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Reliability Analysis of Components Life Based on Copula Model

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Abstract

It is the general character of most engineering systems that failure statistical correlation of elements exists in the fatigue happened process due to the twin loads. Based on randomized Basquin equation, the constitutive relations are established between element life and random variable including twin loads, initial strength,fatigue strength exponent. Product-moment correlation coefficients are derived and used to quantify the dependence of logarithm life of elements. Aim at fatigue life correlation of elements in structural systems, the computation model of the system reliability is preliminarily established by means of using copula function. The new model can express the dependence of elements fatigue life in structural systems, can be used probability prediction of structural systems under common stochastic cyclic load, which gives a new path for reliability-based design and probability assesment in equipment systems with multi-mode damage coupling.

Keywords: *Fatigue reliability, Fatigue life correlation, Basquin equation,Copula*

1. Introduction

The statistical correlation of elements failure mode can be triggered , because elements of the systems share the common dynamic conditions, and bear the twin loads, and are located at the common operating enviroment[1-2]. Thereinto, it is the common characteristic of most engineering systems that failure statistical correlation of elements exists due to the twin loads. Compared with failure independence assumption, the proportion of element jiont-failure probability is changed in probability space of the failure field due to failure dependence, which not convenient for probability analysis and reliability design of structural systems[3]. If failure dependence is ignored in the reliability analysis of systems, the more error is usually brought about[4-5]. with regard to series system, the system reliability is regarded as one of the weakest element ordinarily, which is equivalent to completely failure correlation assumption of elements, thereby underestimating the probability of system risk. In addition, analytical error is also generated because of ignoring failure dependence for the same component with multi-failure mode or multi-damage mechanism. It is the one of important mission for theoretical study of system reliability that the systematic probability analysis is made according to intrinsic mechanism of failure dependence.

As a new and efficient tool of statistical analysis, copula has attracted widespread attension owing to two advantages[6-9]: the one is concise and flexible expression, the other is high capability of describing time-varying and nonlinear feature of statistical dependence of random variables. Correlated structure between random variables is build by Copula function, by which joint distribution function is gained by means of many margin distribution function.

Aim at high cycle fatigue problem in this paper, the degree of statistical correlation among random variables of elements safety margin is described by using product-moment correlation coefficients. Based on randomized Basquin equation, the constitutive relations between element life and twin loads are established. The joint distribution function of elements safety margin is preliminarily built by means of copula function, from which the system fatigue reliability is solved. The new copula model of system fatigue reliability is established, taking into account failure statistical correlation owing to elements bearing common loads, which gives a new path for reliability-based design and probability assesment in equipment systems with multi-mode damage coupling.

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2. Failure dependece mechanism about loading dependence system

Suppose a structure system has n elements, when elements of the system bear twin loads or common loads, the limit-state function of every element is shown in eq.(1)

$$g_i = r_i - s \, , \; i = 1,2,\cdots,n \tag{1}$$

Where random variable r_i is element strength, s express common loads(or stress) of elements. The product-moment correlation coefficient of g_1 and g_2 is given by eq.(2) [5]

$$\rho_{g_1 g_2} = \frac{\rho_{r_1 r_2}\sigma_{r_1}\sigma_{r_2} + \sigma_s^2}{\sqrt{\sigma_{r_1}^2 + \sigma_s^2}\sqrt{\sigma_{r_2}^2 + \sigma_s^2}} \tag{2}$$

Where $\rho_{r_1 r_2}$ is the correlation coefficient of random variable r_1 and r_2 , σ_{x_i} express the mean-square deviation of design variable x_i . If elements strength r_i is independent and the same distribution for symmetrical structural system, eq.(2) is simplified by eq.(3)

$$\rho_{g_1 g_2} = \frac{\sigma_s^{\;2}}{\sqrt{\sigma_r^2 + \sigma^2{}_s}} \tag{3}$$

From eq.(3), it is seen that the correlation coefficient tends to 1 with load variance increase, but to 0 with strength variance increase, which account for failure dependence being from dispersibility of twin loads, nevertheless, the dispersibility of component property contribute to reducing dependence degree of elements or failure mode. When there is failure dependence of system elements due to twin loads, the joint failure probability of elements is increased to some extent, which is bound to affect system reliability analysis.

3. Statistical correlation of elements fatigue life

There are a long-term intensive study on structural fatigue problem for the engineering field since the 19th century, many famous assessment model and formula of fatigue life is put forward successively. With regard to high-cycle fatigue problem, the relation between constant amplitude cyclic stress and the median fatigue life is described by means of Basquin function in eq.(4) [10]

$$s = s_{\mathrm{f}}\left(2N\right)^b \tag{4}$$

Where s is stress amplitude value, N is fatigue life, b is fatigue strength exponent, s_{f} express actual breaking stress of the material. The approximate linear relation of s_{f} and static strength r is expressed as $s_{\mathrm{f}} \approx ar$, thereby eq.(4) take the logarithm in both sides

$$\ln N = \frac{1}{b}\left(\ln s - \ln r - \ln a\right) - \ln 2 \tag{5}$$

Where a is proportional coefficient. For the pull and compression load of a bar, r express the tensile strength of material, a is a constant related to percentage reduction of area.

For metal material, many experiments have shown that fatigue strength exponent b is increased with the increase of materials hardness, which show remarkable statistics correlation, so b is expressed as linear function of $\ln r$ and $\ln a$, which is shown in Figure.1

$$b = K\left(\ln r + \ln a\right) + y \tag{6}$$

Eq.(5) is rewrited as

$$\ln N = \frac{\ln s - (\ln r + \ln a)}{K(\ln r + \ln a) + y} - \ln 2 \quad (7)$$

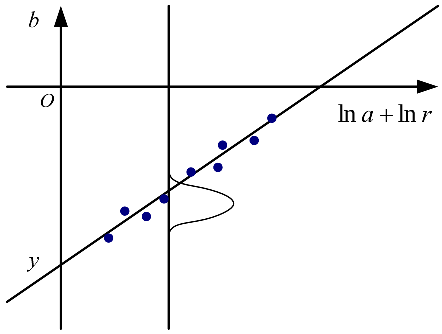

Figure1. linear regression of fatigue strength exponent b

When the load is symmetrical cyclic loading which submit to common independent distribution, in view of variables randomness in Basquin function[16], logarithm life $\ln N$, $\ln s$, $\ln r$, $\ln a$ and intercept y are respectively supposed as normal random variable $X_N \sim N(\mu_{\ln N}, \sigma^2_{\ln N})$, $X_s \sim N(\mu_{\ln s}, \sigma^2_{\ln s})$, $X_r \sim N(\mu_{\ln r}, \sigma^2_{\ln r})$, $X_a \sim N(\mu_{\ln a}, \sigma^2_{\ln a})$ and $\eta \sim N(\mu_y, \sigma^2_y)$. Using above random variables to subsitute for corresponding parameter in eq.(7), and to introduce $X_f = X_r + X_a$, so randoming Basquin logarithmic equation is expressed as

$$X_N = \frac{X_s - X_f}{KX_f + \eta} - \ln 2 \quad (8)$$

Where variable X_f take logarithmic actual breaking stess with dispersibility into account, this dispersibility mainly depends on initial static strength r and proportional coefficient a. Random variables in eq.(8) can be regarded as approximate statistic independence, so, the mathematic relation is established among logarithmic fatigue life and load, strength ,material degenerate uncertainty index, it is shown in Figure.2.

In most cases, normal distribution is able to describe logarithmic life of material under constant amplitude cyclic loading. By center limit theorem, element logarithmic life X_N approximate submit to normal distribution, the mean and variance of X_N can be calculated by two following equation[9]

$$\mu_{X_N} = \frac{\mu_u}{\mu_v} + \frac{\mu_u \sigma^2{}_v}{\mu_v^3} - \ln 2 \quad (9)$$

$$\sigma^2_{X_N} = \frac{\mu_u^2}{\mu_v^2}\left(\frac{\sigma_u^2}{\mu_u^2} + \frac{\sigma_v^2}{\mu_v^2} - 2\rho_{uv}\frac{\sigma_u \sigma_v}{\mu_u \mu_v}\right) \quad (10)$$

Where

$$\mu_u = \mu_{X_s} - \mu_{X_f}, \quad \mu_v = K\mu_{X_f} + \mu_\eta \qquad \sigma_u^2 = \sigma_{X_s}^2 + \sigma_{X_f}^2 \qquad \sigma_v^2 = K^2\sigma_{X_f}^2 + \sigma_\eta^2$$

$$\rho_{uv} = \frac{-K\sigma_f^2}{\sqrt{\sigma_s^2 + \sigma_f^2}\sqrt{K^2\sigma_s^2 + \sigma_\eta^2}}$$

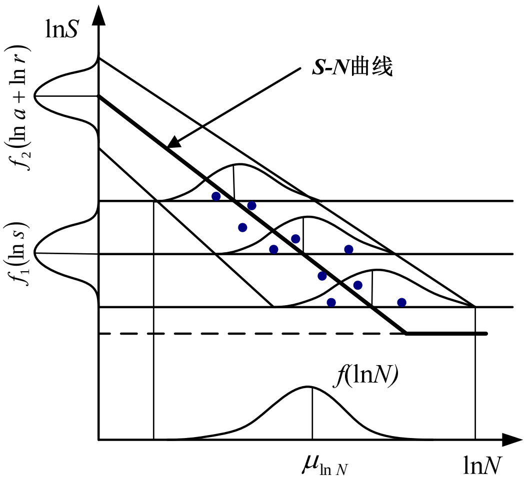

Figure2. Randomize Basquin model

According to eq.(8), for structural system under common cyclic loading s time after time, the logarithmic fatigue lifes of elements are expressed as

$$X_{N_i} = \frac{X_s - X_{f_i}}{KX_{f_i} + \eta_i} - \ln 2 \tag{11}$$

$$X_{N_j} = \frac{X_s - X_{f_j}}{KX_{f_j} + \eta_j} - \ln 2 \tag{12}$$

As shown in eq.(2) and eq.(3), Product-moment correlation coefficient of elements for symmetrical structure is given by

$$\rho_{ij} = \frac{c_{ss}\sigma_{X_s}^2}{c_s^2\sigma_{X_s}^2 + c_f^2\sigma_{X_f}^2 + c_\eta^2\sigma_\eta^2} \tag{13}$$

Where every coefficient can be calculated through partial derivative rules of composite function

$$c_{ss} = \frac{\partial X_{N_i}}{\partial X_s}\Big|_\mu \frac{\partial X_{N_j}}{\partial X_s}\Big|_\mu = \frac{1}{KX_f + \eta}\Big|_\mu^2 \qquad c_s^2 = c_{ss}$$

$$c_f^2 = \frac{\partial X_{N_i}}{\partial X_{f_i}}\Big|_\mu^2 = \frac{(KX_s + \eta)\big|_\mu^2}{(KX_f + \eta)\big|_\mu^4} \qquad c_\eta^2 = \frac{\partial X_{N_i}}{\partial \eta_i}\Big|_\mu^2 = \frac{(X_s - X_f)\big|_\mu^2}{(KX_f + \eta)\big|_\mu^4}$$

Where $\left.\frac{\partial(\cdot)}{\partial(\cdot)}\right|_{\mu}$ mean partial derivative of element logarithmic fatigue life at the mean value,eq.(13) is simplified as follow

$$\rho_{ij} = \frac{\sigma_{X_s}^2}{\sigma_{X_s}^2 + \left(\frac{K\mu_{X_s} + \mu_\eta}{K\mu_{X_f} + \mu_\eta}\right)^2 \sigma_{X_f}^2 + \left(\frac{K\mu_{X_s} - \mu_{X_f}}{K\mu_{X_f} + \mu_\eta}\right)^2 \sigma_\eta^2} \quad (14)$$

From eq.(14), ρ_{ij} tends to 1 with load variance increase, but to 0 with initial static strength variance increase, which explain that the fatigue life correlation mainly come from the dispersibility of twin loading, nevertheless, the dispersibility of material property have contributed to reducing the degree of correlation for elements fatigue life. It is common that the engineering structures bear twin loading or common loading time after time, so the fatigue life is general character correlation in structural systems.

4. Fatigue reliability copula model of the structural system with life correlation

4.1. The concept of copula

A copula is defined as the joint cumulative distribution function of d uniform random variables $(U_1, U_2, \cdots, U_d)$[7]

$$C(u_1, u_2, \cdots, u_d) = P(U_1 < u_1, U_2 < u_2, \cdots, U_d < u_d) \quad (15)$$

Given a joint distribution function $F(x_1, x_2, \cdots, x_d)$ for random variables $(X_1, X_2, \cdots, X_d)$with marginal distribution function $(F_1, F_2, \cdots, F_d)$, F can be written as a function of its distribution function

$$F(x_1, x_2, \cdots, x_d) = C(F_1(x_1), F_2(x_2), \cdots, F_d(x_d)) \quad (16)$$

where copula $C(u_1, u_2, \cdots, u_d)$ is a joint distribution with uniform distribution function. Moreover, if each F_i is continuous, copula C is unique. The dependency structure can be represented by a proper copula function.

There are many family of copula function, a versatile subclass of copulas, called Archimedean copulas, is an important family of multivariate dependence model with attractive stochastic properties. It completely describes the dependency structure of the entire d-dimensional vector X. Some main types of Archimedean family copulas are listed as follows[11]

Gumbel Copula:

$$C(\boldsymbol{u};\theta) = \exp\{-[\textstyle\sum_{i=1}^n (-\ln u_i)^{1/\theta}]^\theta\} \quad (17)$$

Clayton Copula:

$$C(\boldsymbol{u};\theta) = (\textstyle\sum_{i=1}^n u_i^{-\theta} - n + 1)^{-1/\theta} \quad (18)$$

Frank Copula:

$$C(\mu;\theta) = -\frac{1}{\theta}\ln\left\{1+\frac{\prod_{i=1}^{n}\left(e^{-\theta\mu_i}-1\right)}{\left(e^{-\theta}-1\right)^{n-1}}\right\} \tag{19}$$

The probability density plot of above binary Archimedean copulas are listed as Figure 3.

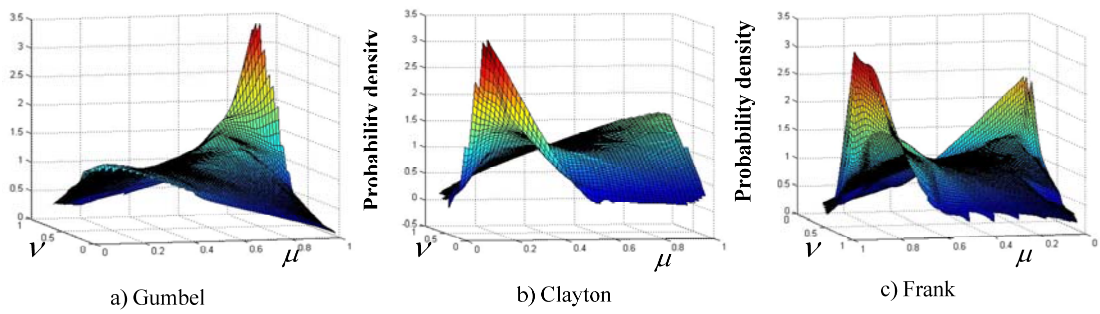

Figure3. Probability density plot of binary Archimedean copula

From Figure.3, we see that Gumbel Copula is susceptible to the change of variable upper tail, so that it is used as describing correlativity bewteen variables with upper tail correlation properties. Clayton Copula is fit for below tail correlation properties. The probability density figure of Frank Copula present ‘U’ shape, which have symmetry, therefore , Frank Copula is used for describing correlativity with symmetrical dependence structures.So, these Archimedean copulas can be applied to describing variables correlativity of different Correlated Failure mode.

5. Fatigue reliability copula model of the structural system

When element logarithmic life submit to normal distribution, the element reliability bearing random load t times can be as followed

$$R_i(t) = 1 - \Phi\left(\frac{\ln t - \mu_{X_{N_i}}}{\sigma_{X_{N_i}}}\right) \tag{20}$$

Where $\Phi(\cdot)$ is standard normal distribution function, $\mu_{X_{N_i}}$ and $\sigma_{X_{N_i}}$ are calculated by eq.(9) and eq.(10) respectively, so the failure probability of the system element i is as followed

$$P_{fi}(t) = 1 - R_i(t) = \Phi\left(\frac{\ln t - \mu_{X_{N_i}}}{\sigma_{X_{N_i}}}\right) \tag{21}$$

Supposing a structural system consist of k elements with initial independent performance, the logarithmic fatigue life of elements is given by

$$X_{N_i} = \frac{X_s - X_{fi}}{KX_{fi} + \eta_i} - \ln 2 \quad i = 1,2,\cdots,k \tag{22}$$

5.1. Series system

When series system have two elements, $i = 1,2$, the failure probability of series system bearing random load t times is

$$\begin{aligned} P_{\mathrm{fs}}(t) &= P\left(X_{N_1} \le \ln t \bigcup X_{N_2} \le \ln t\right) \\ &= P\left(X_{N_1} \le \ln t\right) + P\left(X_{N_2} \le \ln t\right) - P\left(X_{N_1} \le \ln t \bigcap X_{N_2} \le \ln t\right) \\ &= P_{\mathrm{f}1}(t) + P_{\mathrm{f}2}(t) - C\left(P_{\mathrm{f}1}(t), P_{\mathrm{f}2}(t)\right) \end{aligned} \tag{23}$$

So the reliability of the system is

$$R(t) = 1 - P_{\mathrm{fs}}(t) = 1 - P_{\mathrm{f}1}(t) - P_{\mathrm{f}2}(t) + C\left(P_{\mathrm{f}1}(t), P_{\mathrm{f}2}(t)\right) \tag{24}$$

When series system consist of k elements, the reliability of the system is given by

$$\begin{aligned} R(t) &= 1 - P\left(\bigcup_{i=1}^{k} X_{N_i} \le \ln t\right) \\ &= 1 - \sum_{i=1}^{k} P\left(X_{N_i} \le \ln t\right) + \sum_{1\le i<h\le k} P\left(X_{N_i} \le \ln t \bigcap X_{N_h} \le \ln t\right) - \\ &\sum_{1\le i<h<j\le k} P\left(X_{N_i} \le \ln t \bigcap X_{N_h} \le \ln t \bigcap X_{N_j} \le \ln t\right) + \cdots + \\ &(-1)^{k-1} P\left(X_{N_1} \le \ln t \bigcap X_{N_2} \le \ln t \cdots \bigcap X_{N_k} \le \ln t\right) \\ &= 1 - \sum_{i=1}^{k} P_{\mathrm{f}i}(t) + \sum_{1\le i<h\le k} C\left(P_{\mathrm{f}i}(t), P_{\mathrm{f}h}(t)\right) - \sum_{1\le i<h<j\le k} C\left(P_{\mathrm{f}i}(t), P_{\mathrm{f}h}(t), P_{\mathrm{f}j}(t)\right) + \cdots + \\ &(-1)^{k-1} C\left(P_{\mathrm{f}1}(t), P_{\mathrm{f}2}(t), \cdots P_{\mathrm{f}k}(t)\right) \\ &= 1 + \sum_{i=1}^{k} (-1)^i \binom{k}{i} C^{(i)}\left(P_{\mathrm{f}1}(t), \cdots, P_{\mathrm{f}i}(t)\right) \end{aligned} \tag{25}$$

Where superscript of C(·)(·) means the dimension of copula function

5.2. Parallel system

When parallel system consist of k elements, the reliability of the system is given by

$$R(t) = 1 - P_{\mathrm{fs}}(t) = 1 - P\left(\bigcap_{i=1}^{k} X_{N_i} \le \ln t\right) = 1 - C\left(P_{\mathrm{f}1}(t), P_{\mathrm{f}2}(t), \cdots, P_{\mathrm{f}k}(t)\right) \tag{26}$$

5.3. Serial-parallel system

Supposing serial-parallel system consist of 4 elements presented in Figure.4, the reliability of the system is given by

$$\begin{aligned} R(t) &= 1 - P\left(\bigcup_{i=1}^{k}\bigcap_{j=1}^{m^j} X_{N_{I,J}} \le \ln t\right) \\ &= 1 - P\left(X_{N_{11}} \le \ln t \bigcap X_{N_{12}} \le \ln t\right) - P\left(X_{N_{23}} \le \ln t \bigcap X_{N_{24}} \le \ln t\right) \\ &+ P\left[\left(X_{N_{11}} \le \ln t \bigcap X_{N_{12}} \le \ln t\right) \bigcap \left(X_{N_{23}} \le \ln t \bigcap X_{N_{24}} \le \ln t\right)\right] \\ &= 1 - C_1\left(P_{\mathrm{f}11}(t), P_{\mathrm{f}12}(t)\right) - C_2\left(P_{\mathrm{f}23}(t), P_{\mathrm{f}24}(t)\right) + C\left[C_1\left(P_{\mathrm{f}11}(t), P_{\mathrm{f}12}(t)\right), C_2\left(P_{\mathrm{f}23}(t), P_{\mathrm{f}24}(t)\right)\right] \end{aligned} \tag{27}$$

Where $C_1(\cdot)$ express copula function of the first parallel system.

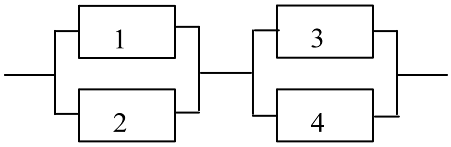

Figure4. reliability block diagram of serial-parallel system

6. Conclusion

Aim at high cycle fatigue problem in this paper, it is ignored that the load sequence has an influence up on fatigue life, it is the rationality and feasibility that the constitutive relations between element life and twin loads, initial strength,fatigue strength exponent are established based on randomized Basquin equation in theory, it showed that the life has correlation due to the system elements bearing common loading.

Copula function can establish the correlation formation of random variables, be associated with the marginal distribution of random variables, describe complex correlativity of variables. Aim at the structural system with fatigue life dependence, the computation model of the system reliability is preliminarily established by means of using copula function. which gives a theory reference for reliability-based design and probability assesment in equipment systems with multi-mode damage coupling.

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