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Iteration Methods for Linear Systems with Positive Definite Matrix

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Abstract

Two classical first order iteration methods, Richardson iteration and HSS iteration for linear systems with positive definite matrix, are demonstrated. Theoretical analyses and computational results show that the HSS iteration has the advantages of fast convergence speed, high computation efficiency, and without requirement of symmetry.

Keywords: *iteration method, Richardson iteration, HSS iteration, symmetric positive definite, generalized positive definite*

1. Introduction

Consider a linear system:

$$Ax = b \tag{1}$$

Where $A \in R^{n\times n}$ is nonsingular. Instead of solving system (1) by a direct method, e.g., by Gaussian elimination, in many cases it may be advantageous to use an iterative method of solution. This is particularly true when the dimension *n* of system (1) is very large. This paper provides some classical iterative methods.

The general scheme of what is known as the first order iteration process consists of successively computing the terms of the sequence:

$$x^{k+1} = Bx^k + \varphi, \quad k = 0,1,2,\cdots \tag{2}$$

Where the initial guess $x^0 \in R^n$ is specified arbitrarily [1]. The matrix B is known as the iteration matrix. Clearly, if the sequence x^k converges, i.e., if there is a limit: $\lim\limits_{k\to\infty} x^k = x$, then x is the solution of system (1).

Iterative method (2) is first order because the next iterate x^{k+1} depends only on one previous iterate, x^k. Following we give two classical first order iteration, Richardson iteration and HSS iteration. The basic contribution of present work is to validate the performance of iteration method for linear systems with positive definite matrix generated randomly.

In section 2, Richardson iteration method and convergence analysis are demonstrated, and HSS iteration method and convergence analysis are developed in Section 3. Optimal convergence speeds of Richardson and HSS iteration are studied in Section 4 for symmetric positive definite matrix. Section 5 gives some numerical example generated randomly. Section 6 concludes the paper.

We now describe our notation. All vectors will be column vectors. The notation $A \in R^{n\times n}$ will signify a real $n\times n$ matrix, $\rho(A)$ be spectral radius of matrix A, and $\|A\|_2$ denote 2-norm. We write I for the identity matrix (I is suitable dimension in context). A vector of zeros in a real space of arbitrary dimension will be denoted by ***0***.

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Definition 1.1 The matrix A is symmetric positive definite (SPD, [2]), i.e., $d^T Ad > 0$ for every $d \in R^n, d \neq \boldsymbol{0}$, and $A^T = A$.

Definition 1.2 The matrix A is generalized positive definite (GPD, [2]), i.e., $d^T Ad > 0$ for every $d \in R^n, d \neq \boldsymbol{0}$.

2. Richardson Iteration for Symmetric Positive Definite Matrix

Richardson iteration was proposed by Lewis Richardson [3]. It is similar to the Jacobi and Gauss-Seidel method.

By recasting system (1) as follows:

$$x = \left(I - \tau A\right) x + \tau b$$

The Richardson iteration is:

$$x^{k+1} = \left(I - \tau A\right) x^k + \tau b = x^k + \tau\left(b - Ax^k\right), k = 0,1,2,\cdots \tag{3}$$

In doing so, the new system will be equivalent to the original one for any value of the parameter $\tau > 0$. System (3) is a particular case of (2) with $B = I - \tau A$ and $\varphi = \tau b$.

In order to prove convergence of Richardson iteration, we first give following lemma.

Lemma 2.1 Let $0 < \lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = \lambda_{\max}$. For any $\tau > 0$, let $\rho(\tau) = \max\limits_{\lambda_j} \left\{ \left|1 - \tau\lambda_j\right| \right\}$.

1. If the parameter τ satisfies the inequalities $0 < \tau < \dfrac{2}{\lambda_{\max}}$, then $\rho(\tau) < 1$.

2. The $\rho(\tau)$ achieves its minimal value $\rho_{\text{opt}}(\tau_{\text{opt}}) = \dfrac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}$ when $\tau = \tau_{\text{opt}} = \dfrac{2}{\lambda_{\min} + \lambda_{\max}}$.

Proof. 1) Since $0 < \lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = \lambda_{\max}$, thus:

$$\max_{\lambda_j} \left\{ \left|1 - \tau\lambda_j\right| \right\} = \max\left\{ \left|1 - \tau\lambda_{\min}\right|, \left|1 - \tau\lambda_{\max}\right| \right\}$$

If the parameter τ satisfies the inequalities $0 < \tau < \dfrac{2}{\lambda_{\max}}$, then:

$$-1 < 1 - \tau\lambda_{\max} < 1 \Rightarrow \left|1 - \tau\lambda_{\max}\right| < 1$$

And,

$$-1 \leq 1 - 2\frac{\lambda_{\min}}{\lambda_{\max}} < 1 - \tau\lambda_{\min} < 1 \Rightarrow \left|1 - \tau\lambda_{\min}\right| < 1$$

Thus,

$$\max_{\lambda_j} \left\{ \left|1 - \tau\lambda_j\right| \right\} = \max\left\{ \left|1 - \tau\lambda_{\min}\right|, \left|1 - \tau\lambda_{\max}\right| \right\} < 1 \cdot$$

2) If $0 < \tau < \dfrac{2}{\lambda_{\max}}$, the value.

$$\rho(\tau) = \max_{\lambda_j} \left\{ \left|1 - \tau\lambda_j\right| \right\} = \max \left\{ \left|1 - \tau\lambda_{\min}\right|, \left|1 - \tau\lambda_{\max}\right| \right\}$$

Is shown by a bold polygonal line in figure 1; it coincides with $\left|1 - \tau\lambda_{\min}\right|$ before the intersection point, and after this point it coincides with $\left|1 - \tau\lambda_{\max}\right|$. Consequently, the minimum value of $\rho = \rho_{\mathrm{opt}}(\tau_{\mathrm{opt}})$ is achieved precisely at the intersection, i.e., at the value of $\tau = \tau_{\mathrm{opt}}$ obtained from the following condition:

$$1 - \tau_{\mathrm{opt}}\lambda_{\min} = \left|1 - \tau_{\mathrm{opt}}\lambda_{\min}\right| = \left|1 - \tau_{\mathrm{opt}}\lambda_{\max}\right| = -(1 - \tau_{\mathrm{opt}}\lambda_{\max})$$

Which yields:

$$\tau_{\mathrm{opt}} = \frac{2}{\lambda_{\min} + \lambda_{\max}}$$

Consequently,

$$\rho_{\mathrm{opt}}(\tau_{\mathrm{opt}}) = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}$$

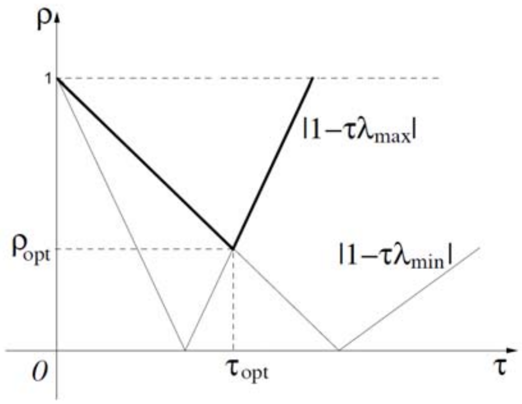


Figure 1. Image of $\rho(\tau) = \max_{\lambda_j} \left\{ \left|1 - \tau\lambda_j\right| \right\} = \max \left\{ \left|1 - \tau\lambda_{\min}\right|, \left|1 - \tau\lambda_{\max}\right| \right\}$

Example 2.1: Let $n = 3, \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 4$. For any $\tau > 0$, let:

$$\rho(\tau) = \max_{\lambda_j} \left\{ \left|1 - \tau\lambda_j\right| \right\} = \max \left\{ \left|1 - 2\tau\right|, \left|1 - 3\tau\right|, \left|1 - 4\tau\right| \right\}$$

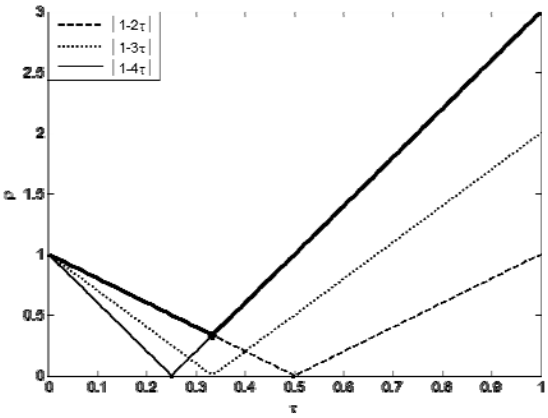


Figure 2. Image of function $|1-2\tau|,|1-3\tau|,|1-4\tau|$, and bold polygonal line

Is image of $\rho(\tau)=\max\{|1-2\tau|,|1-3\tau|,|1-4\tau|\}=\max\{|1-2\tau|,|1-4\tau|\}$.

From Figure 2, if $0<\tau<\frac{2}{\lambda_{\max}}=0.5$, then:

$$\rho(\tau)=\max\{|1-2\tau|,|1-3\tau|,|1-4\tau|\}=\max\{|1-2\tau|,|1-4\tau|\}<1$$

Moreover, if $\tau=\tau_{\text{opt}}=\frac{2}{\lambda_{\min}+\lambda_{\max}}=\frac{1}{3}$, $\rho(\tau)=\max_{\lambda_j}\{|1-\tau\lambda_j|\}$ achieves its minimal value $\rho_{\text{opt}}(\tau_{\text{opt}})=\frac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}=\frac{1}{3}$.

Theorem 2.1: Sppose $A\in R^{n\times n}$ be symmetric positive definite (SPD) matrix. In Richardson iteration (3), if the parameter τ satisfies the inequalities $0<\tau<\frac{2}{\lambda_{\max}}$, then the sequence x^k of iteration (3) converges to the solution of linear system (1) for arbitrary initial point $x^0\in R^n$. Moreover, the best performance of convergence occurs on $\rho=\rho_{\text{opt}}(\tau_{\text{opt}})=\frac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}$ with $\tau=\tau_{\text{opt}}=\frac{2}{\lambda_{\min}+\lambda_{\max}}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and the maximum eigenvalues of the matrix A.

Proof. Let $\rho(B)$ be spectral radius of matrix $B=I-\tau A$. The necessary and sufficient condition for convergence of Richardson iteration is $\rho(B)<1$.

Suppose that, $\lambda_j, j=1,2,\cdots,n$, are the eigenvalues of A arranged in the ascending order:

$$0<\lambda_{\min}=\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_{n-1}\le\lambda_n=\lambda_{\max}$$

Suppose $v_j, j=1,2,\cdots,n$, be the eigenvalues of $B=I-\tau A$. Then $v_j=1-\tau\lambda_j$, $j=1,2,\cdots,n$.

Since $0<\lambda_{\min}=\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_{n-1}\le\lambda_n=\lambda_{\max}$, then:

$$1-\tau\lambda_{\min}=v_1\ge v_2\ge v_3\ge\cdots\ge v_{n-1}\ge v_n=1-\tau\lambda_{\max}$$

Thus,

$$\rho(B)=\max_{v_j}\left\{\left|v_j\right|\right\}=\max_{\lambda_j}\left\{\left|1-\tau\lambda_j\right|\right\}=\max\left\{\left|1-\tau\lambda_{\min}\right|,\left|1-\tau\lambda_{\max}\right|\right\}$$

According to lemma 2.1, if $0<\tau<\dfrac{2}{\lambda_{\max}}$, then:

$$\rho(B)=\max_{v_j}\left\{\left|v_j\right|\right\}=\max_{\lambda_j}\left\{\left|1-\tau\lambda_j\right|\right\}=\max\left\{\left|1-\tau\lambda_{\min}\right|,\left|1-\tau\lambda_{\max}\right|\right\}<1$$

Thus Richardson iteration is convergent.

Moreover, when $\rho(\tau)$ achieves its minimal value $\rho_{\text{opt}}(\tau_{\text{opt}})=\dfrac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}$, Richardson iteration has the best performance of convergence in theory.

3. HSS Iteration for Generalized Positive Definite Matrix

For system of linear equations with generalized positive definite matrix $A\in R^{n\times n}$ (or non-Hermitian positive definite). Splitting coefficient matrix.

$$A=H+S$$

Where $H=\frac{1}{2}(A+A^T)$, $S=\frac{1}{2}(A-A^T)$. Since $H^T=H, S^T=-S$, we call $A=H+S$ Hermitian/skew-Hermitian splitting (HSS, [4-5]).

Lemma 3.1 [6]: Let $A\in R^{n\times n}$, $H=\frac{1}{2}(A+A^T)$, $S=\frac{1}{2}(A-A^T)$.

1) A is generalized positive definite (GPD) if and only if H is symmetric positive definite (SPD).

2) If A is generalized positive definite (GPD), then the determinant $|A|>0$, and A^{-1} is also generalized positive definite.

3) If A is generalized positive definite (GPD), then for any $\alpha>0$, $(\alpha I+H)$, $(\alpha I+S)$, $(\alpha I+H)^{-1}$, and $(\alpha I+S)^{-1}$ are also generalized positive definite.

4) If A is generalized positive definite (GPD), then for any $\alpha>0$, $(\alpha I-S)(\alpha I+S)^{-1}$ is an orthogonal matrix, thus $\left\|(\alpha I-S)(\alpha I+S)^{-1}\right\|_2=1$.

HSS iteration method of system (1) as follows:

$$x^{k+1}=T(\alpha)x^k+G(\alpha)b,\quad k=0,1,2,\cdots, \tag{4}$$

Where $T(\alpha)=\left(\alpha I+S\right)^{-1}\left(\alpha I-H\right)\left(\alpha I+H\right)^{-1}\left(\alpha I-S\right)$, $G(\alpha)=2\alpha\left(\alpha I+S\right)^{-1}\left(\alpha I+H\right)^{-1}$, and parameter $\alpha>0$. For the convergence property of the HSS iteration, we first give following lemma.

Lemma 3.2: Let $0<\lambda_{\min}=\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_{n-1}\le\lambda_n=\lambda_{\max}$. For any $\alpha>0$, let:

$$\rho(\alpha)=\max_{\lambda_j}\left\{\left|\frac{\alpha-\lambda_j}{\alpha+\lambda_j}\right|\right\}$$

1) For any $\alpha>0$, $\rho(\alpha)<1$.

2) The $\rho(\alpha)$ achieves its minimal value $\rho_{\text{opt}}(\alpha_{\text{opt}})=\dfrac{\sqrt{\lambda_{\max}}-\sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}}}$ when $\alpha=\alpha_{\text{opt}}=\sqrt{\lambda_{\min}\lambda_{\max}}$.

Proof. 1) Since $\lambda_j>0,\ j=1,2,\cdots,n$, for any $\alpha>0$

$$|\alpha-\lambda_j|<|\alpha+\lambda_j|,\ \ j=1,2,\cdots,n$$

That is:

$$\left|\frac{\alpha-\lambda_j}{\alpha+\lambda_j}\right|<1,\ \ j=1,2,\cdots,n$$

Thus:

$$\rho(\alpha)=\max_{\lambda_j}\left\{\left|\frac{\alpha-\lambda_j}{\alpha+\lambda_j}\right|\right\}<1$$

2) Since $0<\lambda_{\min}=\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_{n-1}\le\lambda_n=\lambda_{\max}$, thus:

$$\max_{\lambda_j}\left\{\left|\frac{\alpha-\lambda_j}{\alpha+\lambda_j}\right|\right\}=\max\left\{\left|\frac{\alpha-\lambda_{\min}}{\alpha+\lambda_{\min}}\right|,\left|\frac{\alpha-\lambda_{\max}}{\alpha+\lambda_{\max}}\right|\right\}$$

Consequently, the minimum value of $\rho=\rho_{\text{opt}}(\alpha_{\text{opt}})$ is achieved precisely at the intersection, i.e., at the value of $\alpha=\alpha_{\text{opt}}$ obtained from the following condition:

$$\frac{\alpha_{\text{opt}}-\lambda_{\min}}{\alpha_{\text{opt}}+\lambda_{\min}}=\left|\frac{\alpha_{\text{opt}}-\lambda_{\min}}{\alpha_{\text{opt}}+\lambda_{\min}}\right|=\left|\frac{\alpha_{\text{opt}}-\lambda_{\max}}{\alpha_{\text{opt}}+\lambda_{\max}}\right|=-\left(\frac{\alpha_{\text{opt}}-\lambda_{\max}}{\alpha_{\text{opt}}+\lambda_{\max}}\right)$$

Which yields:

$$\alpha_{\text{opt}}=\sqrt{\lambda_{\min}\lambda_{\max}}$$

So,

$$\rho_{\text{opt}}(\alpha_{\text{opt}})=\frac{\sqrt{\lambda_{\max}}-\sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}}}$$

Example 3.1: Let $n=3,\ \lambda_1=2,\lambda_2=3,\lambda_3=4$. For any $\alpha>0$, let:

$$\rho(\alpha)=\max_{\lambda_j}\left\{\left|\frac{\alpha-\lambda_j}{\alpha+\lambda_j}\right|\right\}=\max\left\{\left|\frac{\alpha-2}{\alpha+2}\right|,\left|\frac{\alpha-3}{\alpha+3}\right|,\left|\frac{\alpha-4}{\alpha+4}\right|\right\}$$

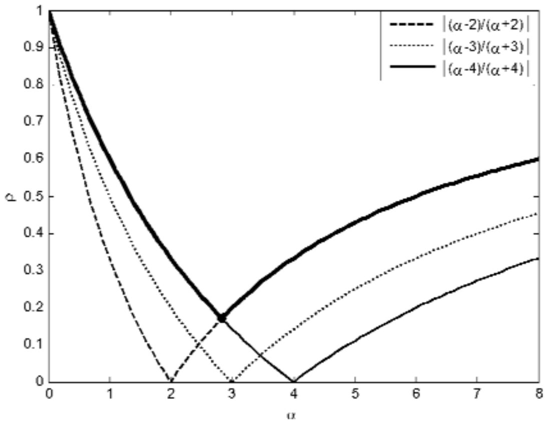


Figure 3. Image of function $\left|\frac{\alpha-2}{\alpha+2}\right|,\left|\frac{\alpha-3}{\alpha+3}\right|,\left|\frac{\alpha-4}{\alpha+4}\right|$, and bold polygonal line

Is image of $\rho(\alpha)=\max\left\{\left|\frac{\alpha-2}{\alpha+2}\right|,\left|\frac{\alpha-3}{\alpha+3}\right|,\left|\frac{\alpha-4}{\alpha+4}\right|\right\}=\max\left\{\left|\frac{\alpha-2}{\alpha+2}\right|,\left|\frac{\alpha-4}{\alpha+4}\right|\right\}$.

From Figure 3, for any $\alpha>0$, then:

$$\rho(\alpha)=\max\left\{\left|\frac{\alpha-2}{\alpha+2}\right|,\left|\frac{\alpha-4}{\alpha+4}\right|\right\}<1$$

Moreover, if $\alpha=\alpha_{\text{opt}}=\sqrt{\lambda_{\min}\lambda_{\max}}=\sqrt{8}$, $\rho(\alpha)$ achieves its minimal value.

$$\rho_{\text{opt}}(\alpha_{\text{opt}})=\frac{\sqrt{\lambda_{\max}}-\sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}}}=\frac{2-\sqrt{2}}{2+\sqrt{2}}$$

Following we apply the above lemmas to obtain the convergence of HSS iteration.

Theorem 3.1: Suppose $A\in R^{n\times n}$ be generalized positive definite (GPD) matrix. In HSS iteration (4), for any $\alpha>0$, $\rho(T(\alpha))\le\rho(\alpha)<1$, so the sequence x^k of iteration (4) converges to the solution of linear system (1) for arbitrary initial point $x^0\in R^n$. Moreover, the best performance of convergence occurs on $\rho_{\text{opt}}(\alpha_{\text{opt}})=\frac{\sqrt{\lambda_{\max}}-\sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}}}$ with $\alpha_{\text{opt}}=\sqrt{\lambda_{\min}\lambda_{\max}}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and the maximum eigenvalues of the matrix H.

Proof:

$$\begin{aligned}\rho(T(\alpha))&=\rho\left((\alpha I+S)T(\alpha)(\alpha I+S)^{-1}\right)\\ &=\rho\left((\alpha I-H)(\alpha I+H)^{-1}(\alpha I-S)(\alpha I+S)^{-1}\right)\\ &=\left\|(\alpha I-H)(\alpha I+H)^{-1}(\alpha I-S)(\alpha I+S)^{-1}\right\|_2\\ &\le\left\|(\alpha I-H)(\alpha I+H)^{-1}\right\|_2\left\|(\alpha I-S)(\alpha I+S)^{-1}\right\|_2\\ &=\left\|\left(\alpha I-H\right)\left(\alpha I+H\right)^{-1}\right\|_2=\max_{\lambda_j}\left\{\left|\frac{\alpha-\lambda_j}{\alpha+\lambda_j}\right|\right\}\end{aligned}$$

According to lemma 3.2, for any $\alpha>0$, $\rho(T(\alpha))\le\rho(\alpha)<1$. Thus Richardson iteration is convergent.

Moreover, when $\rho(\alpha)$ achieves its minimal value $\rho_{\text{opt}}(\alpha_{\text{opt}})=\frac{\sqrt{\lambda_{\max}}-\sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}}}$, HSS iteration has the best performance of convergence in theory.

Remark. In HSS iteration, if A is symmetric positive definite (SPD), then:

$$H=\frac{1}{2}(A+A^T)=A\text{，}S=\frac{1}{2}(A-A^T)=\boldsymbol{O}$$

Thus,

$$T(\alpha)=\left(\alpha I-A\right)\left(\alpha I+A\right)^{-1},\quad G(\alpha)=2\left(\alpha I+A\right)^{-1}$$

Compared with Richardson iteration, HSS iteration reduces the requirement of symmetry. So absolute value equations [7], saddle point problem [8] are also solved by HSS iterative method.

4. Optimal Convergence Speed of Richardson and HSS Iteration

For symmetric positive definite matrix, the optimal convergence speed of Richardson and HSS iteration are contrasted in this section.

Lemma 4.1: Let $0<a<b$. Then:

$$\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}<\frac{b-a}{b+a}$$

Proof:

$$\begin{aligned}&0<a<b\Rightarrow\frac{b}{a}>1\Rightarrow\frac{b}{a}>\sqrt{\frac{b}{a}}\Rightarrow b\sqrt{a}>a\sqrt{b}\\ &\Rightarrow(b-a)\left(\sqrt{b}+\sqrt{a}\right)>(b+a)\left(\sqrt{b}-\sqrt{a}\right)\\ &\Rightarrow\frac{b-a}{b+a}>\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\end{aligned}$$

Theorem 4.1: If A is symmetric positive definite (SPD) matrix, then the optimal convergence speed of HSS iteration is superior to that of Richardson iteration.

Proof. Suppose that, $\lambda_j, j=1,2,\cdots,n$, are the eigenvalues of A arranged in the ascending order:

$$0<\lambda_{\min}=\lambda_1\le\lambda_2\le\lambda_3\le\cdots\le\lambda_{n-1}\le\lambda_n=\lambda_{\max}$$

Then, by Theorem 3.1, the optimal spectral radius of HSS iteration is $\rho_{\text{opt}}(\alpha_{\text{opt}})=\frac{\sqrt{\lambda_{\max}}-\sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}}}$ with $\alpha_{\text{opt}}=\sqrt{\lambda_{\min}\lambda_{\max}}$. While by Theorem 2.1, the optimal spectral radius of Richardson iteration is $\rho_{\text{opt}}(\tau_{\text{opt}})=\frac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}$ with $\tau_{\text{opt}}=\frac{2}{\lambda_{\min}+\lambda_{\max}}$.

According to Lemma 4.1, if $\lambda_{\min}<\lambda_{\max}$, then:

$$\frac{\sqrt{\lambda_{\max}}-\sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}}+\sqrt{\lambda_{\min}}}<\frac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}$$

Thus,

$$\rho_{\text{opt}}(\alpha_{\text{opt}})<\rho_{\text{opt}}(\tau_{\text{opt}})$$

So the corresponding optimal convergence speed of HSS iteration is superior to that of Richardson iteration.

Remark. Power method is conventional way for finding the greatest eigenvalue of a matrix. Since $A\in R^{n\times n}$ be symmetric positive definite matrix. Thus, maximum eigenvalue $\lambda_{\max}$ of matrix A can be obtained by power method. Meanwhile, minimum eigenvalue $\lambda_{\min}$ of matrix A can be obtained by calculating maximum eigenvalue of matrix A^{-1}.

5. Computational Results

In order to illustrate the performance of Richardson iteration and HSS iteration method, we solve linear systems with symmetric positive definite matrix generated randomly. Where the data (***A***, ***b***) are generated by the Matlab scripts:

```
rand('state',0);R=rand(n,n);A=R'*R+n*eye(n);b=A*ones(n,1);
```

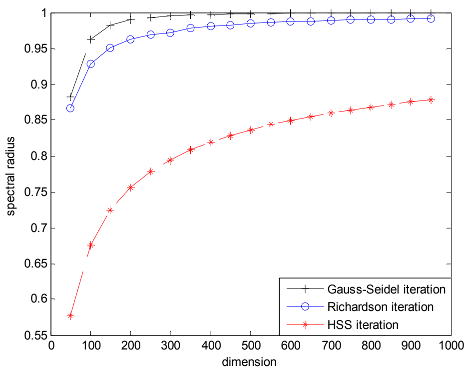
Such that $x=(1,1,\cdots,1)^T$ is the unique solution. We set the random-number generator to the state of 0 so that the same data can be regenerated. Let $x^0=\boldsymbol{0}$, $\varepsilon=1\times10^{-4}$. We use $\|x^{k+1}-x^k\|\le\varepsilon$ as the stopping rule. All experiments were performed on MatlabR2009a with Intel(R) Core(TM) 4×3.3GHz and 2GB RAM.

Simulations were carried out to compare the performance of the Richardson iteration, HSS iteration, and Gauss-Seidel iteration. Spectral radius (ρ), iterations (k), and elapsed time (t) of three iteration methods are listed in Table 1 for different dimension *n*.

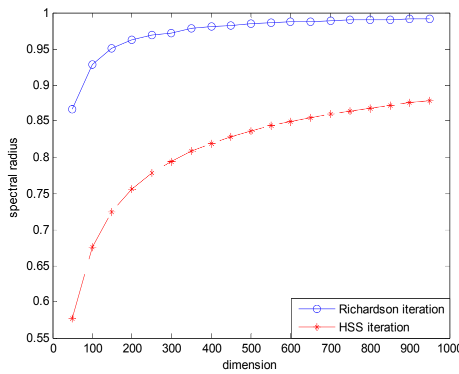
Table 1. Spectral radius, iterations, and elapsed time for different dimension *n*

n	Gauss-Seidel			Richardson			HSS		
	ρ	k	t (s)	ρ	k	t (s)	ρ	k	t (s)
50	0.8822	87	0.0049	0.8659	84	0.0051	0.5771	24	0.0064
100	0.9631	280	0.0108	0.9277	165	0.0047	0.6756	33	0.0075
150	0.9823	577	0.0614	0.9501	244	0.0140	0.7242	40	0.0201
200	0.9897	986	0.2894	0.9620	326	0.0294	0.7558	47	0.0358
250	0.9932	1482	0.9227	0.9692	407	0.0707	0.7777	52	0.0580
300	0.9952	2069	2.1918	0.9714	489	0.1200	0.7946	57	0.0962
350	0.9965	2814	4.9141	0.9778	574	0.2017	0.8084	62	0.1328
400	0.9973	3619	9.5858	0.9805	657	0.3437	0.8194	67	0.1909
450	0.9978	4511	16.9223	0.9826	741	0.4601	0.8287	71	0.2457
500	0.9982	5521	28.3678	0.9843	825	0.6667	0.8367	75	0.3223
550	0.9985	6605	44.8219	0.9857	909	0.9678	0.8436	79	0.4497
600	0.9988	7847	68.9272	0.9869	994	1.2830	0.8497	82	0.5649
650	0.9989	9101	100.3113	0.9879	1079	1.6678	0.8551	86	0.6657
700	0.9991	10450	142.4005	0.9887	1164	2.1944	0.8599	89	0.7852
750	0.9992	11996	200.6673	0.9895	1249	2.6713	0.8643	93	0.9353
800	0.9993	13526	272.2897	0.9901	1334	3.3175	0.8683	96	1.0879
850	0.9994	15337	368.2543	0.9907	1421	4.0451	0.8720	99	1.2743
900	0.9994	17052	475.4076	0.9912	1509	4.8815	0.8754	102	1.4799
950	0.9995	18962	621.7682	0.9917	1595	5.7889	0.8785	105	1.7050

Figure 4(a) shows spectral radius of three methods for different dimension n. Figure 4(b) shows spectral radius of two methods for different dimension n. Figure 5(a) shows iterations of three methods for different dimension n. Figure 5(b) shows iterations of two methods for different dimension n. Figure 6(a) shows elapsed time of three methods for different dimension n. Figure 6(b) shows elapsed time of two methods for different dimension n.

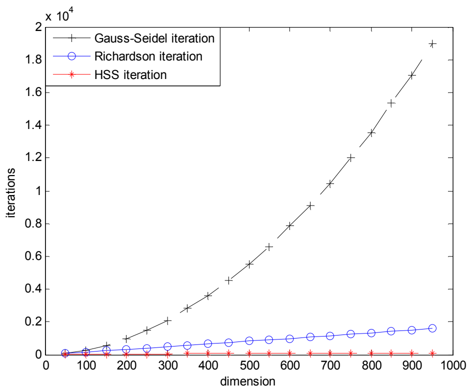


(a)

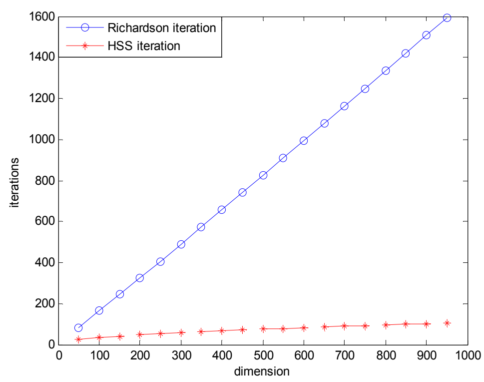


(b)

Figure 4. Spectral radius

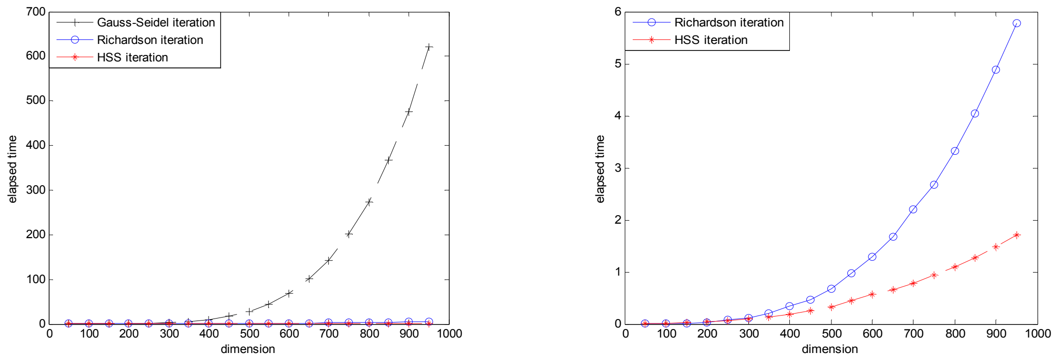


(a)



(b)

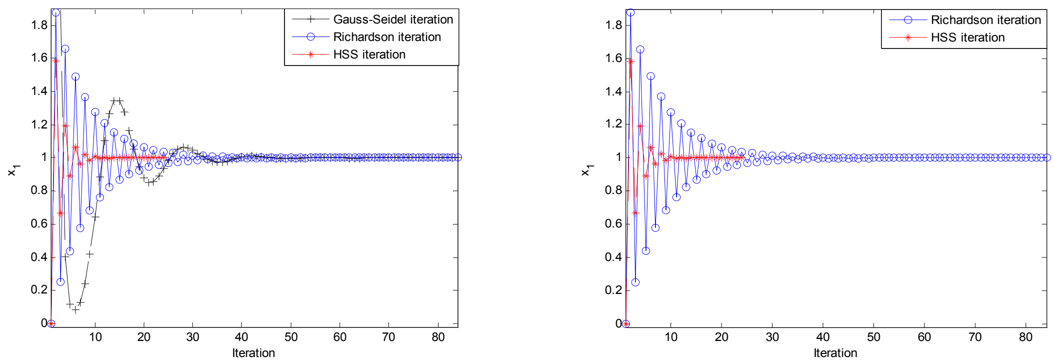
Figure 5. Iterations



(a) (b)

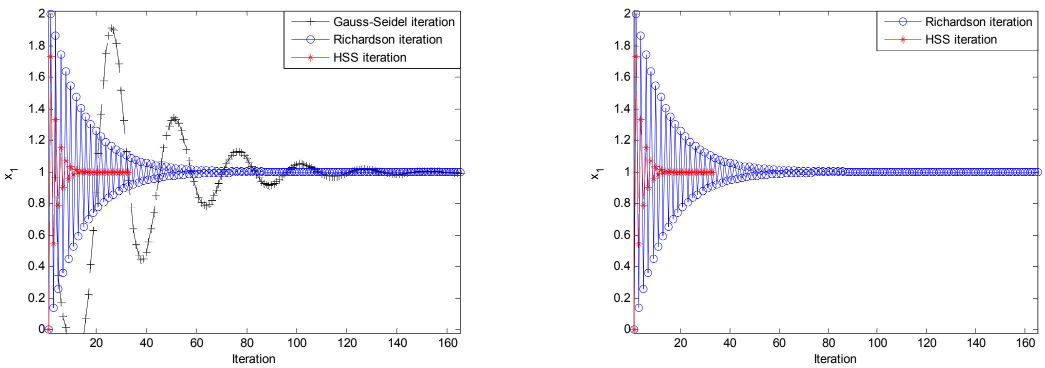
Figure 6. Elapsed time

Figure 7(a) shows convergence process of three methods with n=50. Figure 7(b) shows convergence process of two methods with n=50. Figure 8(a) shows convergence process of three methods with n=100. Figure 8(b) shows convergence process of two methods with n=100.



(a) (b)

Figure 7. Convergence process of x_1 with n=50



(a) (b)

Figure 8. Convergence process of x_1 with n=100

We can see that Gauss-Seidel iteration method is poor, while HSS iteration method is the best.

6. Conclusion

We have demonstrated two iterative methods, Richardson iteration and HSS iteration. Theoretical analyses and computational results show that the HSS iteration method has the advantages of fast convergence speed, high computation efficiency, and without requirement of symmetry. Future works will also focus on studying the applications of HSS iteration on saddle point problems [9], continuous Sylvester equations [10], and choices of parameters in iteration method [11-12].

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