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Bending and free vibration analysis of thick isotropic plates by using exponential shear deformation theory

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Abstract

This paper presents a variationally consistent an exponential shear deformation theory for the bi-directional bending and free vibration analysis of thick plates. The theory presented herein is built upon the classical plate theory. In this displacement-based, refined shear deformation theory, an exponential functions are used in terms of thickness co-ordinate to include the effect of transverse shear deformation and rotary inertia. The number of unknown displacement variables in the proposed theory are same as that in first order shear deformation theory. The transverse shear stress can be obtained directly from the constitutive relations satisfying the shear stress free surface conditions on the top and bottom surfaces of the plate, hence the theory does not require shear correction factor. Governing equations and boundary conditions of the theory are obtained using the dynamic version of principle of virtual work. The simply supported thick isotropic square and rectangular plates are considered for the detailed numerical studies. Results of displacements, stresses and frequencies are compared with those of other refined theories and exact theory to show the efficiency of proposed theory. Results obtained by using proposed theory are found to be agree well with the exact elasticity results. The objective of the paper is to investigate the bending and dynamic response of thick isotropic square and rectangular plates using an exponential shear deformation theory.

Keywords: shear deformation, isotropic plates, shear correction factor, static flexure, transverse shear stresses, free vibration

1. Introduction

The wide spread use of shear flexible materials has stimulated interest in the accurate prediction of structural behavior of thick plates. Thick beams and plates, either isotropic or anisotropic, basically form two and three dimensional problems of elasticity theory. Reduction of these problems to the corresponding one and two dimensional approximate problems for their analysis has always been the main objective of research workers. The shear deformation effects are more pronounced in the thick plates when subjected to transverse loads than in the thin plates under similar loading. These effects are neglected in classical plate theory. In order to describe the correct bending behavior of thick plates including shear deformation effects and the associated cross sectional warping, shear deformation theories are required. This can be accomplished by selection of proper kinematics and constitutive models. These theories can be classified into two major classes on the basis of assumed fields: Stress based theories and displacement based theories. In stress based theories, the stresses are treated as primary variables. In displacement based theories, displacements are treated as primary variables.

Kirchhoff [5, 6] developed the well-known classical plate theory (CPT). It is based on the Kirchhoff hypothesis that straight lines normal to the undeformed midplane remain straight and

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normal to the deformed midplane. In accordance with the kinematic assumptions made in the CPT all the transverse shear and transverse normal strains are zero. The CPT is widely used for static bending, vibrations and stability of thin plates in the area of solid structural mechanics. Since the transverse shear deformation is neglected in CPT, it cannot be applied to thick plates wherein shear deformation effects are more significant. Thus, its suitability is limited to only thin plates. First-order shear deformation theory (FSDT) can be considered as improvement over the CPT. It is based on the hypothesis that the normal to the undeformed midplane remain straight but not necessarily normal to the midplane after deformation. This is known as FSDT because the thicknesswise displacement field for the inplane displacement is linear or of the first order. Reissner [13, 14] has developed a stress based FSDT which incorporates the effect of shear and Mindlin [9] employed displacement based approach. In Mindlin's theory, transverse shear stress is assumed to be constant through the thickness of the plate, but this assumption violates the shear stress free surface conditions on the top and bottom surfaces of the plate. Mindlin's theory satisfies constitutive relations for transverse shear stresses and shear strains by using shear correction factor. The limitations of CPT and FSDTs forced the development of higher order shear deformation theories (HSDTs) to avoid the use of shear correction factors, to include correct cross sectional warping and to get the realistic variation of the transverse shear strains and stresses through the thickness of plate. The higher order theory is developed by Reddy [12] to get the parabolic shear stress distribution through the thickness of plate and to satisfy the shear stress free surface conditions on the top and bottom surfaces of the plate to avoid the need of shear correction factors. Comprehensive reviews of refined theories have been given by Noor and Burton [10] and Vasil'ev [17], whereas Liew et al. [8] surveyed plate theories particularly applied to thick plate vibration problems. A recent review papers are presented by Ghugal and Shimpi [1] and Kreja [7]. The effect of transverse shear and transverse normal strain on the static flexure of thick isotropic plates using trigonometric shear deformation theory is studied by Ghugal and Sayyad [2]. Shimpi and Patel [15] developed two variable refined plate theory for the static flexure and free vibration analysis of isotropic plates; however, theory of these authors yields the frequencies identical to those of Mindlin's theory. Ghugal and Pawar [3] have developed hyperbolic shear deformation theory for the bending, buckling and free vibration analysis of thick shear flexible plates. Karama et al. [4] has proposed exponential shear deformation theory for the multilayered beam structures. Exact elasticity solution for bidirectional bending of plates is provided by Pagano [11], whereas Srinivas et. al. [16] provided an exact analysis for Vibration of simply supported homogeneous thick rectangular plates.

In this paper a displacement based an exponential shear deformation theory (ESDT) is used for the bi-directional bending and free vibration analysis of thick isotropic square and rectangular plates which includes effect of transverse shear deformation and rotary inertia. The displacement field of the theory contains three variables as in the FSDT of plate. The theory is shown to be simple and more effective for the bending and free vibration analysis of isotropic plates.

2. Theoretical formulation

2.1. Isotropic plate under consideration

Consider a plate made up of isotropic material as shown in Fig. 1. The plate occupies a region given by Eq. (1):

$$0 \le x \le a, \qquad 0 \le y \le b, \qquad -h/2 \le z \le h/2. \tag{1}$$

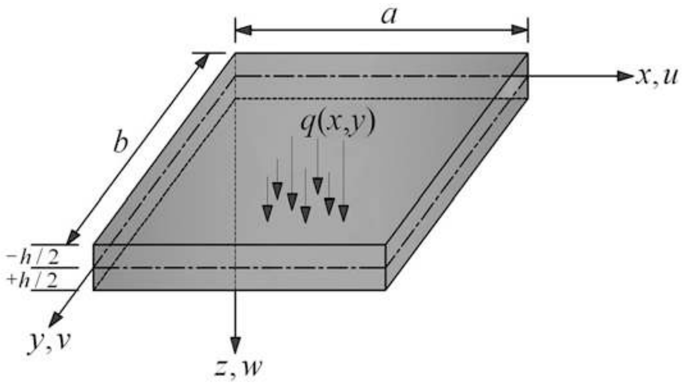

Fig. 1. Plate geometry and co-ordinate system

2.2. *Assumptions made in the proposed theory*

1. The displacement components u and v are the inplane displacements in x and y — directions respectively and w is the transverse displacement in z-direction. These displacements are small in comparison with the plate thickness.

2. The in-plane displacement u in x-direction and v in y-direction each consist of two parts:

   (a) a displacement component analogous to displacement in classical plate theory of bending;

   (b) displacement component due to shear deformation which is assumed to be exponential in nature with respect to thickness coordinate.

3. The transverse displacement w in z-direction is assumed to be a function of x and y coordinates.

4. The plate is subjected to transverse load only.

2.3. *The proposed plate theory*

Based upon the before mentioned assumptions, the displacement field of the proposed plate theory is given as below:

$$\begin{aligned} u(x,y,z,t) &= -z\frac{\partial w(x,y,t)}{\partial x} + f(z)\phi(x,y,t), \\ v(x,y,z,t) &= -z\frac{\partial w(x,y,t)}{\partial y} + f(z)\psi(x,y,t), \\ w(x,y,z,t) &= w(x,y,t), \end{aligned} \tag{2}$$

where $f(z) = z\exp\left[-2\left(\frac{z}{h}\right)^2\right]$.

Here u, v and w are the displacements in the x, y and z-directions respectively. The exponential function in terms of thickness coordinate $[f(z)]$ in both the inplane displacements u and v is associated with the transverse shear stress distribution through the thickness of plate. The functions ϕ and ψ are the unknown functions associated with the shear slopes.

2.4. Superiority of the present theory

The present theory is a displacement-based refined theory, and refined shear deformation theories are known to be successful techniques for improving the accuracy of displacement and stresses. The kinematics of the present theory is much richer than those of the higher order shear deformation theories available in the literature, because if the exponential term is expanded in power series, the kinematics of higher order theories are implicitly taken into account to good deal of extent. Exponential function has all even and odd powers in its expansion unlike sine function which have only odd powers.

2.5. Strain-displacement relationships

Normal strains (ε_x and ε_y) and shear strains $(\gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ are obtained within the framework of linear theory of elasticity using the displacement field given by Eq. (2).

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + f(z)\frac{\partial \phi}{\partial x},\\
\varepsilon_y &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} + f(z)\frac{\partial \psi}{\partial y},\\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} + f(z)\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right),\\
\gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\mathrm{d}f(z)}{\mathrm{d}z}\phi,\\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\mathrm{d}f(z)}{\mathrm{d}z}\psi.
\end{aligned} \tag{3}
$$

2.6. Stress-strain relationships

For a plate of constant thickness, composed of isotropic material, the effect of transverse normal stress σ_z on the gross response of the plate is assumed to be negligible in comparison with inplane stresses σ_x and σ_y. Therefore, for a linearly elastic material, stresses σ_x and σ_y are related to normal strains ε_x and ε_y and shear stresses τ_{xy}, τ_{yz} and τ_{zx} are related to shear strains γ_{xy}, γ_{yz} and γ_{zx} by the following constitutive relations:

$$
\begin{aligned}
\sigma_x &= \frac{E}{1-\mu^2}(\varepsilon_x + \mu\varepsilon_y) = \frac{E}{1-\mu^2}\left[-z\frac{\partial^2 w}{\partial x^2} + f(z)\frac{\partial \phi}{\partial x}\right] + \frac{\mu E}{1-\mu^2}\left[-z\frac{\partial^2 w}{\partial y^2} + f(z)\frac{\partial \psi}{\partial y}\right],\\
\sigma_y &= \frac{E}{1-\mu^2}(\varepsilon_y + \mu\varepsilon_x) = \frac{\mu E}{1-\mu^2}\left[-z\frac{\partial^2 w}{\partial x^2} + f(z)\frac{\partial \phi}{\partial x}\right] + \frac{E}{1-\mu^2}\left[-z\frac{\partial^2 w}{\partial y^2} + f(z)\frac{\partial \psi}{\partial y}\right],\\
\tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\mu)}\left[-2z\frac{\partial^2 w}{\partial x \partial y} + f(z)\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right)\right],\\
\tau_{zx} &= G\gamma_{zx} = \frac{E}{2(1+\mu)}\frac{\mathrm{d}f(z)}{\mathrm{d}z}\phi, \qquad \tau_{yz} = G\gamma_{yz} = \frac{E}{2(1+\mu)}\frac{\mathrm{d}f(z)}{\mathrm{d}z}\psi.
\end{aligned} \tag{4}
$$

3. Governing equations and boundary conditions

Using the Eqs. (2)–(4) and the principle of virtual work, variationally consistent governing differential equations and associated boundary conditions for the plate under consideration can be obtained. The dynamic version of principle of virtual work when applied to the plate leads to

$$\int\limits_{z=-h/2}^{z=h/2}\int\limits_{y=0}^{y=b}\int\limits_{x=0}^{x=a}[\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{yz}\delta\gamma_{yz}+\tau_{zx}\delta\gamma_{zx}+\tau_{xy}\delta\gamma_{xy}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z- \tag{5}$$
$$\int\limits_{y=0}^{y=b}\int\limits_{x=0}^{x=a} q(x,y)\,\delta w\,\mathrm{d}x\,\mathrm{d}y+\rho\int\limits_{z=-h/2}^{z=h/2}\int\limits_{y=0}^{y=b}\int\limits_{x=0}^{x=a}\left[\frac{\partial^2 u}{\partial t^2}\delta u+\frac{\partial^2 v}{\partial t^2}\delta v+\frac{\partial^2 w}{\partial t^2}\delta w\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0,$$

where symbol δ denotes the variational operator. Employing Green's theorem in Eq. (5) successively, we obtain the coupled Euler-Lagrange equations, which are the governing equations and the associated boundary conditions of the plate. The governing differential equations in-terms of stress resultants are as follows:

$$\begin{aligned}
&\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=I_1\frac{\partial^2 w}{\partial t^2}-I_2\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)+I_3\left(\frac{\partial^3\phi}{\partial x\partial t^2}+\frac{\partial^3\psi}{\partial y\partial t^2}\right),\\
&\frac{\partial N_{sx}}{\partial x}+\frac{\partial N_{sxy}}{\partial y}-N_{Tcx}=-I_3\frac{\partial^3 w}{\partial x\partial t^2}+I_4\frac{\partial^2\phi}{\partial t^2},\\
&\frac{\partial N_{sy}}{\partial y}+\frac{\partial N_{sxy}}{\partial x}-N_{Tcy}=-I_3\frac{\partial^3 w}{\partial y\partial t^2}+I_4\frac{\partial^2\psi}{\partial t^2}.
\end{aligned} \tag{6}$$

The boundary conditions at $x=0$ and $x=a$ obtained are of the following form:

$$\begin{array}{lll}
\text{either} & V_x=0 & \text{or } w \text{ is prescribed,}\\
\text{either} & M_x=0 & \text{or } \frac{\partial w}{\partial x} \text{ is prescribed,}\\
\text{either} & N_{sx}=0 & \text{or } \phi \text{ is prescribed,}\\
\text{either} & N_{sxy}=0 & \text{or } \psi \text{ is prescribed.}
\end{array} \tag{7}$$

The boundary conditions at $y=0$ and $y=b$ obtained are of the following form:

$$\begin{array}{lll}
\text{either} & V_y=0 & \text{or } w \text{ is prescribed,}\\
\text{either} & M_y=0 & \text{or } \frac{\partial w}{\partial y} \text{ is prescribed,}\\
\text{either} & N_{sxy}=0 & \text{or } \phi \text{ is prescribed,}\\
\text{either} & N_{sy}=0 & \text{or} \psi \text{ is prescribed.}
\end{array} \tag{8}$$

Reaction at the corners of the plate is of the following form:

$$\text{either} \quad M_{xy}=0 \qquad \text{or } w \text{ is prescribed.} \tag{9}$$

The stress resultants in the governing equations [Eq. (6)] and boundary conditions [Eqs. (7)–(9)] are given as:

$$\begin{aligned}
(M_x,M_y,M_{xy})&=\int_{-h/2}^{h/2}(\sigma_x,\sigma_y,\tau_{xy})z\,\mathrm{d}z, \quad (N_{sx},N_{sy},N_{sxy})=\int_{-h/2}^{h/2}(\sigma_x,\sigma_y,\tau_{xy})f(z)\,\mathrm{d}z,\\
(N_{Tcx},N_{Tcy})&=\int_{-h/2}^{h/2}(\tau_{zx},\tau_{yz})\frac{\mathrm{d}f(z)}{\mathrm{d}z}\,\mathrm{d}z,\\
V_x&=\frac{\partial M_x}{\partial x}+2\frac{\partial M_{xy}}{\partial y}, \quad V_y=\frac{\partial M_y}{\partial y}+2\frac{\partial M_{xy}}{\partial x}.
\end{aligned} \tag{10}$$

The governing differential equations in-terms of unknown displacement variables used in the displacement field (w, ϕ and ψ) obtained are as follows:

$$
\begin{aligned}
D_1\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-D_2\left(\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^3\phi}{\partial x\partial y^2}+\frac{\partial^3\psi}{\partial y^3}+\frac{\partial^3\psi}{\partial x^2\partial y}\right)+&\\
I_1\frac{\partial^2 w}{\partial t^2}-I_2\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)+I_3\left(\frac{\partial^3\phi}{\partial x\partial t^2}+\frac{\partial^3\psi}{\partial y\partial t^2}\right)&=q,\\
D_2\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-D_3\left(\frac{\partial^2\phi}{\partial x^2}+\frac{(1-\mu)}{2}\frac{\partial^2\phi}{\partial y^2}\right)+&\\
D_4\phi-D_3\frac{(1+\mu)}{2}\frac{\partial^2\psi}{\partial x\partial y}-I_3\frac{\partial^3 w}{\partial x\partial t^2}+I_4\frac{\partial^2\phi}{\partial t^2}&=0,\\
D_2\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-D_3\left(\frac{(1-\mu)}{2}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+&\\
D_4\psi-D_3\frac{(1+\mu)}{2}\frac{\partial^2\phi}{\partial x\partial y}-I_3\frac{\partial^3 w}{\partial y\partial t^2}+I_4\frac{\partial^2\psi}{\partial t^2}&=0.
\end{aligned}
\tag{11}
$$

The associated consistent boundary conditions in-terms of unknown displacement variables obtained along the edges $x = 0$ and $x = a$ are as below:

$$
\begin{aligned}
&\text{either}\quad D_1\left[\frac{\partial^3 w}{\partial x^3}+(2-\mu)\frac{\partial^3 w}{\partial x\partial y^2}\right]-\\
&D_2\left[\frac{\partial^2\phi}{\partial x^2}+(1-\mu)\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\psi}{\partial x\partial y}\right]-I_2\frac{\partial^3 w}{\partial x\partial t^2}-I_3\frac{\partial^3\phi}{\partial t^2}=0 \quad \text{or } w \text{ is prescribed,}\\
&\text{either}\quad D_1\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)-D_2\left(\frac{\partial\phi}{\partial x}+\mu\frac{\partial\psi}{\partial y}\right)=0 \quad \text{or } \frac{\partial w}{\partial x} \text{ is prescribed,}\\
&\text{either}\quad D_2\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)-2D_3\left(\frac{\partial\phi}{\partial x}+\mu\frac{\partial\psi}{\partial y}\right)=0 \quad \text{or } \phi \text{ is prescribed,}\\
&\text{either}\quad D_3\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right)-D_2\frac{\partial^2 w}{\partial x\partial y}=0 \quad \text{or } \psi \text{ is prescribed.}
\end{aligned}
\tag{12}
$$

The associated consistent boundary conditions in-terms of unknown displacement variables obtained along the edges $y = 0$ and $y = b$ are as below:

$$
\begin{aligned}
&\text{either}\quad D_1\left[\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2\partial y}\right]-\\
&D_2\left[\frac{\partial^2\psi}{\partial y^2}+(1-\mu)\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\phi}{\partial x\partial y}\right]-I_2\frac{\partial^3 w}{\partial y\partial t^2}-I_3\frac{\partial^3\psi}{\partial t^2}=0 \quad \text{or } w \text{ is prescribed,}\\
&\text{either}\quad D_1\left(\mu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-D_2\left(\mu\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}\right)=0 \quad \text{or } \frac{\partial w}{\partial y} \text{ is prescribed,}\\
&\text{either}\quad D_3\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right)-D_2\frac{\partial^2 w}{\partial x\partial y}=0 \quad \text{or } \phi \text{ is prescribed,}\\
&\text{either}\quad D_2\left(\mu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-2D_3\left(\mu\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}\right)=0 \quad \text{or } \psi \text{ is prescribed.}
\end{aligned}
\tag{13}
$$

The boundary condition in-terms of unknown displacement variables (w, ϕ and ψ) obtained along the corners of plate is:

$$\text{either}\quad 2D_1\frac{\partial^2 w}{\partial x\partial y} - D_2\left(\frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial x}\right) = 0 \qquad \text{or } w \text{ is prescribed,} \tag{14}$$

where constants D_i and I_i appeared in governing equations and boundary conditions are as follows:

$$D_1 = \frac{Eh^3}{12(1-\mu^2)},\qquad D_2 = \frac{A_0E}{(1-\mu^2)},\qquad D_3 = \frac{B_0E}{(1-\mu^2)},\qquad D_4 = \frac{C_0E}{2(1+\mu)}, \tag{15}$$

$$I_1 = \rho h,\qquad I_2 = \frac{\rho h^3}{12},\qquad I_3 = \rho A_0,\qquad I_4 = \rho B_0 \tag{16}$$

and

$$A_0 = \int_{-h/2}^{h/2} zf(z)\,\mathrm{d}z,\qquad B_0 = \int_{-h/2}^{h/2} f^2(z)\,\mathrm{d}z,\qquad C_0 = \int_{-h/2}^{h/2}\left[\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right]^2\mathrm{d}z. \tag{17}$$

4. Illustrative examples

Example 1: Bending analysis of isotropic plates subjected to uniformly distributed load

A simply supported isotropic square plates occupying the region given by the Eq. (1) is considered. The plate is subjected to uniformly distributed transverse load, $q(x,y)$ on surface $z=-h/2$ acting in the downward z-direction as given below:

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{18}$$

where q_{mn} are the coefficients of Fourier expansion of load, which are given by

$$\begin{aligned} q_{mn} &= \tfrac{16q_0}{mn\pi^2} && \text{for } m=1,3,5,\ldots, \text{ and } n=1,3,5,\ldots,\\ q_{mn} &= 0 && \text{for } m=2,4,6,\ldots, \text{ and } n=2,4,6,\ldots. \end{aligned} \tag{19}$$

The governing differential equations and the associated boundary conditions for static flexure of square plate under consideration can be obtained directly from Eqs. (6)–(9). The following are the boundary conditions of the simply supported isotropic plate.

$$w = \psi = M_x = N_{sx} = 0 \quad \text{at } x=0 \text{ and } x=a, \tag{20}$$

$$w = \phi = M_y = N_{sy} = 0 \quad \text{at } y=0 \text{ and } y=b. \tag{21}$$

Example 2: Bending analysis of isotropic plates subjected to sinusoidal load progress

A simply supported square plates is subjected to sinusoidal load progress in both x and y directions, on surface $z=-h/2$, acting in the downward z direction. The load is expressed as:

$$q(x,y) = q_0\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right), \tag{22}$$

where q_0 is the magnitude of the sinusoidal loading at the centre.

Example 3: Bending analysis of isotropic plate subjected to linearly varying load

A simply supported square plate is subjected to linearly varying transverse load (q_0x/a). The intensity of load is zero at the edge $x=0$ and maximum (q_0) at the edge $x=a$. The magnitude of coefficient of Fourier expansion of load in the Eq. (18) is given by $q_{mn} = -(8q_0/mn\pi^2)\cos(m\pi)$.

4.1. The closed-form solution

The governing equations for bending analysis of plate (static flexure), discarding all the terms containing time derivatives becomes:

$$\begin{aligned}
D_1\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-D_2\left(\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^3\phi}{\partial x\partial y^2}+\frac{\partial^3\psi}{\partial y^3}+\frac{\partial^3\psi}{\partial x^2\partial y}\right)&=q,\\
D_2\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-D_3\left(\frac{\partial^2\phi}{\partial x^2}+\frac{(1-\mu)}{2}\frac{\partial^2\phi}{\partial y^2}\right)+D_4\phi-D_3\frac{(1+\mu)}{2}\frac{\partial^2\psi}{\partial x\partial y}&=0,\\
D_2\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-D_3\left(\frac{(1-\mu)}{2}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+D_4\psi-D_3\frac{(1+\mu)}{2}\frac{\partial^2\phi}{\partial x\partial y}&=0.
\end{aligned} \tag{23}$$

The following is the solution form for $w(x,y), \phi(x,y)$, and $\psi(x,y)$ satisfying the boundary conditions perfectly for a plate with all the edges simply supported:

$$\begin{aligned}
w(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\\
\phi(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\\
\psi(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \psi_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),
\end{aligned} \tag{24}$$

where w_{mn}, ϕ_{mn} and ψ_{mn} are unknown coefficients, which can be easily evaluated after substitution of Eq. (24) in the set of three governing differential Eq. (23) resulting in following three simultaneous equations, in case of sinusoidal load $m=1$ and $n=1$,

$$\begin{aligned}
K_{11}w_{mn}+K_{12}\phi_{mn}+K_{13}\psi_{mn}&=q_{mn},\\
K_{12}w_{mn}+K_{22}\phi_{mn}+K_{23}\psi_{mn}&=0,\\
K_{13}w_{mn}+K_{23}\phi_{mn}+K_{33}\psi_{mn}&=0,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
K_{11}&=D_1\pi^4\left(\frac{m^4}{a^4}+\frac{n^4}{b^4}+2\frac{m^2n^2}{a^2b^2}\right), & K_{12}&=-D_2\pi^3\left(\frac{m^3}{a^3}+\frac{mn^2}{ab^2}\right),\\
K_{13}&=-D_2\pi^3\left(\frac{n^3}{b^3}+\frac{m^2n}{a^2b}\right), & K_{22}&=D_3\pi^2\left(\frac{(1-\mu)}{2}\frac{n^2}{b^2}+\frac{m^2}{a^2}\right)+D_4,\\
K_{23}&=D_3\frac{(1+\mu)}{2}\frac{mn\pi^2}{ab}, & K_{33}&=D_3\pi^2\left(\frac{(1-\mu)}{2}\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)+D_4.
\end{aligned} \tag{26}$$

Having obtained the values of w_{mn}, ϕ_{mn} and ψ_{mn} from above set of Eqs. (25) and (24), one can then calculate all the displacement and stress components within the plate using displacement field given by Eq. (2) and stress strain relationships given by Eq. (4).

4.2. Computation of displacements and inplane stresses

Substituting the final solution for $w(x,y)$, $\phi(x,y)$ and $\psi(x,y)$ in the displacement field, the final displacements (u, v and w) can obtained and using strain-displacement relations, final strains (ε_x, ε_y, γ_{xy}, γ_{yz} and γ_{zx}) can be obtained. Finally, the inplane stresses (σ_x, σ_y and τ_{xy}) could be obtained by using stress-strain relations (constitutive relations) as given by the Eq. (5). Non-dimensional displacements are represented as $\bar{u}$, $\bar{v}$ and $\bar{w}$, whereas non-dimensional inplane stresses are represented as $\bar{\sigma}_x$, $\bar{\sigma}_y$ and $\bar{\tau}_{xy}$.

4.3. Computation of transverse shear stresses

The transverse shear stresses τ_{zx} and τ_{yz} can be obtained either by using the constitutive relations [Eq. (4)] or by integrating equilibrium equations with respect to the thickness coordinate. Equilibrium equations of three-dimensional elasticity, ignoring body forces, can be used to obtain transverse shear stresses. These equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0. \tag{27}$$

Integrating Eq. (27) both w.r.t the thickness coordinate z and imposing the following boundary conditions at top and bottom surfaces of the plate

$$[\tau_{zx}]_{z=\pm h/2} = 0, \qquad [\tau_{yz}]_{z=\pm h/2} = 0, \tag{28}$$

expressions for τ_{zx} and τ_{yz} can be obtained satisfying the requirements of zero shear stress conditions on the top and bottom surfaces of the plate. Non-dimensional transverse shear stresses are represented as $\bar{\tau}_{zx}$ and $\bar{\tau}_{yz}$. Further it may be noted that τ_{zx} and $\bar{\tau}_{zx}$ obtained by constitutive relations are indicated by τ_{zx}^{CR} and $\bar{\tau}_{zx}^{CR}$ and when they are obtained by using equilibrium equations, are indicated by τ_{zx}^{EE} and $\bar{\tau}_{zx}^{EE}$. In case of isotropic plate $u = v$, $\sigma_x = \sigma_y$ and $\tau_{zx} = \tau_{yz}$.

Example 4: Free vibration analysis of isotropic plates

The following is the solution form of $w(x, y, t)$, $\phi(x, y, t)$, and $\psi(x, y, t)$ for free vibration analysis satisfying the boundary conditions (time dependent), perfectly for a plate with all the edges simply supported:

$$\begin{aligned} w &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin \omega_{mn} t, \\ \phi &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin \omega_{mn} t, \\ \psi &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \psi_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin \omega_{mn} t, \end{aligned} \tag{29}$$

where w_{mn} is the amplitude of translation and ϕ_{mn} and ψ_{mn} are the amplitudes of rotations. ω_{mn} is the natural frequency. The governing equations for free vibration of simply supported square and rectangular plate can be obtained by setting the applied transverse load $q(x, y)$ equal to zero in the set of Eq. (11). Substitution of solution form [Eq. (29)] into the governing equations of free vibration [Eqs. (11)] of plate results in following three simultaneous equations:

$$\begin{aligned} D_1\left(\frac{m^4\pi^4}{a^4} + 2\frac{m^2n^2\pi^4}{a^2b^2} + \frac{n^4\pi^4}{b^4}\right) w_{mn} - \\ D_2\left(\frac{m^3\pi^3}{a^3} + \frac{mn^2\pi^3}{ab^2}\right)\phi_{mn} - D_2\left(\frac{n^3\pi^3}{b^3} + \frac{m^2n\pi^3}{a^2b}\right)\psi_{mn} - \\ \omega^2\left(I_1 + I_2\frac{m^2\pi^2}{a^2} + I_2\frac{n^2\pi^2}{b^2}\right) w_{mn} + \omega_{mn}^2 I_3 \frac{m\pi}{a}\phi_{mn} + \omega_{mn}^2 I_3 \frac{n\pi}{b}\psi_{mn} = 0, \end{aligned} \tag{30}$$

$$-D_2\left(\frac{m^3\pi^3}{a^3}+\frac{mn^2\pi^3}{ab^2}\right)w_{mn}+\left(D_3\frac{m^2\pi^2}{a^2}+D_3\frac{(1-\mu)}{2}\frac{n^2\pi^2}{b^2}+D_4\right)\phi_{mn}+ \tag{31}$$
$$D_3\frac{(1+\mu)}{2}\frac{mn\pi^2}{ab}\psi_{mn}+\omega_{mn}^2 I_3\frac{m\pi}{a}w_{mn}-\omega_{mn}^2 I_4\phi_{mn}=0,$$
$$-D_2\left(\frac{n^3\pi^3}{b^3}+\frac{m^2n\pi^3}{a^2b}\right)w_{mn}+D_3\frac{(1+\mu)}{2}\frac{mn\pi^2}{ab}\phi_{mn}+ \tag{32}$$
$$\left(D_3\frac{n^2\pi^2}{b^2}+D_3\frac{(1-\mu)}{2}\frac{m^2\pi^2}{a^2}+D\right)\psi_{mn}+\omega_{mn}^2 I_3\frac{n\pi}{b}w_{mn}-\omega_{mn}^2 I_4\psi_{mn}=0.$$

Eqs. (30)–(32) are written in following matrix form:

$$\left([K]-\omega_{mn}^2[M]\right)\{\Delta\}=0, \tag{33}$$

where $[K]$ is stiffness matix, $[M]$ is mass matix and $\{\Delta\}$ is amplitude vector. The elements of stiffness matrix are given in Eq. (26). Elements of mass matrix and amplitude vector are given below:

$$M_{11}=\left(I_1+I_2\frac{m^2\pi^2}{a^2}+I_2\frac{n^2\pi^2}{b^2}\right),\quad M_{12}=-I_3\frac{m\pi}{a},\quad M_{13}=-I_3\frac{n\pi}{b}, \tag{34}$$
$$M_{22}=I_4,\quad M_{23}=0,\quad M_{33}=I_4,\quad M_{21}=M_{12},\quad M_{31}=M_{13},$$
$$\{\Delta\}^T=\{w_{mn}\ \phi_{mn}\ \psi_{mn}\}. \tag{35}$$

Following material properties of isotropic plates are used:

$$E=210\,\text{GPa},\qquad \mu=0.3,\qquad G=\frac{E}{2(1+\mu)}\quad\text{and}\quad \rho=7\,800\,\text{kg/m}^3, \tag{36}$$

where E is the Young's modulus, G is the shear modulus, μ is the Poisson's ratio and ρ is density of the material.

5. Numerical results and discussion

5.1. Numerical results

Results obtained for displacements, stresses and natural frequencies will now be compared and discussed with the corresponding results of higher order shear deformation theory (HSDT) of Reddy [12], trigonometric shear deformation theory (TSDT) of Ghugal and Sayyad [2], hyperbolic shear deformation theory (HPSDT) of Ghugal and Pawar [3], classical plate theory (CPT) of Kirchhoff [5,6], first order shear deformation theory (FSDT) of Mindlin [9], the exact elasticity solution for bidirectional bending of plate Pagano [11] and exact elasticity solution for free vibrational analysis of plate Srinivas et. al. [16]. The numerical results are presented in the following non-dimensional form for the purpose of presenting the results in this paper.

$$\bar{u}=\frac{uE_2}{qhS^3},\ \bar{w}=\frac{100Ew}{qhS^4},\ (\bar{\sigma}_x,\bar{\tau}_{xy})=\frac{(\sigma_x,\tau_{xy})}{qS^2},\ (\bar{\tau}_{zx})=\frac{(\tau_{zx})}{qS},\ \bar{\omega}_{mn}=\omega_{mn}h\sqrt{\rho/G}, \tag{37}$$

where $S(a/h)=$ Aspect Ratio. The percentage error in result of a particular theory with respect to the result of exact elasticity solution is calculated as follows:

$$\%\ \text{error}=\frac{\text{value by a particular theory-valueby exact elasticity solution}}{\text{value by exact elasticity solution}}\times 100. \tag{38}$$

Table 1. Comparison of non-dimensional inplane displacement ($\bar{u}$) at ($x = 0$, $y = b/2$, $z = \pm h/2$), transverse displacement ($\bar{w}$) at ($x = a/2$, $y = b/2$, $z = 0$), inplane normal stress ($\bar{\sigma}_x$) at ($x = a/2$, $y = b/2$, $z = \pm h/2$), inplane shear stress ($\bar{\tau}_{xy}$) at ($x = 0$, $y = 0$, $z = \pm h/2$) and transverse shear stress $\bar{\tau}_{zx}$ at ($x = 0$, $y = b/2$, $z = 0$) in isotropic square plate subjected to uniformly distributed load

S	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
4	Present	ESDT	0.079	5.816	0.300	0.223	0.481	0.472
	Reddy [12]	HSDT	0.079	5.869	0.299	0.218	0.482	0.452
	Ghugal and Sayyad [2]	TSDT	0.074	5.680	0.318	0.208	0.483	0.420
	Ghugal and Pawar [3]	HPSDT	0.079	5.858	0.297	0.185	0.477	0.451
	Mindlin [9]	FSDT	0.074	5.633	0.287	0.195	0.330	0.495
	Kirchhoff [5, 6]	CPT	0.074	4.436	0.287	0.195	–	0.495
	Pagano [11]	Elasticity	0.072	5.694	0.307	–	0.460	–
10	Present	ESDT	0.075	4.658	0.289	0.204	0.494	0.490
	Reddy [12]	HSDT	0.075	4.666	0.289	0.203	0.492	0.486
	Ghugal and Sayyad [2]	TSDT	0.073	4.625	0.307	0.195	0.504	0.481
	Ghugal and Pawar [3]	HPSDT	0.074	4.665	0.289	0.193	0.489	0.486
	Mindlin [9]	FSDT	0.074	4.670	0.287	0.195	0.330	0.495
	Kirchhoff [5, 6]	CPT	0.074	4.436	0.287	0.195	–	0.495
	Pagano [11]	Elasticity	0.073	4.639	0.289	–	0.487	–

Table 2. Comparison of non-dimensional inplane displacement ($\bar{u}$) at ($x = 0$, $y = b/2$, $z = \pm h/2$), transverse displacement ($\bar{w}$) at ($x = a/2$, $y = b/2$, $z = 0$), inplane normal stress ($\bar{\sigma}_x$) at ($x = a/2$, $y = b/2$, $z = \pm h/2$), inplane shear stress ($\bar{\tau}_{xy}$) at ($x = 0$, $y = 0$, $z = \pm h/2$) and transverse shear stress $\bar{\tau}_{zx}$ at ($x = 0$, $y = b/2$, $z = 0$) in isotropic square plate subjected to sinusoidal load

S	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
4	Present	ESDT	0.046	3.748	0.213	0.114	0.238	0.236
	Reddy [12]	HSDT	0.046	3.787	0.209	0.112	0.237	0.226
	Ghugal and Sayyad [2]	TSDT	0.044	3.653	0.226	0.133	0.244	0.232
	Ghugal and Pawar [3]	HPSDT	0.047	3.779	0.209	0.112	0.236	0.235
	Mindlin [9]	FSDT	0.044	3.626	0.197	0.106	0.159	0.239
	Kirchhoff [5, 6]	CPT	0.044	2.803	0.197	0.106	–	0.238
	Pagano [11]	Elasticity	0.049	3.662	0.217	–	0.236	–
10	Present	ESDT	0.044	2.954	0.200	0.108	0.239	0.238
	Reddy [12]	HSDT	0.044	2.961	0.199	0.107	0.238	0.229
	Ghugal and Sayyad [2]	TSDT	0.044	2.933	0.212	0.110	0.245	0.235
	Ghugal and Pawar [3]	HPSDT	0.044	2.959	0.199	0.107	0.237	0.238
	Mindlin [9]	FSDT	0.044	2.934	0.197	0.106	0.169	0.239
	Kirchhoff [5, 6]	CPT	0.044	2.802	0.197	0.106	–	0.238
	Pagano [11]	Elasticity	0.044	2.942	0.200	–	0.238	–

Table 3. Comparison of inplane displacement ($\bar{u}$) at ($x = 0$, $y = b/2$, $z = \pm h/2$), transverse displacement ($\bar{w}$) at ($x = a/2$, $y = b/2$, $z = 0$), inplane normal stress ($\bar{\sigma}_x$) at ($x = a/2$, $y = b/2$, $z = \pm h/2$), inplane shear stress ($\bar{\tau}_{xy}$) at ($x = 0$, $y = 0$, $z = \pm h/2$) and transverse shear stress $\bar{\tau}_{zx}$ at ($x = 0$, $y = b/2$, $z = 0$) in isotropic square plate subjected to linearly varying load

S	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
4	Present	ESDT	0.0396	2.908	0.150	0.111	0.240	0.236
	Reddy [12]	HSDT	0.039 5	2.935	0.150	0.109	0.241	0.226
	Ghugal and Sayyad [2]	TSDT	0.037 0	2.840	0.159	0.104	0.241	0.210
	Ghugal and Pawar [3]	HPSDT	0.039 5	2.929	0.148	0.092	0.239	0.225
	Mindlin [9]	FSDT	0.037 0	2.817	0.144	0.097	0.165	0.247
	Kirchhoff [5, 6]	CPT	0.037 0	2.218	0.144	0.097	–	0.247
	Pagano [11]	Elasticity	0.036 0	2.847	0.153	–	0.230	–
10	Present	ESDT	0.037 5	2.329	0.144	0.102	0.247	0.245
	Reddy [12]	HSDT	0.037 5	2.333	0.144	0.101	0.246	0.243
	Ghugal and Sayyad [2]	TSDT	0.036 5	2.313	0.153	0.097	0.252	0.241
	Ghugal and Pawar [3]	HPSDT	0.037 0	2.332	0.144	0.096	0.245	0.243
	Mindlin [9]	FSDT	0.037 0	2.335	0.143	0.097	0.165	0.248
	Kirchhoff [5, 6]	CPT	0.037 0	2.213	0.143	0.097	–	0.248
	Pagano [11]	Elasticity	0.036 5	2.320	0.144	–	0.244	–

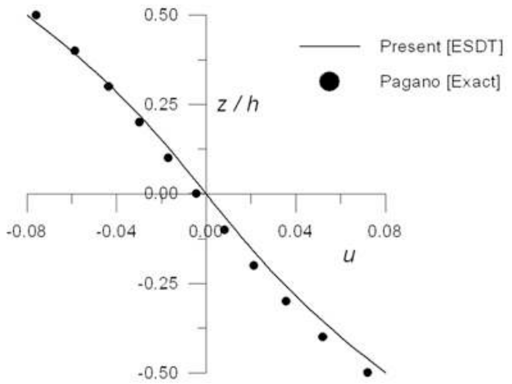

Fig. 2. Through thickness variation of inplane displacement of isotropic plate subjected to uniformly distributed load for aspect ratio 4

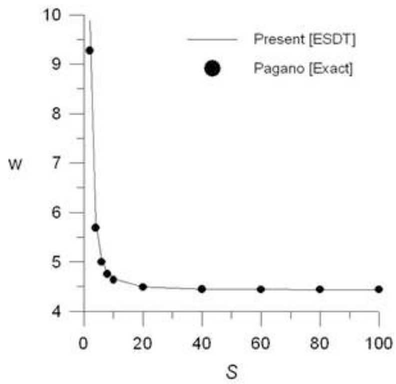

Fig. 3. Through thickness variation of transverse displacement of isotropic plate subjected to uniformly distributed load for aspect ratio 4

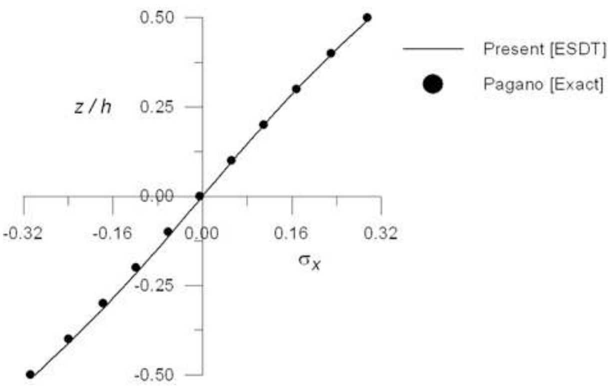

Fig. 4. Through thickness variation of inplane normal stress of isotropic plate subjected to uniformly distributed load for aspect ratio 4

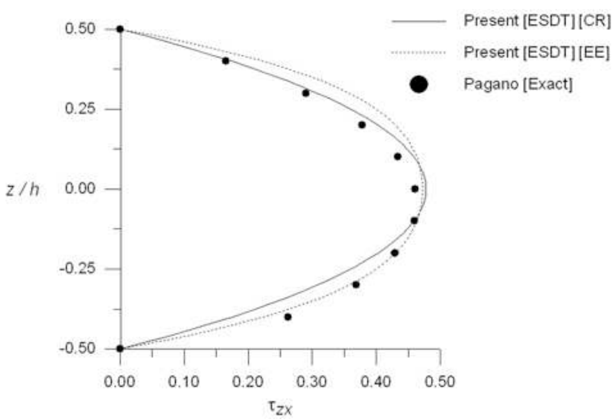

Fig. 5. Through thickness variation of transverse shear stress of isotropic plate subjected to uniformly distributed load for aspect ratio 4

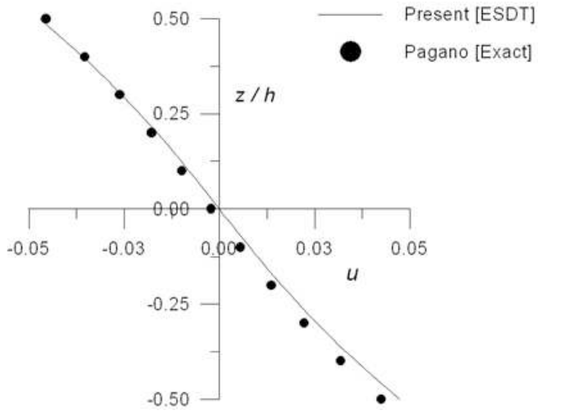

Fig. 6. Through thickness variation of inplane displacement of isotropic plate subjected to single sine load for aspect ratio 4

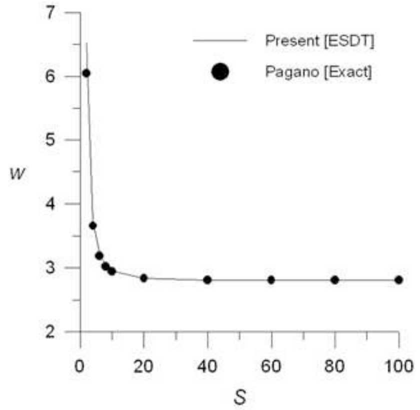

Fig. 7. Through thickness variation of transverse displacement of isotropic plate subjected to single sine load for aspect ratio 4

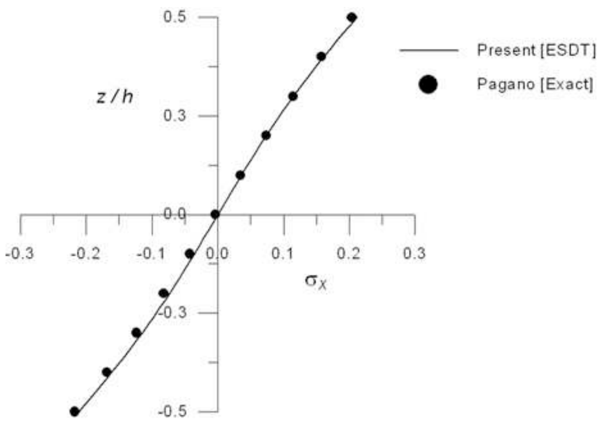

Fig. 8. Through thickness variation of inplane normal stress of isotropic plate subjected to single sine load for aspect ratio 4

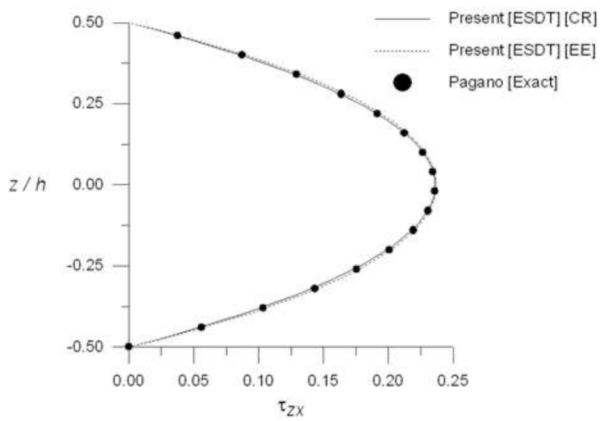

Fig. 9. Through thickness variation of transverse shear stress of isotropic plate subjected to single sine load for aspect ratio 4

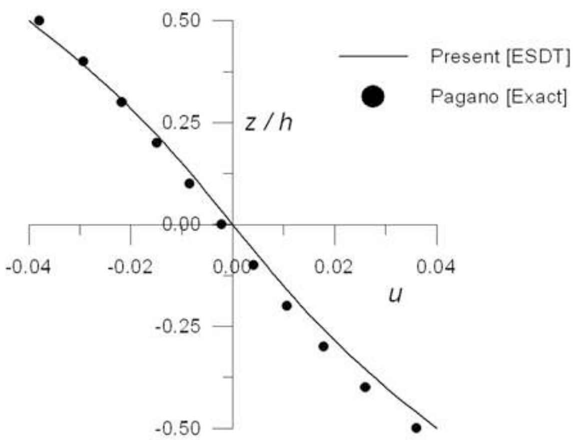

Fig. 10. Through thickness variation of inplane displacement of isotropic plate subjected to linearly varying load for aspect ratio 4

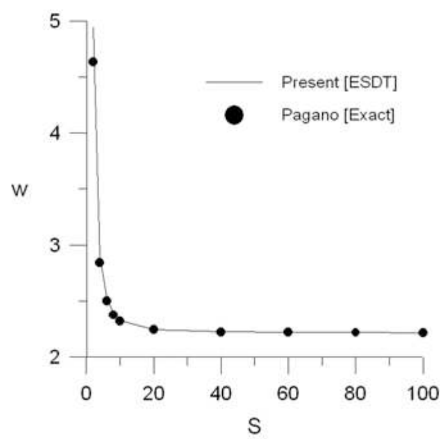

Fig. 11. Through thickness variation of transverse displacement of isotropic plate subjected to linearly varying load for aspect ratio 4

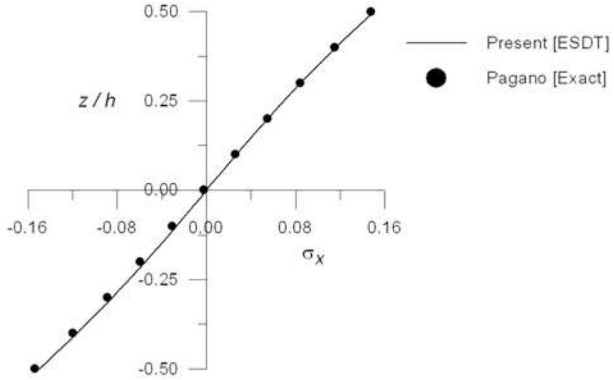

Fig. 12. Through thickness variation of inplane normal stress of isotropic plate subjected to linearly varying load for aspect ratio 4

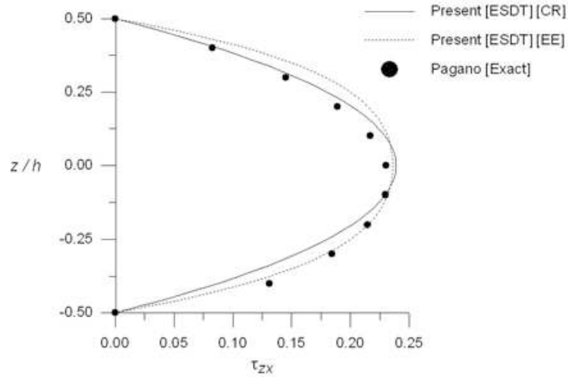

Fig. 13. Through thickness variation of transverse shear stress of isotropic plate subjected to linearly varying load for aspect ratio 4

5.2. Discussion of results

Example 1: Table 1 shows the comparison of maximum displacements and stresses for the isotropic plate subjected to uniformly distributed load. The present theory and other higher order theories overestimate the results of inplane displacement as compared to those of exact solution. Through thickness variation of inplane displacement for isotropic plate subjected to uniformly distributed load is shown in Fig. 2. The value of maximum transverse displacement by present theory (ESDT), HPSDT and HSDT overestimate it by 2.142%, 2.880% and 3.073% for aspect ratio 4 and 0.409%, 0.560% and 0.582% for aspect ratio 10 respectively. TSDT gives the results which are in close agreement with exact value as compared to the theories of Kirchhoff and Mindlin for both aspect ratios. Variation of maximum transverse displacement with aspect ratio is shown in Fig. 3. Maximum values of inplane normal stress obtained by Present theory and HSDT are in close agreement with exact solution for aspect ratio 4, whereas yields exact value of it for aspect ratio 10. However CPT and FSDT underestimate the result by 6.51% for aspect ratio 4.

Its through thickness distribution is shown in Fig. 4. As exact elasticity solutions for inplane shear stress are not available, the results are compared with the other higher order theories, and corresponding values of FSDT and CPT. Present theory is in close agreement with the available solution in the literature. The transverse shear stress can be obtained directly by constitutive relations and equilibrium equations. The examination of Table 1 also reveals that the present theory overestimates the transverse shear stress by 4.656% than the exact elasticity solution when obtained using constitutive relation and underestimates the same by 2.608% when obtained using equilibrium equation for aspect ratio 4 (see Fig. 5). For aspect ratio 10, the results of transverse shear stresses, obtained by constitutive relations and equilibrium equation are in close agreement with the elasticity solution.

Example 2: The displacements and stresses for isotropic plate subjected to sinusoidal load are presented in Tables 2. The result of inplane displacement predicted by present theory and HSDT is identical for the aspect ratio 4 (overestimated by 6.976%), whereas HPSDT overestimates it by 9.302%. Inplane displacement predicted by TSDT is in excellent agreement for both the aspect ratios. Its variation through thickness of the plate is shown in Fig. 6.

Table 4. Comparison of natural bending mode frequencies ($\bar{\omega}_w$) and thickness shear mode frequencies ($\bar{\omega}_\phi$ and $\bar{\omega}_\psi$) of simply supported isotropic square plates ($S = 10$)

a/b	$\bar{\omega}$	(m,n)	Exact [16]	Present [ESDT]	Ghugal and Sayyad [2]	Reddy [12]	Mindlin [9]	CPT [5,6]
1.0	$\bar{\omega}_w$	$(1,1)$	0.0932	0.0931	0.0933	0.0931	0.0930	0.0955
		$(1,2)$	0.2226	0.2223	0.2231	0.2219	0.2219	0.2360
		$(1,3)$	0.4171	0.4163	0.4184	0.4150	0.4149	0.4629
		$(2,2)$	0.3421	0.3415	0.3431	0.3406	0.3406	0.3732
		$(2,3)$	0.5239	0.5228	0.5258	0.5208	0.5206	0.5951
		$(2,4)$	0.7511	0.7499	0.7542	0.7453	0.7446	0.8926
		$(3,3)$	0.6889	0.6874	0.6917	0.6839	0.6834	0.8090
		$(4,4)$	1.0889	1.0872	1.0945	1.0785	1.0764	1.3716
	$\bar{\omega}_\phi$	$(1,1)$	3.1729	3.1626	3.1729	3.1749	3.1730	–
		$(1,2)$	3.2192	3.2091	3.2191	3.2212	3.2193	–
		$(1,3)$	3.2949	3.2851	3.2949	3.2969	3.2951	–
		$(2,2)$	3.2648	3.2549	3.2648	3.2668	3.2650	–
		$(2,3)$	3.3396	3.3299	3.3396	3.3415	3.3397	–
		$(2,4)$	3.4414	3.4320	3.4414	3.4433	3.4416	–
		$(3,3)$	3.4126	3.4031	3.4126	3.4145	3.4128	–
		$(4,4)$	3.6094	3.6004	3.6094	3.6112	3.6096	–
	$\bar{\omega}_\psi$	$(1,1)$	3.2465	3.2428	3.2469	3.2555	3.2538	–
		$(1,2)$	3.3933	3.3994	3.3940	3.4125	3.4112	–
		$(1,3)$	3.6160	3.6381	3.6178	3.6517	3.6510	–
		$(2,2)$	3.5298	3.5455	3.5312	3.5589	3.5580	–
		$(2,3)$	3.7393	3.7709	3.7414	3.7848	3.7842	–
		$(2,4)$	4.0037	4.0576	4.0082	4.0720	4.0720	–
		$(3,3)$	3.9310	3.9786	3.9351	3.9928	3.9926	–
		$(4,4)$	4.4013	4.4944	4.4102	4.5092	4.5098	–

Table 5. Comparison of natural bending mode frequencies ($\bar{\omega}_w$) of simply supported isotropic rectangular plates ($S = 10$)

a/b	$\bar{\omega}$	(m,n)	Exact [16]	Present [ESDT]	Ghugal and Sayyad [2]	Reddy [12]	Mindlin [9]	CPT [5,6]
$\sqrt{2}$	$\bar{\omega}_w$	$(1,1)$	0.0704	0.0704	0.0705	0.0704	0.0703	0.0718
		$(1,2)$	0.1376	0.1376	0.1393	0.1374	0.1373	0.1427
		$(1,3)$	0.2431	0.2433	0.2438	0.2426	0.2424	0.2591
		$(1,4)$	0.3800	0.3803	0.3811	0.3789	0.3782	0.4182
		$(2,1)$	0.2018	0.2017	0.2023	0.2041	0.2012	0.2128
		$(2,2)$	0.2634	0.2639	0.2642	0.2628	0.2625	0.2821
		$(2,3)$	0.3612	0.3639	0.3623	0.3601	0.3595	0.3958
		$(2,4)$	0.4890	0.4928	0.4906	0.4874	0.4861	0.5513
		$(3,1)$	0.3987	0.3985	0.3999	0.3975	0.3967	0.4406
		$(3,2)$	0.4535	0.4552	0.4550	0.4520	0.4509	0.5073
		$(3,3)$	0.5411	0.5465	0.5431	0.5392	0.5375	0.6168

The maximum central deflection for single sine load obtained by the present theory overestimates the value by 2.348% than the exact. TSDT yields the value very closed to the exact value, whereas HPSDT overestimates it by 3.194%. HSDT is in error by 3.413%. The FSDT underestimates the value of maximum transverse deflection by 0.983%, whereas CPT underestimates the same by 23.45% as compared to exact value due to the neglect of shear deformation for aspect ratio 4. For aspect ratio 10, the value of maximum transverse displacement by present theory overestimate it by 0.407%, HPSDT overestimate it by 0.577%, whereas TSDT underestimates it by 0.306%, and HSDT and FSDT overestimate it by 0.645% and 0.271% respectively. For the same aspect ratio, CPT underestimates the value by 4.758%. Fig. 7 shows the variation of transverse displacement with aspect ratio of the plate. The present theory underestimates inplane normal stress by 1.483%, HPSDT underestimates it by 3.686%. TSDT overestimates it by 4.147%, whereas HSDT and FSDT underestimate it by 3.686%, and 9.216% respectively for aspect ratio 4. It is observed that the values of present theory and other theories are in close agreement with those of exact solution for aspect ratio 10. The through thickness distribution of this stress is shown in Fig. 8. The transverse shear stress obtained by constitutive relations are much closed to those of elasticity solution for aspect ratio 4 and 10. The present theory predicts exact value of transverse shear stress for aspect ratios 4 and 10 using equations of equilibrium. HSDT and TSDT underestimate the transverse shear stress for both the aspect ratios. HPSDT underestimate the transverse shear stress for aspect ratio 4 and yield exact value of it for aspect ratio 10. The variation of this stress through the thickness is presented in Fig. 9.

Example 3: The numerical results of displacements and stresses of simply supported square plate subjected to linearly varying load are presented in Table 3 and found in excellent agreement with exact solution. Through thickness variations of displacements and stresses are shown in Figs. 10–13.

Example 4: Table 4 shows comparison of non-dimensional bending mode frequencies and thickness shear mode frequencies for simply supported isotropic square plates. The non-dimensional frequency corresponding to bending mode is denoted by $\bar{\omega}_w$. From the examination of Table 4, it is observed that the present theory (ESDT) yields excellent values of bending frequencies for all modes of vibration as compared to those of exact results. The value of bending frequencies for fundamental mode predicted by ESDT and HSDT are identical. TSDT overestimates the bending frequency for fundamental mode. HSDT underestimates the bending frequencies for higher modes. FSDT yields the lower values of bending frequency for all modes of vibration compared to those of other higher order and exact results, whereas CPT yields the higher values for this frequency. The non-dimensional frequency corresponding to thickness-shear modes are denoted by $\bar{\omega}_\phi$ and $\bar{\omega}_\psi$. The proposed ESDT shows excellent results for thickness shear mode frequency $\bar{\omega}_\phi$ for higher modes. HSDT yields the higher values of this frequency compared to those of present and exact theories. Results for thickness shear mode frequency ($\bar{\omega}_\phi$) obtained by FSDT is not satisfactory for higher modes. Thickness-shear mode frequencies ($\bar{\omega}_\psi$) for square plate predicted by ESDT shows good accuracy of results, whereas HSDT and FSDT overestimates the same. Comparison of non-dimensional bending mode frequency ($\bar{\omega}_w$) of simply supported isotropic rectangular plate is presented in Table 5. For rectangular plate, ESDT and HSDT show exact value for the bending frequency when $m = 1$, $n = 1$. FSDT underestimates the bending frequencies for rectangular plate, whereas CPT overestimates the same for fundamental mode.

6. Conclusions

From the study of bending and free vibration analysis of thick isotropic plates by using exponential shear deformation theory (ESDT), following conclusions are drawn:

1. The results of displacements and stresses obtained by present theory for the all loading cases are in excellent agreement with those of exact solution.

2. The results of displacements and stresses when plate subjected to linearly varying load are exactly half of those when plate subjected to uniformly distributed load

3. The frequencies obtained by the present theory for bending and thickness shear modes of vibration for all modes of vibration are in excellent agreement with the exact values of frequencies for the simply supported square plate.

4. The frequencies of bending and thickness shear modes of vibration according to present theory are in good agreement with those of higher order shear deformation theory for simply supported rectangular plate. This validates the efficacy and credibility of the proposed theory.

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