Classifying Edges and Faces as Manifold or Non-Manifold Elements in 4D Orthogonal Pseudo-Polytopes

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ABSTRACT

This article presents our experimental results for classifying edges and faces as manifold or non-manifold elements in 4D Orthogonal Pseudo-Polytopes (4D-OPP's). For faces in 4D-OPP's we propose a condition to classify them as manifold or non-manifold. For the edges' analysis in 4D-OPP's we have developed two approaches: 1) The analogy between incident (manifold and non-manifold) edges to a vertex in 3D Orthogonal Pseudo-Polyhedra (3D-OPP's) with incident (manifold and non-manifold) faces to a edge in 4D-OPP's; and 2) The extension of the concept of "cones of faces" (which is applied for classifying a vertex in 3D-OPP's as manifold or non-manifold) to "hypercones of volumes" for classifying an edge as manifold or non-manifold in 4D-OPP's. Both approaches have provided the same results, which present that there are eight types of edges in 4D-OPP's. Finally, the generalizations for classifying the n-3 and the n-2 dimensional boundary elements for n-dimensional Orthogonal Pseudo-Polytopes as manifold or non-manifold elements is also presented.

Keywords: Computational geometry, Geometric interrogations and reasoning, Geometric and topological representations.

1. INTRODUCTION

Recent interest has been growing in studying multidimensional polytopes (4D and beyond) for representing phenomena in n-dimensional spaces. Some examples include the works described in [Fei90], [Weg97] and [Lee99]. These previous works show how some of these phenomena's features rely on the polytopes' geometric and topologic relations. However, due to the need of visualizing and analyzing these phenomena (i.e. multidimensional data), it is essential first to analyze these polytopes and their boundaries that compose them [Her98]. So, this article covers that first step, in our research, with the boundary's analysis for classifying edges and faces as manifold or non-manifold elements in 4D Orthogonal Pseudo-Polytopes.

2. THE 4D ORTHOGONAL POLYTOPES

[Cox63] defines an Euclidean polytope Π_n as a finite region of n-dimensional space enclosed by a finite number of (n-1)-dimensional hyperplanes. The finiteness of the region implies that the number N_{n-1} of bounding hyperplanes satisfies the inequality $N_{n-1}>n$. The part of the polytope that lies on one of these hyperplanes is called a cell. Each cell of a Π_n is an (n-1)-dimensional polytope, Π_{n-1}. The cells of a Π_{n-1} are Π_{n-2}'s, and so on; we thus obtain a descending sequence of elements Π_{n-3}, Π_{n-4},..., Π_1 (an edge), Π_0 (a vertex).

Orthogonal Polyhedra (3D-OP) are defined as polyhedra with all their edges (Π_1's) and faces (Π_2's) oriented in three orthogonal directions ([Jua88] & [Pre85]). Orthogonal Pseudo-Polyhedra (3D-OPP) will refer to regular and orthogonal polyhedra with non-manifold boundary [Agu98].

Similarly, **4D Orthogonal Polytopes (4D-OP)** are defined as 4D polytopes with all their edges (Π_1's), faces (Π_2's) and volumes (Π_3's) oriented in four orthogonal directions and **4D Orthogonal Pseudo-Polytopes (4D-OPP)** will refer to 4D regular and orthogonal polytopes with non-manifold boundary. Because the 4D-OPP's definition is an extension from the 3D-OPP's, is easy to generalize the concept to define **n-dimensional Orthogonal Polytopes (nD-OP)** as n-dimensional polytopes with all their Π_{n-1}'s, Π_{n-2}'s,..., Π_1's oriented in n orthogonal directions. Finally, **n-dimensional Orthogonal Pseudo-Polytopes (nD-OPP)** are defined as n-dimensional regular and orthogonal polytopes with non-manifold boundary.

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3. THE Π_{n-2} ANALYSIS FOR 2D, 3D AND 4D-OPP'S

The Π_0 Analysis for 2D-OPP's

A set of quasi-disjoint rectangles determines a 2D-OPP whose vertices must coincide with some of the rectangles' vertices [Agu98]. Each of these rectangles' vertices can be considered as the origin of a 2D local coordinate system, and they may belong to up to four rectangles, one for each local quadrant. The two possible adjacency relations between the four possible rectangles can be of edge or vertex. There are $2^4 = 16$ possible combinations which, by applying symmetries and rotations, may be grouped into six equivalence classes, also called configurations [Sri81].

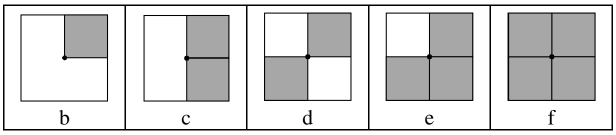

Table 1. The 2D configurations with all their rectangles incident to the origin.

Because we are interested in the vertex analysis, we will consider only those configurations where all their rectangles are incident to the origin. According to the configurations' nomenclature presented in [Agu98], the studied configurations are b, c, d, e and f (see Table 1). There are only two types of vertices in the 2D-OPP's: **the manifold vertex with two incident edges** (configurations b and e), and **the non-manifold vertex with four incident edges** (configuration d) [Agu98]. The remaining configurations represent no vertex because configuration c has only two incident and collinear edges, and in configuration f there are no incident edges.

The Π_1 Analysis for 3D-OPP's

A set of quasi-disjoint boxes determines a 3D-OPP whose vertices must coincide with some of the boxes' vertices [Agu98]. Each of these boxes' vertices can be considered as the origin of a 3D local coordinate system, and they may belong to up to eight boxes, one for each local octant. There are $2^8 = 256$ possible combinations which, by applying symmetries and rotations, may be grouped into 22 equivalence classes [Lor87], also called configurations [Sri81]. Each configuration has its complementary configuration which is the class that contains the complementary combinations of all the combinations in the given class [Agu98]. Grouping complementary configurations leads to the 14 major cases [Van94]. The configurations with 5, 6, 7 and 8 surrounding boxes are complementary, and thus analogous, to combinations with 3, 2, 1 and 0 surrounding boxes, respectively [Agu98]. Finally, each configuration, with four surrounding boxes is self-complementary.

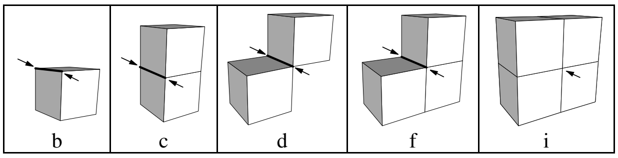

Table 2. The 3D configurations where all their boxes are incident to a same edge (the arrows show the analyzed edge).

Because we are interested in the edge analysis, we will consider only those configurations where all their boxes are incident to a same edge. According to the configurations' associated nomenclature presented in [Agu98], the studied configurations are b, c, d, f and i (see Table 2). [Agu98] concluded that there are only two types of edges in the 3D-OPP's:

- **The manifold edge with two incident faces.** This type of edges is found in configurations b and f. The edge's two incident faces in configuration b belong to one cube's boundary and they are perpendicular to each other. The edge's two incident faces in configuration f belong to two different cubes with edge adjacency and they result perpendicular to each other.
- **The non-manifold edge with four incident faces**. This type of edges is found in configuration d, where two of its faces belongs to a cube and the remaining belong to a second cube with edge adjacency.
- The remaining configurations represent no edge because in configuration c there are only two incident and coplanar faces, and in configuration i there are no incident faces.

The Π_2 Analysis For 4D-OPP's

A set of quasi-disjoint *hyper-boxes* (i.e., hypercubes, which in this paper will be represented using Claude Bragdon's projection [Ruc84]) determines a 4D-OPP whose vertices must coincide with some of the *hyper-boxes'* vertices. We will consider the *hyper-boxes'* vertices as the origin of a 4D local coordinate system, and they may belong to up to 16 *hyper-boxes*, one for each local *hyper-octant*. The 4D-OPP's vertices are determined according to the presence of absence of each of these 16 surrounding *hyper-boxes*. The four possible adjacency relations between the 16 possible *hyper-boxes* can be of volume, face, edge or vertex. There are 2^{16}=65,536 possible combinations of vertices in 4D-OPP's which can be grouped, applying symmetries and rotations, into 253 equivalence classes, also called configurations [Pér01]. Each configuration has its complementary configuration which is the class that contains the complementary combinations of all the combinations in the given class. Grouping complementary configurations leads to the 145 major cases [Pér01]. The combinations with 9, 10, 11, 12, 13, 14, 15 and 16 surrounding hyper-boxes are complementary, and thus analogous, to combinations with 7, 6, 5, 4, 3, 2, 1 and 0 surrounding hyper-boxes, respectively. Finally, each configuration, with eight surrounding hyper-boxes is self-complementary [Pér01].

We will consider only those configurations whose *hyper-boxes* are incident to a same face. According to the configurations' associated nomenclature presented in [Pér01], the studied configurations are 2, 3, 4, 7 and 13 (Table 3). In [Pér01] is concluded that there are only two types of faces in the 4D-OPP's:

- **The manifold faces with two incident volumes.** The face's two incident volumes in configuration 2 belong to the boundary of only one hypercube and they are perpendicular to each other. While in configuration 7, The face's two incident volumes belong to two different hypercubes with face adjacency and they result perpendicular to each other.
- **The non-manifold faces with four incident volumes.** This type of faces is found in configuration 4, where two of its incident volumes belongs to a hypercube and the remaining two belong to a second hypercube with face adjacency.
- The remaining configurations represent no face because in configuration 3 there are only two incident and *co-hyperplanar* volumes, and in configuration 13 there are no incident volumes (analogous to 3D configurations c and i in Table 2).

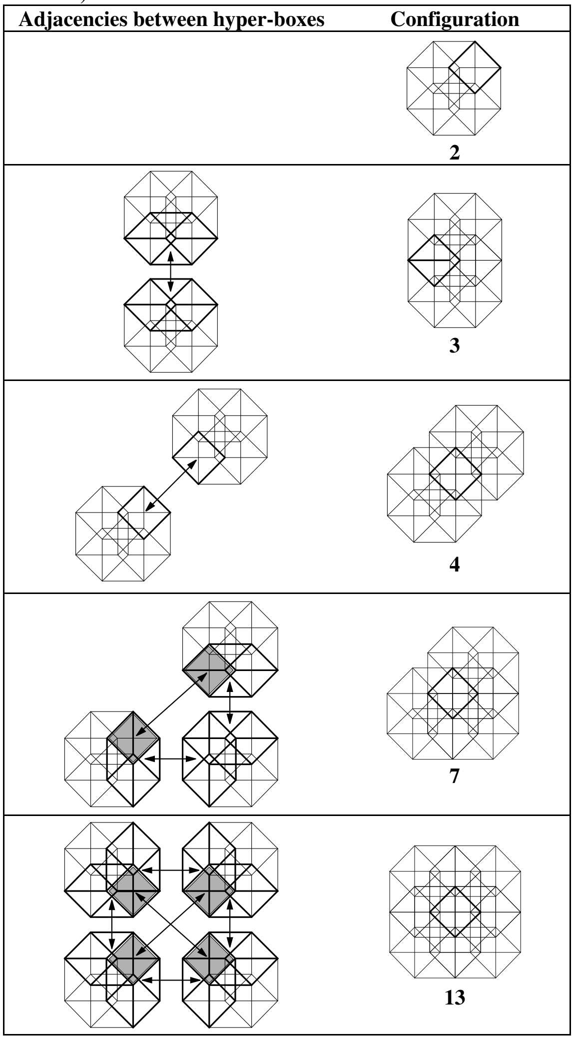

Table 3. Configurations 2, 3, 4, 7 and 13 for 4D-OPP's

Classifying the Π_{n-2}'s in nD-OPP's

Finally, the generalized conditions to classify a Π_{n-2} as manifold or non-manifold in a nD-OPP are:

- If two perpendicular Π_{n-1}'s are incident to a Π_{n-2} then it must be classified as **manifold**.
- If four Π_{n-1}'s are incident to a Π_{n-2} then it must be classified as **non-manifold**.

4. THE Π_{n-3} ANALYSIS FOR 3D AND 4D-OPP'S

The Π_0 Analysis for 3D-OPP's

There are eight types of vertices (also two non valid vertices are identified) for 3D-OPP's [Agu98]. These vertices can be classified depending on the number of two-manifold and non-manifold edges incident to them and they are referred as V3, V4, V4N1, V4N2, V5N, V6, V6N1 and V6N2 [Agu98] (Table 4). In this nomenclature "V" means vertex, the first digit shows the number of incident edges, the "N" is present if at least one non-manifold edge is incident to the vertex and the second digit is included to distinguish between two different types that otherwise could receive the same name.

Each vertex has the following properties [Agu98]:

- V3: all three incident edges are two-manifold and perpendicular to each other.

- V4: all four incident edges are two-manifold, they lie on a plane, and can be grouped in two couples of collinear edges.
- V4N1: three of its four incident edges are perpendicular to each other and also two-manifold ones, while the fourth is non-manifold and collinear to one of the other three.
- V4N2: two of its four incident edges are two-manifold and collinear, while each of its other two is non-manifold and perpendicular to the other three.
- V5N: four of its five incident edges are two-manifold and lie in a plane, while the fifth is non-manifold and perpendicular to the rest of them.
- V6: all six incident edges are two-manifold.
- V6N1: three of its six incident edges are perpendicular to each other and also two-manifold ones, while each of its remaining three edges is non-manifold and collinear to one of the first three.
- V6N2: all of its six incident edges are non-manifold.
- Non valid vertex 1: its two manifold edges are collinear.
- Non valid vertex 2: its two non-manifold edges are collinear.

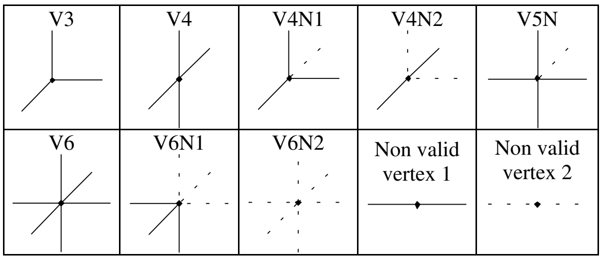

Table 4. Vertices present in 3D-OPP's (dotted lines indicate non-manifold edges and continuous lines indicate manifold edges).

The Π_1 Analysis for 4D-OPP's

Vertices can be defined in terms of the manifold or non-manifold edges that are incident to these vertices in 3D-OPP's [Agu98]. The same process will be extended to describe edges in terms of the manifold or non-manifold faces that are incident to those edges in 4D-OPP's. In this way, we have identified eight types of edges and two non valid edges. We will also extend the nomenclature used by [Agu98] to describe them. Such edges will be referred as E3, E4, E4N1, E4N2, E5N, E6, E6N1 and E6N2 (Table 5). The only difference with the nomenclature used to describe the vertices is that "E" means edge instead of "V" that means vertex. Each edge has the following properties:

- E3: all three incident faces are two-manifold and perpendicular to each other.
- E4: all four incident faces are manifold and lie on a hyperplane, and they can be grouped in two couples of coplanar faces.
- E4N1: three of its four incident faces are perpendicular to each other and also two-manifold ones, while the fourth is non-manifold and coplanar to one of the other three.
- E4N2: two of its four incident faces are two-manifold and coplanar, while each of its other two is non-manifold and perpendicular to the other three.
- E5N: four of its five incident faces are two-manifold and lie in a hyperplane, while the fifth is non-manifold and perpendicular to the rest of them.
- E6: all six incident faces are two-manifold.
- E6N1: three of its six incident manifold faces are perpendicular to each other, while each of its remaining three faces is non-manifold and coplanar to one of the first three.
- E6N2: all of its six incident faces are non-manifold.
- Non valid edge 1: its two manifold faces are coplanar.
- Non valid edge 2: its two non-manifold faces are coplanar.

It results interesting that the number, classifications and positions of the incident faces to an edge in 4D-OPP's are analogous to the way that a set of edges are incident to a vertex in 3D-OPP's.

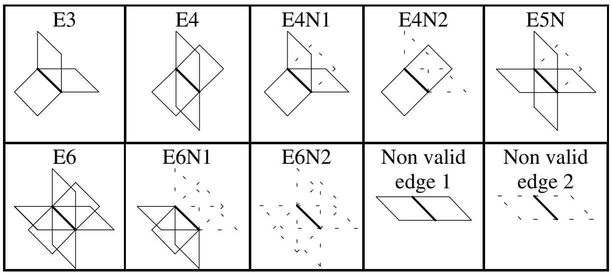

Table 5. Edges present in 4D-OPP's (dotted lines indicate non-manifold faces and continuous lines indicate manifold faces).

Classifying the Π_0's in Polyhedra Through its Cones of Faces

A **polyhedron** is a bounded subset of the 3D Euclidean Space enclosed by a finite set of plane polygons such that every edge of a polygon is shared by exactly one other polygon (adjacent polygons) [Pre85]. A **pseudo-polyhedron** is a bounded subset of the 3D Euclidean Space enclosed by a finite collection of planar faces such that every edge has at least two adjacent faces, and if any two faces meet, they meet at a common edge [Tan91]. Edges and vertices, as boundary elements for polyhedra, may be either two-manifold (or just manifold) or non-manifold elements. In the case of edges, they are (non) manifold elements when every points of it is also a (non) manifold point, except that either or both of its ending vertices might be a point of the opposite type [Agu98]. A manifold edge is adjacent to exactly two faces, and a manifold vertex is the apex (i.e., the common vertex) of only one **cone of faces**. Conversely, a non-manifold edge is adjacent to more than two faces, and a non-manifold vertex is the apex (i.e., the common vertex) of more than one **cone of faces** [Ros91].

3D vertex	Classification
V3	Manifold
V4	Manifold
V4N1	Non-manifold
V4N2	Non-manifold
V5N	Non-manifold
V6	Non-manifold or manifold according to its geometric context.
V6N1	Non-manifold
V6N2	Non-manifold

Table 6. 3D-OPP's vertices classification.

Using the concept of cones of faces it is easy to construct an algorithm to determine the classification of a vertex as manifold or non-manifold in any polyhedron or pseudo-polyhedron. Using this algorithm over the possible vertices in 3D-OPP's we have the results presented in Table 6 which coincide with those presented by [Agu98].

Classifying the Π_1's in 4D Polytopes Through its Hyper-Cones of Volumes

Due to the analogy between 3D-OPP's vertices described in terms of their incident manifold or non-manifold edges, and 4D-OPP's edges described in terms of their incident manifold or non-manifold faces, the next logical step is to extend the concept of cones of faces presented in the previous section to classify 4D polytopes' edges as manifold or non-manifold.

Faces, edges and vertices, as boundary elements for 4D polytopes, may be either manifold or non-manifold elements. [Cox63] and [Han93] have stated that a manifold face is adjacent to exactly two volumes, and now we suggest that a manifold edge is the common edge (apex) of only one **hyper-cone of volumes**. Conversely, we have suggested that a non-manifold face is adjacent to more than two volumes, and now we suggest that a non-manifold edge is the common edge (apex) of more than one **hyper-cone of volumes**.

Using the concept of hyper-cones of volumes, it is easy to extend the algorithm for obtaining the vertex classification for 3D-OPP's used for previous section, to allow us classifying an edge, as manifold or non-manifold, in any 4D polytope or 4D pseudo-polytope. The algorithm is defined with the following steps:

1 Get the set of Π_3's that are incident to edge *A* (a Π_1).
2 From the set of Π_3's select one of them.
3 The selected Π_3 has two Π_2's that are incident to *A*, get one of them and label it as *START* and *ANOTHER*.
4 Repeat
4.1 If the number of Π_3's to *ANOTHER* is more than one, then A is a non-manifold Π_1. End.
4.2 The *ANOTHER* Π_2 is common to another Π_3, find it.
4.3 The Π_3 has another Π_2 that is common to *A*, find it and label it as *ANOTHER*.
4.4 Until *START = ANOTHER* (it has been found a hyper-cone of volumes).
5 If there are more Π_3's to analyze then *A* is non-manifold (there are more hyper-cones of volumes). End.
6 Otherwise, *A* is manifold (*A* is the common edge of only one hyper-cone of volumes). End.

5. RESULTS

Using the algorithm presented in the previous section over the possible edges in 4D-OPP's we have that the edges' classifications are analogous to the 3D-OPP's vertices' classifications. Table 7 shows the edges' classifications given by the extended algorithm and their analogous 3D results.

4D edge	Classification through hyper-cones of volumes	3D vertex	Classification through cones of faces
E3	Manifold	V3	Manifold
E4	Manifold	V4	Manifold
E4N1	Non-manifold	V4N1	Non-manifold
E4N2	Non-manifold	V4N2	Non-manifold
E5N	Non-manifold	V5N	Non-manifold
E6	Non-manifold or manifold according to its geometric context.	V6	Non-manifold or manifold according to its geometric context.
E6N1	Non-manifold	V6N1	Non-manifold
E6N2	Non-manifold	V6N2	Non-manifold

Table 7. 4D-OPP's edges classifications and their analogy with 3D-OPP's vertices.

Classifying the Π_{n-3} in nD Polytopes Through its nD Hyper-Cones of Π_{n-1}'s

Due to the analogy found between 3D vertices and 4D edges with the extension of the concept of cones of faces, is feasible to generalize the last presented algorithm to classify the Π_{n-3} as manifold or non-manifold in nD polytopes through their nD hyper-cones of Π_{n-1}'s. The proposed general algorithm is the following:

1 Get the set of Π_{n-1}'s that are incident to Π_{n-3} *A*.
2 From the set of Π_{n-1}'s select one of them.
3 The selected Π_{n-1} has two Π_{n-2}'s that are incident to Π_{n-3} A, get one of them and label it as *START* and *ANOTHER*.
4 Repeat
4.1 If the number of incident Π_{n-1}'s to *ANOTHER* is more than one, then *A* is a non-manifold Π_{n-3}.
4.2 The *ANOTHER* Π_{n-2} is common to another Π_{n-1}, find it.
4.3 The Π_{n-1} has another Π_{n-2} that is common to *A*, find it and label it as *ANOTHER*.
4.4 Until *START = ANOTHER* (it has been found a nD hyper-cone of Π_{n-1}'s).
5 If there are more Π_{n-1}'s to analyze then Π_{n-3} *A* is non-manifold (there are more nD hyper-cones of Π_{n-1}'s).
6 Otherwise, Π_{n-3} *A* is manifold (A is the common Π_{n-3} of only one nD hyper-cone of Π_{n-1}'s).

The Eight Types of Π_{n-3}'s in nD Orthogonal Pseudo-Polytopes

Due to the analogy between vertices in 3D-OPP's and edges in 4D-OPP's (Table 7), we can extend their properties to propose the eight types of Π_{n-3}'s in nD-OPP's. Such Π_{n-3}'s will be referred as Π_{n-3}3, Π_{n-3}4, Π_{n-3}4N1, Π_{n-3}4N2, Π_{n-3}5N, Π_{n-3}6, Π_{n-3}6N1 and Π_{n-3}6N2. In this nomenclature 'Π_{n-3}' indicates the (n-3)-dimensional element (i.e. vertices in 3D-OPP's and edges in 4D-OPP's), the first digit shows the number of incident Π_{n-2} (i.e. edges in 3D-OPP's and faces in 4D-OPP's), the 'N' is present if at least one non-manifold Π_{n-2} is incident to the Π_{n-3} and the second digit is included to distinguish between two different types that otherwise could receive the same name.

6. FUTURE WORK

The results of this article are being used in studying the extension for the Extreme Vertices Model (EVM) [Agu98] to the four dimensional space (EVM-4D). The EVM-4D will be a representation model for 4D-OPP's that will allow queries and operations over them. However, the fact related to a model purely geometric (four geometric dimensions) is not restrictive for our research, because it will be used under geometries as the 4D spacetime. The first main application for the EVM-4D will cover the visualization and analysis for multidimensional data under the context of a Geographical Information System (GIS).

7. ACKNOWLEDGEMENTS

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