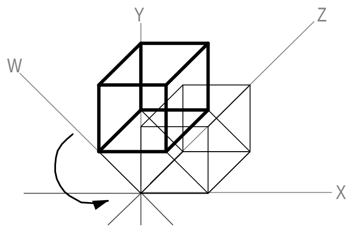
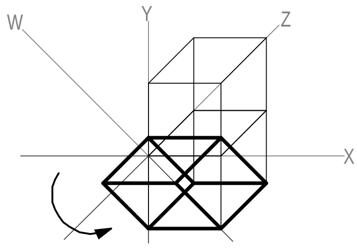
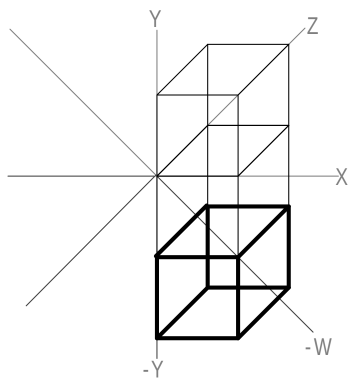
- The supporting hyperplane for volume E is parallel to the supporting hyperplane for the central volume. Consequently, there are no adjacencies between volume E and central volume (this is the reason for not calling "adjacent volume" to volume E).
- In the tesseract, we still have an empty position. This position corresponds to the most distant volume from the central volume (the inferior position, Fig. 5). This position will be occupied by volume E. This is the reason for calling E the "satellite volume".

At the beginning of this document is mentioned the need for maintaining a face adjacency between all the volumes while they rotate towards the selected hyperplane. Volumes B, C, D, F, G and H share a face with central volume (remember that central volume is static during the whole unraveling process). In order to determine the needed transformations for the satellite volume, we must first select the volume which will share a face with it. Any volume, except the central one, can be selected for this. In this work, volume D will be selected to share a face with satellite volume through the hyper-flattening process.

The direction and the rotation plane for volume D was determined before (ZX plane +90°). These transformations will take it to its final position. During the beginning of the unraveling process, the same transformations will be applied to satellite volume. In this way, we ensure that volumes E and satellite will share a face.

When volume D has finished its movement, it will be placed in its final position in the tesseract. At this moment, the satellite volume's supporting hyperplane will be perpendicular to the selected hyperplane and the shared face will be parallel to ZX plane. The last movement to apply to the satellite volume will be a +90° rotation around the supporting plane of the shared face with volume D.

The set of movements to be executed for the satellite volume are resumed in the Table 4 (Central volume and volume D are shown too).

Current position	***Transformations***
	Rotation of volumes D and satellite around the plane ZX (+90°).
	Volume D is in its final position. Rotation of satellite volume of +90° around the shared face with volume D (parallel plane to ZX).
	Satellite volume in its final position (inferior position in the 3D cross on -Y axis).

Associated transformations to satellite volume.
Table 4

Now, all the transformations to unravel the hypercube have been determined. To ravel it back, the same process must be applied in an inverse way (the angles' signs must be changed).

4. IMPLEMENTATION

4.1 Rotations in the 4D Space

[Banks94] and [Holla91] have identified that if a rotation in the 2D space is given around a point, and a rotation in our 3D space is given around a line, then a rotation in the 4D space, in analogous way, must be given around a plane.

[Holla91] considers that rotations in the 3D space must be considered as rotations parallel to a 2D plane instead of rotations around an axis. [Holla91] supports this idea considering that given an origin of rotation and a destination point in the 3D space, the set of all rotated points for a given rotation matrix lie in a single plane, which is called the rotation plane. Moreover, the rotation axis in the 3D space is perpendicular to the rotation plane. The concept of rotation plane is consistent with the 2D space because all the rotated points lie in the same and only plane. Finally, with the above ideas, [Holla91] constructs the six basic 4D rotation matrices around the main planes in the 4D space (namely XY, YZ, XZ, XW, YW and ZW planes) based in the fact that only two coordinates change for a given rotation (these changing coordinates correspond to the rotation plane).

Using these ideas, [Duffi94] generalize the concept of rotation in an *n*D space ($n \geq 2$) as the rotation of an axis Xa in direction to an axis Xb. The plane described by axis Xa and Xb is what [Holla91] defined as rotation plane. [Duffi94] presents the following general rotation matrix:

$$R_{ab}(\theta) = \left[r_{ij} \left| \begin{array}{ll} r_{ii} = 1 & i \neq a, i \neq b \\ r_{aa} = \cos\theta & \\ r_{bb} = \cos\theta & \\ r_{ab} = -\sin\theta & \\ r_{ba} = \sin\theta & \\ r_{ij} = 0 & elsewhere \end{array} \right. \right]$$

Matrix $R_{ab}(\theta)$ is an identity matrix except in the intersection of columns a and b and rows a and b. Because in an *n*D space there are $C(n,2)$ main planes, this is precisely the number of main rotations for such space.

From these concepts, we must consider that a rotation can be referenced by using two notations: using the axis that describe the rotation plane or using the axis that describe the (*n-2*)D subspace that is fixed during the rotation. In this work we have referred to rotations in the 4D space using the second notation.

4.2 The 4D-3D-2D Projections

[Banks94] establishes that the same techniques used to project 3D objects onto 2D planes can be applied to project 4D polytopes onto 3D hyperplanes (our 3D space for example). Then we have that a 4D-3D parallel projection (or just removing *W* coordinate from the polytope's points) is:

$$P(x, \quad y, \quad z, \quad w) \mapsto P'(x, \quad y, \quad z)$$

And a 4D-3D perspective projection is defined when the center of projection is on *W* axis at a distance *pw* from the origin. If the projection hyperplane is $W = 0$ then we have a point *P* can be projected as:

$$P(x \quad y \quad z \quad w) \mapsto P'\left(\frac{x \cdot pw}{pw - w}, \quad \frac{y \cdot pw}{pw - w}, \quad \frac{z \cdot pw}{pw - w} \right)$$

Because a 4D-3D projection will produce a volume as the "shadow" of a 4D polytope, [Holla91] considers valid to process this volume with some of the 3D-2D projections (parallel or perspective) to be finally projected onto a computer screen. Then we have four possible 4D-3D-2D projections:

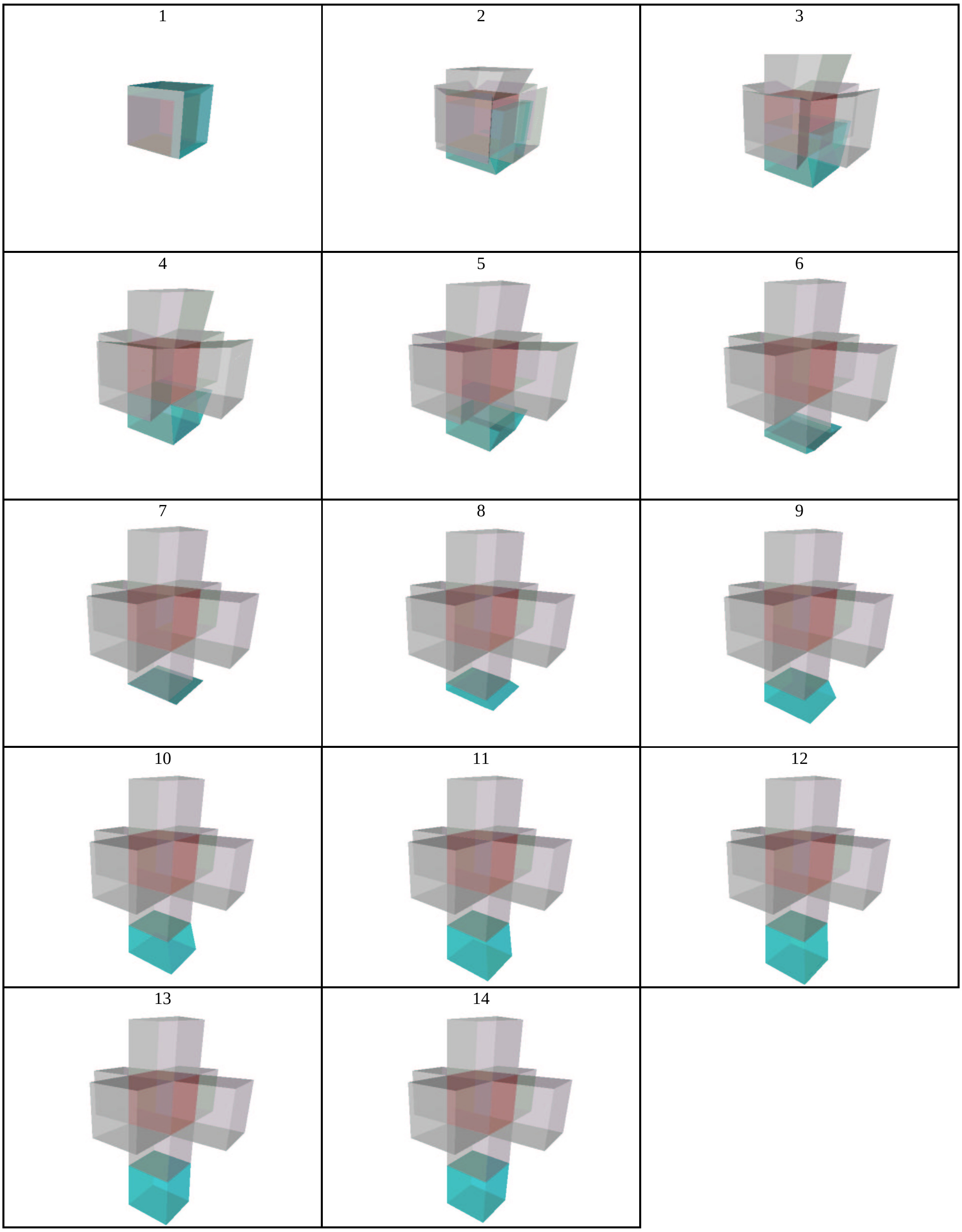
- 4D-3D Perspective - 3D-2D Perspective Projection.
- 4D-3D Perspective - 3D-2D Parallel Projection.
- 4D-3D Parallel - 3D-2D Perspective Projection.
- 4D-3D Parallel - 3D-2D Parallel Projection.

For example, for the hypercube presented in Fig. 1, it was used 4D-3D Perspective Projection and 3D-2D Perspective Projection.

5. RESULTS

Table 5 presents some snapshots from the hypercube's unraveling sequence. In snapshots 1 to 6, the applied rotations are ±0°, ±15°, ±30°, ±45°, ±60° and ±75° (the rotation’s sign depends of the adjacent volume). In snapshot 7, the applied rotation is ±82°; the satellite volume looks like a plane -an effect due to the 4D-3D projection. In snapshot 8, the applied rotation is ±90°; the adjacent volumes finish their movements. In snapshots 9 to 14, the satellite volume moves independently and the applied rotations are +15°, +30°, +45°, +60°, +75° and +90°.

At the present time, the results of this research are used with efficiency as didactic material at the Universidad de las Américas - Puebla, México.



Unraveling the hypercube (See text for details).
Table 5

6. FUTURE WORK

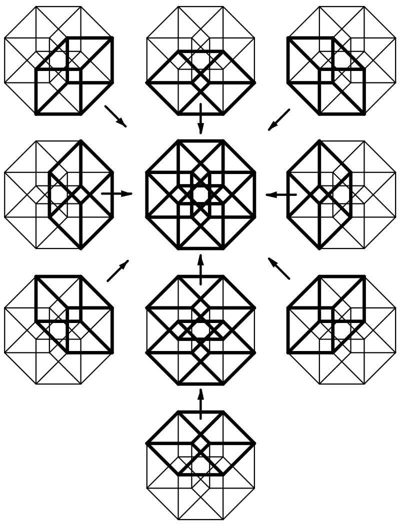
The n-dimensional hyper-tesseract

Observing the unravelings for the square (a 2D cube), the cube and the 4D hypercube we can generalize the ***n-dimensional hyper-tesseract*** ($n \geq 1$) as the result of the (n+1)-dimensional hypercube's unraveling with the following properties:

- The (n+1)-dimensional hypercube will have 2(n+1) n-dimensional cells on its boundary [Banch96].

- A central cell will be static during the unraveling/raveling process.
- 2(n+1)-2 cells are adjacent to central cell. All of them will share a (n-1)-dimensional cell with central cell.
- A satellite cell won't be adjacent to central cell because their supporting hyperplanes are parallel. It will be adjacent to any of the adjacent cells (it will share a (n-1)-dimensional cell with the selected adjacent cell).
- All the adjacent cells and satellite cell during the unraveling/raveling process will rotate ±90° around the supporting hyperplane of the (n-1)-dimensional shared cells.

For example, the 4D hyper-tesseract is the result of the 5D hypercube's unraveling. The 4D hyper-tesseract will be composed by 10 hypervolumes, where one of them will be the central hypervolume (static), eight of them are adjacent to central hypervolume (they share a volume) and the last one will be the satellite hypervolume (it shares a volume with any of the adjacent hypervolumes). See Fig. 9. The adjacent hypervolumes and the satellite hypervolume will rotate around a volume or a hyperplane during the unraveling/raveling process.



The possible adjacency relations between the 4D *hyper-tesseract's* central hypervolume and the adjacent hypervolumes.
Figure 9

In this research we found a method to unravel a hypercube to obtain the tesseract. Also, we have proposed a generalization to describe the properties of the n-dimensional hyper-tesseract, the result of the (n+1)-dimensional hypercube's unraveling. For the 5D space the rotations will be around a volume, for the 6D space they will be around a hypervolume and so forth. This is the direction to follow in our research to get the parameters to unravel the 5D hypercube and to obtain the 4D hyper-tesseract (and to unravel hypercubes in higher dimensional spaces). Also, another direction to follow will be the related to rotations around arbitrary planes in the 4D space (analogously to rotations around an arbitrary axis in the 3D space). Finding the procedures to rotate around arbitrary planes, the hypercube's position may be not relevant.

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