TELKOMNIKA, Vol.16, No.6, December 2018, pp. 3024–3033
ISSN: 1693-6930, accredited First Grade by Kemenristekdikti, Decree No: 21/E/KPT/2018
DOI: 10.12928/TELKOMNIKA.v16i6.10112

Transfer Function, Stabilizability, and Detectability of Non-Autonomous Riesz-spectral Systems

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Abstract

Stability of a state linear system can be identified by controllability, observability, stabilizability, detectability, and transfer function. The approximate controllability and observability of non-autonomous Riesz-spectral systems have been established as well as non-autonomous Sturm-Liouville systems. As a continuation of the establishments, this paper concern on the analysis of the transfer function, stabilizability, and detectability of the non-autonomous Riesz-spectral systems. A strongly continuous quasi semigroup approach is implemented. The results show that the transfer function, stabilizability, and detectability can be established comprehensively in the non-autonomous Riesz-spectral systems. In particular, sufficient and necessary conditions for the stabilizability and detectability can be constructed. These results are parallel with infinite dimensional of autonomous systems.

Keywords: *detectability, non-autonomous Riesz-spectral system, stabilizability, transfer function*

1. Introduction

Let X, U, and Y be complex Hilbert spaces. This paper concerns on the linear non-autonomous control systems with state x, input u, and output y:

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0, \\ &y(t) = C(t)x(t), \quad t \geq 0, \end{aligned} \tag{1}$$

where $A(t)$ is a linear closed operator in X with domain $\mathcal{D}(A(t)) = \mathcal{D}$, independent of t and dense in X; $B(t) : U \to X$ and $C(t) : X \to Y$ are bounded operators such that $B(\cdot) \in L_\infty(\mathbb{R}^+, \mathcal{L}_s(U, X))$ and $C(\cdot) \in L_\infty(\mathbb{R}^+, \mathcal{L}_s(X, Y))$, where $\mathcal{L}_s(V, W)$ and $L_\infty(\Omega, W)$ denote the space of bounded operators from V to W equipped with strong operator topology and the space of bounded measurable functions from Ω to W provided with essential supremum norm, respectively. The state linear system (1) is denoted by $(A(t), B(t), C(t))$. Symbol $(A(t), B(t), -)$ and $(A(t), -, C(t))$ denote the state linear system (1) if $C(t) = 0$ and $B(t) = 0$, respectively. In particular, this paper shall investigate stability of the state linear system (1) where each $A(t)$ is a generalized Riesz-spectral operator [1] using transfer function, stabilizability, and detectability . The following explanations give some reasons why these investigations are important for the systems. In the autonomous control system (A, B, C), where A is an infinitesimal generator of a C_0-semigroup $T(t)$, there exist special relationships among input, state, and output [2]. Controllability map specifies the relationship between the input and the state, observability map specifies the relationship between an initial state and the output. The relationship between the input and the output can be characterized by transfer function, a linear map that specifies relationship between the Laplace transform of the inputs and outputs. The following studies show how urgency of the transfer function is in the linear system. In the infinite-dimensional autonomous system, external stability is indicated by boundedness of its transfer function [3]. Weiss [4] have proved a formula for the transfer function of a regular linear system, which is similar to the formula in the finite dimensions. As an extension of results of [4], Staffans and Weiss [5] have generalized the results

Received May 31, 2018; Revised October 5, 2018; Accepted November 3, 2018

of regular linear systems to well-posed linear systems. They have also introduced the Lax-Phillips semigroup induced by a well-posed linear system. Weiss [6] have studied four transformations which lead from one well-posed linear system to another: time-inversion, flow-inversion, time-flow-inversion, and duality. In particular, a well-posed linear system is flow-invertible if and only if the transfer function of the system has a uniformly bounded inverse on some right half-plane. Finally, Partington [7] have given simple sufficient conditions for a space of functions on $(0, \infty)$ such that all shift-invariant operators defined on the space are represented by transfer functions. Controllability and stability are qualitative control problems which are important aspects of the theory of control systems. Kalman *et al.* [8] had initiated the theory for the finite dimensional of autonomous systems. Recently, the theory was generalized into controllability and stabilizability of the non-autonomous control systems of various applications, see; e.g. [9, 10, 11], and the references therein. The concept of the stabilizability is to find an admissible control $u(t)$ such that the corresponding solution $x(t)$ of the system has some required properties. If the stabilizability is identified by null controllability, system (1) is said to be stabilizable if there exists a control $u(t) = F(t)x(t)$ such that the zero solution of the closed-loop system

$$\dot{x}(t) = [A(t) + B(t)F(t)]x(t), \quad t \geq 0,$$

is asymptotically stable in the Lyapunov sense. In this case, $u(t) = F(t)x(t)$ is called the stabilizing feedback control. In particular for autonomous system $(A, B, -)$, where A is an infinitesimal generator of an exponentially stable C_0-semigroup $T(t)$, $(A, B, -)$ is stabilizable if operator $A+BF$ is an infinitesimal generator of an exponentially stable C_0-semigroup $T_{BF}(t)$ [2]. In the finite-dimensional autonomous control system, Kalman *et al.* [8] and Wonham [12] had shown that the system is stabilizable if it is null controllable in a finite time. But, it does not hold for the converse. Furthermore, if the system is completely stabilizable, then it is null controllable in a finite time. For finite-dimensional non-autonomous control systems, Ikeda *et al.* [13] proved that the system is completely stabilizable whence it is null controllable. Generalizations of the results of the stabilizability for the finite-dimensional systems into infinite-dimensional systems have been successfully done. Extending the Lyapunov equation in Banach spaces, Phat and Kiet [14] specified the relationship between stability and exact null controllability in the autonomous systems. Guo *et al.* [15] proved the existence of the infinitesimal generator of the perturbation semigroup. For neutral type linear systems in Hilbert spaces, Rabah *et al.* [16] proved that exact null controllability implies the complete stabilizability. In the paper, unbounded feedback is also investigated. In the non-autonomous systems, Hinrichsen and Pritchard [17] investigated radius stability for the systems under structured non-autonomous perturbations. Niamsup and Phat [18] proved that exact null controllability implies the complete stabilizability for linear non-autonomous systems in Hilbert spaces. Leiva and Barcenas [19] have introduced a C_0-quasi semigroup as a new approach to investigate the non-autonomous systems. In this context, $A(t)$ is an infinitesimal generator of a C_0-quasi semigroup on a Banach space. Sutrima *et al.* [20] and Sutrima *et al.* [21] investigated the advanced properties and some types of stabilities of the C_0-quasi semigroups in Banach spaces, respectively. Even Barcenas *et al.* [22] have characterized the controllability of the non-autonomous control systems using the quasi semigroup approach, although it is still limited to the autonomous controls. In particular, Sutrima *et al.* [1] characterized the controllability and observability of non-autonomous Riesz-spectral systems. The references explain that the transfer function, stabilizability, and detectability of the control systems are important indicators for the stability. Unfortunately, there are no studies of these concepts in the non-autonomous Riesz-spectral systems. Therefore, this paper concerns on investigations of the transfer function, stabilizability, and detectability of the non-autonomous Riesz-spectral systems that have not been investgated at this time yet. These investigations use the C_0-quasi semigroup approach.

2. Proposed Methods and Discussion

2.1. Transfer Function

We recall the definition of a non-autonomous Riesz-spectral system that refers to Sutrima *et al.* [1]. The definition of a Riesz-spectral operator follows [2].

Definition 1. The linear non-autonomous system $(A(t), B(t), C(t))$ is called a non-autonomous Riesz-spectral system if $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup which has an expression

$$A(t) = a(t)A \tag{2}$$

where A is a Riesz-spectral operator on X and a is a bounded continuous function such that $a(t) > 0$ for all $t \geq 0$.

By expression (2), for every $t \geq 0$, we see that $A(t)$ and A have the common domain and eigenvectors. In fact, if λ_n, $n \in \mathbb{N}$, is an eigenvalue of A, then $a(t)\lambda_n$ is the eigenvalue of $A(t)$ of (2). Hence, in general $A(t)$ may have the non-simple eigenvalues. In the non-autonomous Riesz-spectral system $(A(t), B(t), C(t))$, where $A(t)$ is an infinitesimal generator of a C_0-quasi semigroup $R(t, s)$, relationships among input, state, and output are determined by $R(t, s)$. The relationships of state-input and state-output of the system have been discussed by Sutrima *et al.* [1]. In this subsection, we focus on characterizing the relationship between the input and output of the system using the transfer function, where $B(t) \in L_2(U, X)$ and $C(t) \in L_2(X, Y)$ for all $t \geq 0$. We define multiplicative operators $\mathcal{B} : L_2(\mathbb{R}^+, U) \to L_2(\mathbb{R}^+, X)$ and $\mathcal{C} : L_2(\mathbb{R}^+, X) \to L_2(\mathbb{R}^+, Y)$ by:

$$(\mathcal{B}u)(t) = B(t)u(t) \quad \text{and} \quad (\mathcal{C}x)(t) = C(t)x(t), \quad t \geq 0,$$

respectively. We see that the operators $\mathcal{B}$ and $\mathcal{C}$ are bounded. The definition of the transfer function of the system $(A(t), B(t), C(t))$ follows the definition for the autonomous systems of [2].

Definition 2. Let $(A(t), B(t), C(t))$ be the non-autonomous Riesz-spectral system with zero initial state. If there exists a real α such that $\hat{y}(s) = G(s)\hat{u}(s)$ for $\mathrm{Re}\,(s) > \alpha$, where $\hat{u}(s)$ and $\hat{y}(s)$ denote the Laplace transforms of u and y, respectively, and $G(s)$ is a $\mathcal{L}(U, Y)$-valued function of complex variable defined for $\mathrm{Re}\,(s) > \alpha$, then G is called transfer function of the system $(A(t), B(t), C(t))$. The impulse response h of $(A(t), B(t), C(t))$ is defined as the Laplace inverse transform of G.

The transfer function of the non-autonomous Riesz-spectral systems with finite-rank inputs and outputs can be stated in eigenvalues and eigenvectors of the Riesz-spectral operator.

Theorem 3. The transfer function G and impulse response h of the non-autonomous Riesz-spectral system $(A(t), B(t), C(t))$ exist and are given by

$$G(s) = \mathcal{C}a(\cdot)\,(sI - a(\cdot)A)^{-1}\,\mathcal{B} \quad \text{on} \quad \rho(A(\cdot)),$$

and

$$h(t) = \begin{cases} \mathcal{C}T(t)\mathcal{B}, & t \geq 0 \\ 0, & t < 0, \end{cases}$$

where $T(t)$ is a C_0-semigroup with the infinitesimal generator A.

Proof. By definition of Riesz-spectral operator and Theorem 3 of [1], we have that the resolvent set of A is connected, so $\rho_\infty(A) = \rho(A)$. We shall verify the existences of the transfer function and impulse response. Let $R(t, s)$ be a C_0-quasi semigroup with infinitesimal generator $A(t)$ of the form (2). For $\mathrm{Re}\,(s) > \omega_0 a_1$, where ω_0 is the growth bound of a C_0-semigroup $T(t)$ with infinitesimal generator A and $a_1 := \sup_{t\geq 0} a(t)$. By Lemma 2.1.11 of [2], we have

$$\mathcal{C}a(\cdot)\,(sI - a(\cdot)A)^{-1}\,\mathcal{B}u = \mathcal{C}\int_0^\infty e^{-\frac{s}{a(\cdot)}\sigma}T(\sigma)\mathcal{B}ud\sigma = \int_0^\infty e^{-\frac{s}{a(\cdot)}\sigma}\mathcal{C}T(\sigma)\mathcal{B}ud\sigma$$

for all $u \in L_2(\mathbb{R}^+, U)$ and $\mathrm{Re}\,(s) > \omega_0 a_1$. As in the proof of Lemma 2.5.6 of [2], these s can be extended to all $s \in \rho(a(\cdot)A)$.

Corollary 4. Let $A(t)$ be an operator of the form (2), where A is a Riesz-spectral operator with eigenvalues $\{\lambda_n \in \mathbb{C} : n \in \mathbb{N}\}$. If $B : \mathbb{R}^+ \to \mathcal{L}(\mathbb{C}^m, X)$ and $C : \mathbb{R}^+ \to \mathcal{L}(X, \mathbb{C}^k)$ such that

$B(t) \in \mathcal{L}(\mathbb{C}^m, X)$ and $C(t) \in \mathcal{L}(X, \mathbb{C}^k)$, then the transfer function and impulse response of the system $(A(t), B(t), C(t))$ are given by

$$G(s) = \sum_{n=1}^{\infty} \frac{a(\cdot)}{s - a(\cdot)\lambda_n} \mathcal{C}\phi_n \left(\overline{\mathcal{B}^*\psi_n}\right)^{tr} \quad \text{for} \quad s \in \rho(A(\cdot)) \tag{3}$$

$$h(t) = \begin{cases} \sum_{n=1}^{\infty} e^{\frac{\lambda_n}{a(\cdot)}t} \mathcal{C}\phi_n \left(\overline{\mathcal{B}^*\psi_n}\right)^{tr}, & t \geq 0 \\ 0, & t < 0, \end{cases} \tag{4}$$

where ϕ_n and ψ_n are the corresponding eigenvectors of A and A^*, respectively. Symbol W^{tr} denotes transpose of W.
Proof. By representation of $(sI - a(\cdot)A)^{-1}$ of Theorem 3 of [1] and facts that operators $B(t)$ and $C(t)$ are bounded for all $t \geq 0$, then from Theorem 3 for $s \in \rho(A(\cdot))$ we have

$$\begin{aligned} G(s)u &= \mathcal{C}a(\cdot)\left(sI - a(\cdot)A\right)^{-1}\mathcal{B}u \\ &= \mathcal{C}\left[\lim_{N\to\infty} \sum_{n=1}^{N} \frac{a(\cdot)}{s - a(\cdot)\lambda_n} \langle \mathcal{B}u, \psi_n \rangle \phi_n\right] = \sum_{n=1}^{\infty} \frac{a(\cdot)}{s - a(\cdot)\lambda_n} \mathcal{C}\phi_n \left(\overline{\mathcal{B}^*\psi_n}\right)^{tr}. \end{aligned}$$

Here, we use the property $\langle v, w \rangle_{\mathbb{C}^m} = \overline{w}^{tr} v$. Thus, expression (3) is proved. By a similar argument and condition (c) of Theorem 3 of [1] we have expression (4) for $h(t)$. The following example illustrates the transfer function of a non-autonomous Riesz-spectral system. The example is modified from Example 4.3.11 of [2].

Example 5. Consider the controlled non-autonomous heat equation on the interval $[0, 1]$,

$$\begin{aligned} \frac{\partial x}{\partial t}(t,\xi) &= a(t)\frac{\partial^2 x}{\partial \xi^2}(t,\xi) + 2b(t)u(t)\chi_{[\frac{1}{2},1]}(\xi), \quad 0 < \xi < 1, \quad t \geq 0, \\ \frac{\partial x}{\partial \xi}(t,0) &= \frac{\partial x}{\partial \xi}(t,1) = 0, \\ y(t) &= 2c(t)\int_0^{1/2} x(t,\xi)d\xi, \end{aligned} \tag{5}$$

where $a : \mathbb{R}^+ \to \mathbb{R}$ is a boundedly uniformly continuous positive function and $b : \mathbb{R}^+ \to \mathbb{C}$ is a bounded continuous function.

We verify the transfer function and impulse response of the governed system. Setting $X = L_2[0, 1]$ and $U = Y = \mathbb{C}$, the problem (5) is a non-autonomous Riesz-spectral system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \quad t \geq 0, \end{aligned} \tag{6}$$

where $A(t) = a(t)A$ with $Ax = \dfrac{d^2x}{d\xi^2}$ on domain

$$\mathcal{D} = \mathcal{D}(A) = \{x \in X : x,\ \tfrac{dx}{d\xi} \text{ are absolutely continuous},\ \tfrac{d^2x}{d\xi^2} \in X,\ \tfrac{dx}{d\xi}(0) = \tfrac{dx}{d\xi}x(1) = 0\},$$

$$B(t)u(t) = 2b(t)u(t)\chi_{[\frac{1}{2},1]}(\xi), \quad \text{and} \quad C(t)x(t) = 2c(t)\int_0^{1/2} x(t,\xi)d\xi.$$

The eigenvalues and eigenvectors of A are $\{0, -n^2\pi^2 : n \in \mathbb{N}\}$ and $\{1, \sqrt{2}\cos(n\pi\xi) : n \in \mathbb{N}\}$, respectively. It is easy to show that A is a self-adjoint Riesz-spectral operator with its Riesz basis $\{1, \sqrt{2}\cos(n\pi\xi) : n \in \mathbb{N}\}$. In this case we have

$$B^*(t)x = \int_0^1 x\chi_{[\frac{1}{2},1]}(\xi)d\xi,$$

for all $\in X$ and $t \geq 0$. In virtue of (3) evaluating at t, the transfer function of the system (6) is given by

$$G(s)(t) = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{2a(t)[\cos(n\pi) - 1]}{[s + a(t)n^2\pi^2](n\pi)^2},$$

and evaluating at s in (4), we have the impulse response

$$h(t)(s) = 1 + \sum_{n=1}^{\infty} \frac{2[\cos(n\pi) - 1]}{(n\pi)^2} e^{-\frac{(n\pi)^2}{a(s)} t},$$

on $t \geq 0$ and zero elsewhere, respectively.

2.2. Stabilizability

We first recall the definition of uniformly exponentially stable C_0-quasi semigroup that refers to [21, 23]. The concept play an important role in characterizing stabilizability of the non-autonomous Riesz-spectral systems.

Definition 6. A C_0-quasi semigroup $R(t, s)$ is said to be:

(a) uniformly exponentially stable on a Banach space X if there exist constants $\alpha > 0$ and $N \geq 1$ such that

$$\|R(t, s)x\| \leq Ne^{-\alpha s}\|x\|, \tag{7}$$

for all $t, s \geq 0$ and $x \in X$;

(b) β-uniformly exponentially stable on a Banach space X if (7) holds for $-\alpha < \beta$.

A constant α is called decay rate and the supremum over all possible values of α is called stability margin of $R(t, s)$. Indeed, the stability margin is minus of the uniform growth bound $\omega_0(R)$ defined

$$\omega_0(R) = \inf_{t \geq 0} \omega_0(t),$$

where $\omega_0(t) = \inf\limits_{s>0} \left(\frac{1}{s} \log \|R(t, s)\|\right)$. We see that $R(t, s)$ is β-uniformly exponentially stable if its stability margin is at least $-\beta$. We give two preliminary results which are urgent in discussing the stabilizability.

Theorem 7. Let $R(t, s)$ be a C_0-quasi semigroup on a Banach space X. The $R(t, s)$ is uniformly exponentially stable on X if and only if $\omega_0(R) < 0$.

Proof. By taking log on (7), we have the assertion.

Theorem 8. Let $A(t)$ be an infinitesimal generator of C_0-quasi semigroup $R(t, s)$ on a Banach space X. If $B(\cdot) \in L_\infty(\mathbb{R}^+, \mathcal{L}_s(X))$, then there exists a uniquely C_0-quasi semigroup $R_B(t, s)$ with its infinitesimal generator $A(t) + B(t)$ such that

$$R_B(r, t)x = R(r, t)x + \int_0^t R(r + s, t - s)B(r + s)R_B(r, s)xds, \tag{8}$$

for all $t, r, s \geq 0$ with $t \geq s$ and $x \in X$. Moreover, if $\|R(r, t)\| \leq M(t)$, then

$$\|R_B(r, t)\| \leq M(t)e^{\|B\|M(t)t}.$$

Proof. We define

$$R_0(r, t)x = R(r, t)x,$$
$$R_n(r, t)x = \int_0^t R(r + s, t - s)B(r + s)R_{n-1}(r, s)xds, \tag{9}$$

for all $t, r, s \geq 0$ with $t \geq s$, $x \in X$, and $n \in \mathbb{N}$, and

$$R_B(r,t) = \sum_{n=0}^{\infty} R_n(r,t), \tag{10}$$

Following the proof of Theorem 2.4 of [19], we obtain the assertions. The investigations of stabilizability and detectability of the non-autonomous Riesz-spectral systems are generalizations of the concepts of stabilizability and detectability for the autonomous systems that had been developed by Curtain and Zwart [2].

Definition 9. The non-autonomous Riesz-spectral system $(A(t), B(t), C(t))$ is said to be:

(a) stabilizable if there exsits an operator $F \in L_\infty(\mathbb{R}^+, \mathcal{L}_s(X,U))$ such that $A(t) + B(t)F(t)$, $t \geq 0$, is an infinitesimal generator of a uniformly exponentially stable C_0-quasi semigroup $R_{BF}(t,s)$. The operator F is called a stabilizing feedback operator;

(b) detectable if there exists an operator $K \in L_\infty(\mathbb{R}^+, \mathcal{L}_s(Y,X))$ such that $A(t) + K(t)C(t)$, $t \geq 0$, is an infinitesimal generator of a uniformly exponentially stable C_0-quasi semigroup $R_{KC}(t,s)$. The operator K is called an output injection operator.

If the quasi semigroup $R_{BF}(t,s)$ is β-uniformly exponentially stable, we say that the system $(A(t), B(t), -)$ is β-stablizable. If $R_{KC}(t,s)$ is β-uniformly exponentially stable, we say that the system $(A(t), -, C(t))$ is β-detectable.

For $\delta \in \mathbb{R}$, we can decompose the spectrum of A in complex plane into two distinct parts:

$$\sigma_\delta^+(A) := \sigma(A) \cap \overline{\mathbb{C}_\delta^+};\ \ \mathbb{C}_\delta^+ = \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) > \delta\}$$
$$\sigma_\delta^-(A) := \sigma(A) \cap \mathbb{C}_\delta^-;\ \ \mathbb{C}_\delta^- = \{\lambda \in \mathbb{C} : \mathrm{Re}\,(\lambda) < \delta\}.$$

In the autonomous case, if B has finite-rank and the system $(A, B, -)$ is stabilizable, then we can decompose the spectrum of A into a δ-stable part and a δ-unstable part which comprises eigenvalues with finite multiplicity. In other word, A has at most finitely many eigenvalues in $\overline{\mathbb{C}_\delta^+}$. We shall apply the decomposition of the spectrum to the non-autonomous Riesz-spectral systems.

Definition 10. An operator A is said to be satisfying the spectrum decomposition assumption at δ if $\sigma_\delta^+(A)$ is bounded and separated from $\sigma_\delta^-(A)$ in such way that a rectifiable, simple, closed curve, Γ_δ, can be drawn so as to enclose an open set containing $\sigma_\delta^+(A)$ in its interior and $\sigma_\delta^-(A)$ in its exterior.

According to Definition 10, we have that classes of Riesz-spectral operators with a pure point spectrum and only finitely many eigenvalues in $\sigma_\delta^+(A)$ satisfy the spectrum decomposition assumption. For every $t \geq 0$ we define the spectral projection $P_\delta(t)$ on X by

$$P_\delta(t)x = \frac{1}{2\pi i}\int_{\Gamma_\delta} (\lambda I - A(t))^{-1} x \, d\lambda, \tag{11}$$

for all $x \in X$, where Γ_δ is traversed once in the positive direction (counterclockwise). By this operator, we can decompose any Hilbert space X to be:

$$X = X_\delta^+ \oplus X_\delta^-, \ \text{ where } \ X_\delta^+ := P_\delta(t)X \ \text{ and } \ X_\delta^- := (I - P_\delta(t))X. \tag{12}$$

By this decomposition, we denote

$$A(t) = \begin{pmatrix} A_\delta^+(t) & 0 \\ 0 & A_\delta^-(t) \end{pmatrix}, \qquad R(t,s) = \begin{pmatrix} R_\delta^+(t,s) & 0 \\ 0 & R_\delta^-(t,s) \end{pmatrix} \tag{13}$$

$$B(t) = \begin{pmatrix} B_\delta^+(t) \\ B_\delta^-(t) \end{pmatrix}, \qquad C(t) = \begin{pmatrix} C_\delta^+(t) & C_\delta^-(t) \end{pmatrix}, \tag{14}$$

where $B_\delta^+(t) = P_\delta(t)B(t) \in \mathcal{L}(U, X_\delta^+)$, $B_\delta^-(t) = (I - P_\delta(t))B(t) \in \mathcal{L}(U, X_\delta^-)$, $C_\delta^+(t) = C(t)P_\delta(t) \in \mathcal{L}(X_\delta^+, Y)$, and $C_\delta^-(t) = C(t)(I - P_\delta(t)) \in \mathcal{L}(X_\delta^-, Y)$. In virtue of the decomposition, we can express the system $(A(t), B(t), C(t))$ as the vector sum of the two subsystems: $(A_\delta^+(t), B_\delta^+(t), C_\delta^+(t))$ on X_δ^+ and $(A_\delta^-(t), B_\delta^-(t), C_\delta^-(t))$ on X_δ^-. In particular, if the input operator $B(\cdot)$ has finite-rank, then the subsystem $(A_\delta^+(t), B_\delta^+(t), -)$ has a finite dimension.

Theorem 11. Let $(A(t), B(t), -)$ be a non-autonomous Riesz-spectral system on the state space X where $B(\cdot)$ has a finite rank. If A satisfies the spectrum decomposition assumption at β, X_β^+ has a finite dimension, $R_\beta^-(t,s)$ is β-uniformly exponentially stable, and subsystem $(A_\beta^+(t), B_\beta^+(t), -)$ is controllable, then the system $(A(t), B(t), -)$ β-stabilizable. In this case, a β-stabilizing feedback operator is given by $F(t) = F_0(t)P_\beta(t)$, where F_0 is a β-stabilizing feedback operator for $(A_\beta^+(t), B_\beta^+(t), -)$.

Proof. Since $(A_\beta^+(t), B_\beta^+(t), -)$ is controllable, there exists a feedback operator balik $F_0(\cdot) \in L_\infty(\mathbb{R}^+, \mathcal{L}_s(X_\beta^+, U)$ such that the spectrum of $A_\beta^+(t) + B_\beta^+(t)F_0(t)$ in $\mathbb{C}_\beta^-$ for all $t \geq 0$. We choose a feedback operator $F(\cdot) = (F_0(\cdot), 0) \in L_\infty(\mathbb{R}^+, \mathcal{L}_s(X, U))$ for the system $(A(t), B(t), -)$. According to Theorem 8, the perturbed operator $A(t) + B(t)F(t) = \begin{pmatrix} A_\beta^+(t) + B_\beta^+(t)F_0(t) & 0 \\ B_\beta^-(t)F_0(t) & A_\beta^-(t) \end{pmatrix}$ is the infinitesimal generator of a C_0-quasi semigroup $R(t,s)$. Moreover, if β_1 is the growth bound of $R(t,s)$, then β_1 is the maximum of that of the quasi semigroups generated by $A_\beta^+(t) + B_\beta^+(t)F_0(t)$ and $A_\beta^-(t)$. Therefore, $\beta_1 < \beta$, i.e. the system $(A(t), B(t), -)$ is β-stabilizable. By the dual property between the stabilizability and detectability, Theorem 11 provides the sufficiency for the β-detectability.

Theorem 12. Let $(A(t), -, C(t))$ be the non-autonomous Riesz-spectral system on the state space X where $C(\cdot)$ has a finite rank. If A satisfies the spectrum decomposition assumption at β, X_β^+ has a finite dimension, $R_\beta^-(t,s)$ is β-uniformly exponentially stable, and $(A_\beta^+(t), -, C_\beta^+(t))$ is observable, then $(A(t), -, C(t))$ is β-detectable. In this case, β-stabilizing output injection operator is given by $K(t) = i_\beta(t)K_0(t)$, where K_0 is an operator such that $A_\beta^+(t) + K_0(t)C_\beta^+(t)$ is the infinitesimal generator of the β-uniformly exponentially stable C_0-quasi semigroup and $i_\beta(t)$ is an injection operator from X_β^+ to X.

Proof. By the dual concept, we have that the system $(A(t), -, C(t))$ is β-detectability if and only if $(A^*(t), C^*(t), -)$ is β-stabilizability. From Theorem 11, A^* satisfies the spectrum decomposition assumption at β. The corresponding spectral projection is given by

$$P_\beta^*(t)x = \frac{1}{2\pi i}\int_{\Gamma_\beta} (\lambda I - A^*(t))^{-1} x d\lambda,$$

for all $x \in X$. We can choose such that Γ_β is symmetric with respect to the real axis. Hence, the decomposition of $(A^*(t), C^*(t), -)$ is the adjoint of the decomposition of $(A(t), -, C(t))$. By the dual argument, we have the required results.

Example 13. Consider the non-autonomous Riesz-spectral system in Example 5. Using all of the agreement there, we show that the system is stabilizable and detectable.

We have the spectral decomposition of $A(t)$:

$$A(t)x = a(t)\sum_{n=1}^{\infty} -(n\pi)^2\langle x, \phi_n\rangle\phi_n \quad \text{for} \quad x \in \mathcal{D},$$

where $\phi_n(\xi) = \sqrt{2}\cos(n\pi\xi)$, and the family of the operators generates a C_0-quasi semigroup $R(t,s)$ given by

$$R(t,s)x = \langle x, 1\rangle 1 + \sum_{n=1}^{\infty} e^{-(n\pi)^2(g(t+s)-g(t))}\langle x, \phi_n\rangle\phi_n,$$

where $g(t) = \int_0^t a(s)ds$. Since the set of eigenvalues of A has an upper bound, then A satisfies the spectrum assumption at any real β. Suppose we choose $\beta = -2$. In this case we have $\sigma^+_{-2}(A) = \{0\}$ and so there exists a closed simple curve Γ_{-2} that encloses the eigenvalue 0. For any $x \in X$ and $t \geq 0$, Cauchy's Theorem gives

$$P_{-2}(t)x = \frac{1}{2\pi i}\int_{\Gamma_{-2}} (\lambda I - a(t)A)^{-1}x d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{-2}} \frac{1}{\lambda}\langle x, 1\rangle d\lambda = \langle x, 1\rangle.$$

In virtue of (12), we have that X^+_{-2} has dimension 1. Moreover, in the subspace we have $(A^+_{-2}(t), B^+_{-2}(t), C^+_{-2}(t)) = (0, I, I)$ which are controllable and observable. There exists $F_0(\cdot) \in L_\infty(\mathbb{R}^+, \mathcal{L}_s(X^+_{-2}, U))$ such that the spectrum of $A^+_{-2}(t) + B^+_{-2}(t)F_0(t)$ in $\mathbb{C}^-_{-2}$ for all $t \geq 0$. We can choose $F_0(t)x = -3\langle x, 1\rangle$ for all $x \in X^+_{-2}$. Theorem 11 concludes that $(A(t), B(t), C(t))$ is (-2)-stabilizable with the feedback operator $u = F(t)z$, where $F(t) = \begin{pmatrix} F_0(t) & 0 \end{pmatrix} = \begin{pmatrix} -3I & 0 \end{pmatrix}$. By the duality, Theorem 12 shows that the system $(A(t), -, C(t))$ is (-2)-detectable with output injection operator $K(t) = \begin{pmatrix} -3I \\ 0 \end{pmatrix}$ such that $K(t)y = -3y1$. The decay constants of the quasi semigroup generated by $A(t) + B(t)F(t)$ and $A(t) + L(t)C(t)$ are 3. The following theorem is a similar result with Theorem 13 of [1] for controllability and observability of the non-autonomous Riesz-spectral systems on Hilbert spaces.

Theorem 14. Let $(A(t), B(t), C(t))$ be a non-autonomous Riesz-spectral system. Necessary and sufficient conditions for $(A(t), B(t), -)$ is β-stabilizable are that there exists an $\epsilon > 0$ such that $\sigma^+_{\beta-\epsilon}(A)$ comprises at most finitely many eigenvalues and

$$\operatorname{rank}(\langle b_1(t), \varphi_n\rangle, \ldots, \langle b_m(t), \varphi_n\rangle) = 1, \tag{15}$$

for all n such that $\lambda_n \in \sigma^+_{\beta-\epsilon}(A)$ and $t \geq 0$. Necessary and sufficient conditions for $(A(t), -, C(t))$ is β-detectable are that there exists an $\epsilon > 0$ such that $\sigma^+_{\beta-\epsilon}(A)$ comprises at most finitely many eigenvalues and

$$\operatorname{rank}(\langle \phi_n, c_1(t)\rangle, \ldots, \langle \phi_n, c_k(t)\rangle) = 1, \tag{16}$$

for all n such that $\lambda_n \in \sigma^+_{\beta-\epsilon}(A)$ and $t \geq 0$.

Proof. We only need to prove necessary and sufficient conditions for β-stabilizability. For β-detectability follows the dual argument of the system. *Sufficiency for β-stabilizability.* Since $\sigma^+_{\beta-\epsilon}(A)$ only contains at most finitely many eigenvalues of A, then $X^+_{\beta-\epsilon}$ has a finite dimension and A satisfies the spectrum decomposition assumption at $\beta - \epsilon$. By condition (a) of Theorem 3 of [1] and Cauchy's Theorem, for any $t \geq 0$ we have

$$P_{\beta-\epsilon}(t)x = \frac{1}{a(t)}\sum_{\lambda_n \in \sigma^+_{\beta-\epsilon}} \langle x, \psi_n\rangle \phi_n.$$

Since $A(t)$ and A have common eigenvectors for every $t \geq 0$, again Theorem 3 of [1] gives

$$X^+_{\beta-\epsilon} = \operatorname*{span}_{\lambda_n \in \sigma^+_{\beta-\epsilon}}\{\phi_n\}, \quad X^-_{\beta-\epsilon} = \overline{\operatorname*{span}}_{\lambda_n \in \sigma^-_{\beta-\epsilon}}\{\phi_n\}, \quad R^-_{\beta-\epsilon}(t,s) = \sum_{\lambda_n \in \sigma^-_{\beta-\epsilon}} e^{\lambda_n(g(t+s)-g(t))}\langle x, \psi_n\rangle \phi_n,$$

$$A^+_{\beta-\epsilon}(t)x = a(t)\sum_{\lambda_n \in \sigma^+_{\beta-\epsilon}} \lambda_n\langle x, \psi_n\rangle \phi_n, \quad \text{and} \quad B^+_{\beta-\epsilon}(t)u = \sum_{\lambda_n \in \sigma^+_{\beta-\epsilon}} \langle B(t)u, \psi_n\rangle \phi_n, \tag{17}$$

where $g(t) = \int_0^t a(s)ds$. This result shows that $R^-_{\beta-\epsilon}(t,s)$ is a C_0-quasi semigroup corresponding to the Riesz-spectral operator $A^-_{\beta-\epsilon}$ on $X^-_{\beta-\epsilon}$. Consequently, $\omega_0(R^-_{\beta-\epsilon}) < 0$ and Theorem 7 states that $R^-_{\beta-\epsilon}(t,s)$ is β-uniformly exponentially stable. We need prove that $(A^+_{\beta-\epsilon}(t), B^+_{\beta-\epsilon}(t), -)$ is controllable. The reachibility subspace of $(A^+_{\beta-\epsilon}(t), B^+_{\beta-\epsilon}(t), -)$ is the smallest $A^+_{\beta-\epsilon}$-invariant subspace of $X^+_{\beta-\epsilon}$ which contains $\operatorname{ran} B^+_{\beta-\epsilon}(t)$. In virtue of Lemma 2.5.6 of [2], this subspace is spanned by the eigenvectors of $A^+_{\beta-\epsilon}$. Hence, if this subspace does not equal with the state space $X^+_{\beta-\epsilon}$, then there exists a $\lambda_j \in \sigma^+_{\beta-\epsilon}(A)$ such that $\phi_j \notin X^+_{\beta-\epsilon}$. Therefore, its biorthogonal element ψ_j is orthogonal to the reachibility subspace, in particular $\langle B(t)u, \psi_j\rangle = 0$ for every $u \in \mathbb{C}^m$ and $t \geq 0$. This contradicts to (15), and so $(A^+_{\beta-\epsilon}(t), B^+_{\beta-\epsilon}(t), -)$ is controllable. Thus, Theorem 11 shows that the system $(A(t), B(t), -)$ is β-stabilizable. *Necessity for β-stabilizability*. From Definition 6 and Definition 9, if $(A(t), B(t), -)$ is β-stabilizable, then there exists $\epsilon > 0$ such that the system is also $(\beta - \epsilon)$-stabilizable. In virtue of Theorem 11, A satisfies the spectrum decomposition assumption at $\beta - \epsilon$. Moreover, the subspace $X^+_{\beta-\epsilon}$ is $R(t, s)$-invariant. Lemma 2.5.8 of [2] gives

$$X^+_{\beta-\epsilon} = \overline{\operatorname{span}}_{n\in\mathbb{J}}\{\phi_n\}.$$

Since $X^+_{\beta-\epsilon}$ has a finite dimension, then $\mathbb{J}$ contains at most finitely many elements. So the spectrum of $A^+_{\beta-\epsilon} = A|_{X^+_{\beta-\epsilon}}$ is contained in $\mathbb{C}^+_{\beta-\epsilon}$ and the spectrum of $A^-_{\beta-\epsilon} = A|_{X^-_{\beta-\epsilon}}$ is contained in $\mathbb{C}^-_{\beta-\epsilon}$. This concludes that the index set $\mathbb{J}$ equals with the set $\{n \in \mathbb{N} : \lambda_n \in \sigma^+_{\beta-\epsilon}(A)\}$. Thus, $\sigma^+_{\beta-\epsilon}(A)\}$ comprises at most finitely many eigenvalues. By (17) and Theorem 11, we conclude that $(A^+_{\beta-\epsilon}(t), B^+_{\beta-\epsilon}(t), -)$ is controllable. Suppose that the condition (15) does not hold. There exists $\lambda_n \in \sigma^+_{\beta-\epsilon}(A)$ such that $\langle B(t)u, \psi_n\rangle = 0$ for all $u \in \mathbb{C}^m$. This states that the reachibility subspace of $(A^+_{\beta-\epsilon}(t), B^+_{\beta-\epsilon}(t), -)$ does not equal to $X^+_{\beta-\epsilon}$, that is $(A^+_{\beta-\epsilon}(t), B^+_{\beta-\epsilon}(t), -)$ is not controllable. This gives a contradiction. In practice, Theorem 14 is more applicable than Theorem 11 and 12 in characterizing the stabilizability and detectability of the non-autonomous Riesz-spectral systems. We return to Example 13. For $\beta = -2$, we can choose $\epsilon = 1$ such that $\sigma^+_{-3}(A) = \{0\}$ and the corresponding eigenvector is $\phi_0(\xi) = 1$. In this case, it is obvious that

$$\int_0^1 b(t)\phi_0(\xi)d\xi \neq 0 \quad \text{and} \quad \int_0^1 c(t)\phi_0(\xi)d\xi \neq 0,$$

for all $t \geq 0$, i.e. the conditions (15) and (16) are confirmed. Hence, the system $(A(t), B(t), C(t))$ are stabilizable and detectable.

3. Conclusion

The concepts of the transfer function, stabilizability, and detectability of the autonomous systems can be generalized to the non-autonomous Riesz-spectral systems. The results are alternative considerations in analyzing the related control problems. There are opportunities to generalize these results to the generally non-autonomous systems including the time-dependent domain.

Acknowledgement

The authors are grateful to Research Institution and Community Service of Universitas Sebelas Maret Surakarta for funding and to the reviewers for helpful comments.

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