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Cobb-Douglass Utility Function in Optimizing the Internet Pricing Scheme Model

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Abstract

The greater numbers of internet users the greater challenge will be tackled by ISP to provide good services but gain maximum profit. By analyzing Cobb-Douglass utility function we will obtain optimal pricing scheme. This research is based on previous research conducted by [1]. Wu and Banker [1] analyzed modified Cobb-douglass utility function and obtained optimal model of flat fee and two part tariff for homogen consumers meanwhile we focus on getting optimal pricing scheme model by using original Cobb-Douglass utility function. The first step to conduct this research is by formulating Cobb-Douglass utility function then analyzing that function. The results show that we obtain optimal pricing scheme model for homogenous and heterogeneous consumer cases. The two-part tariff pricing scheme yield better optimal solution rather than flat fee and two-part tariff pricing scheme regarding with homogen consumers and heterogen consumers based on willingness to pay. For heterogeneous consumers based on consumption level, the optimal pricing scheme is on two-part tariff pricing scheme.

Keywords: *original Cobb-Douglass utility function, homogenous consumers, heterogeneous consumers*

1. Introduction

Nowadays, internet development increases fast so this condition increases the users to use the internet. Internet Service Provider (ISP) has to provide best services with optimal prices for the consumers. ISP needs not the best utility function not only to gain profit for itself but also to pursue the consumers applying the service provided by ISP. Previous works regarding the pricing strategies to maximize the ISP profit are due to [2, 3]. According to [4], the utility function usually connects with level of consumer satisfaction to information service consumption which can maximize the profit to achieve certain objectives.

There is a lot of assumption to be applied to the utility function but the researchers usually use the bandwidth function with fixed loss and delay and follow the rules that marginal utility as bandwidth function diminishing with increasing bandwidth [5-15]. The other reason dealing with the choices of utility function is that the utility function should be differentiable and easily to be analyzed the homogeneity and heterogeneity that impacts the choice of pricing structure for the companies. Kelly [16] also contends that the utility function also can be assumed to be increasing function, strictly concave and continuously differentiable.

There exist some utility functions, but we need the utility function that can fulfill consumer satisfactions. Previous work on utility function is due to [1]. In their explanation, the results show that flat fee and two-part tariff pricing scheme yield the optimal solutions using the modified Cobb-Douglass utility function.

The contribution of this paper basically is to analyze the original Cobb-Douglass utility function to obtain optimal solution of information service pricing scheme. The optimal solutions can be different from what the previous work done by previous researches in terms of the utility function chosen and the analysis in finding the best of three pricing strategies. Comparison of the results between original utility function and modified will be conducted to observe which utility function offers best maximum prices to consumers.

2. Research Method

The steps in conducting the research are as follows.

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1. Apply the original Cobb-Douglas utility function on three internet pricing schemes which are flat fee, usage-based and two-part tariff for homogen and heterogen consumers that previously described by [1].
2. Analyze the utility function forms analytically.
3. Compare the obtained model analysis of those three pricing schemes.
4. Compare the obtained model analysis using original Cobb-Douglas utility function with modified Cobb-Douglas utility function proposed by [1].
5. Conclude and obtain the optimal model of information service pricing scheme.

3. Results and Analysis

The parameter and decision variables are adopted in [1, 17]. We use the original Cobb-Douglass as follows.

$$U(X\,,Y) = X^aY^b$$

The following are the analysis of Cobb-Douglass utility function for three pricing strategies. The analysis follows the steps of [1].

3.1 Homogen Consumer

Consumers Optimization problem (adopted in [1])

$$maks_{X,Y,Z}\, X^a + Y^b - P_xX - P_yY - PZ \tag{1}$$

Such that

$$X \leq \bar{X}Z \tag{2}$$

$$Y \leq \bar{Y}Z \tag{3}$$

$$X^a + Y^b - P_xX - P_yY - PZ \geq 0 \tag{4}$$

$$Z = 0\ or\ 1 \tag{5}$$

ISP optimization problem:

$$max_{P,P_X,P_Y} \sum_i (P_XX^* + P_YY^* + PZ^*) \tag{6}$$

where (X^*,Y^*,Z^*) = argmaks $X^a + Y^b\ - P_xX - P_yY - PZ$
such that Constraint (3)-(5)
We proceed to Lemma 1a-9a as the lemma improved from [1] using original Cobb Douglass utility function.

Case 1a. if providers apply the flat fee rate by setting up $P_x = 0, P_y = 0$ dan $P > 0$, it means that the price set up by providers does not have an impact to time usage (peak or nonpeak hours) so, our homogenous optimization problem will be

$$\underset{X,Y,Z}{maks} = X^aY^b - P_xX - P_yY - PZ = X^aY^b - (0)X - (0)Y - P(1) = X^aY^b - P$$

Using Eq. (4), we obtain $X^aY^b - P_xX - P_yY - PZ \geq 0 \Leftrightarrow P \leq X^aY^b$

Then the provider problem will be

$\max_{P,P_X,P_Y} = \sum_{i=1}^{j}(P_x X^* + P_y Y^* + PZ^*) = \sum_{i=1}^{j}((0)X^* + (0)Y^* + P(1)) = \sum_{i=1}^{j}((0)X^* + (0)Y^* +$ *XaYb=i=1jXaYb*

When the function can maximize *X,Y,* and *Z* then the consumers will fully utilize the services by choosing consumption level $X = \bar{X}$ dan $Y = \bar{Y}$, which is maximum usage level with maximum utility, so that the consumer can get the price $\bar{X}^a\bar{Y}^b$. Maximum flat fee rate provider can charge is $\bar{X}^a\bar{Y}^b$ with maximum profit of $\sum_{i=1}^{j}[\bar{X}^a\bar{Y}^b]$; where *i* shows the number of consumers. According to this case, we obtain the Lemma 1a.

Lemma 1a: If the providers apply flat fee rate, then the charges will be $\bar{X}^a\bar{Y}^b$ and maximum profit obtained will be $\sum_{i=1}^{j}[\bar{X}^a\bar{Y}^b]$; where *i* shows the number of consumers.

Case 2a. If the providers apply the pure usage base price by setting up $P_x > 0, P_y > 0$ and $P = 0$ then the providers give different prices which are peak and nonpeak hour prices. Given the function $\underset{X,Y,Z}{maks} = X^aY^b - P_xX - P_yY$. To maximize that function we will apply necessary and sufficient conditions as follows.

(i) The necessary condition. $\frac{\partial(X^aY^b - P_xX - P_yY)}{\partial X} = 0$, so $a\,X^{a-1}Y^b = P_x$ (7)

$$\Leftrightarrow X^{a-1} = \frac{P_x}{aY^b} \Leftrightarrow X^* = \left(\frac{P_x}{aY^b}\right)^{\frac{1}{a-1}}$$

(ii) The sufficient condition:

$$\frac{\partial^2(X^aY^b - P_xX - P_yY)}{\partial X^2} > 0 \Leftrightarrow \frac{\partial(a\,X^{a-1}Y^b - P_x)}{\partial X} = a(a-1)\,X^{a-2}Y^b > 0\,; a,b > 0$$

It means that $P_x = a\,X^{a-1}Y^b$ is minimum price.

and

(i) The necessary condition. $\frac{\partial(X^aY^b - P_xX - P_yY)}{\partial Y} = 0$, so $b\,X^aY^{b-1} = P_y$ (8)

$$\Leftrightarrow Y^{b-1} = \frac{P_y}{bX^a} \Leftrightarrow Y^* = \left(\frac{P_y}{bX^a}\right)^{\frac{1}{b-1}}$$

(ii) The sufficient condition: $\frac{\partial^2(X^aY^b - P_xX - P_yY)}{\partial Y^2} > 0$

$$\Leftrightarrow \frac{\partial(b\,X^aY^{b-1} - P_y)}{\partial Y} = b(b-1)\,X^aY^{b-2} > 0\,; a,b > 0$$

It means that $P_y = b\,X^aY^{b-1}$ is minimum price.

Then the provider problem will be

$$\max_{P,P_X,P_Y} \sum_{i=1}^{j}(P_X X^* + P_Y Y^*) = \sum_{i=1}^{j}\left[P_X\left(\frac{P_x^{\frac{1}{a-1}}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right) + P_Y\left(\frac{P_y^{\left(\frac{1}{b-1}\right)}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right)\right]$$

$$= \sum_{i=1}^{j}\left[\left(\frac{P_x^{\left(1+\frac{1}{a-1}\right)}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right) + P_Y\left(\frac{P_y^{\left(1+\frac{1}{b-1}\right)}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right)\right] = \sum_{i=1}^{j}\left[\left\{\frac{(a\,X^{a-1}Y^b)^{1+\frac{1}{a-1}}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right\} +\right.$$

b XaYb−11 + 1b−1b1b−1Xab−1

$$= \sum_{i=1}^{j}\left[\left\{\frac{a^{\left(1+\frac{1}{a-1}\right)}X^{a}Y^{\left(b+\frac{b}{a-1}\right)}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right\}+\left\{\frac{b^{\left(1+\frac{1}{b-1}\right)}X^{\left(a+\frac{a}{b-1}\right)}Y^{b}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right\}\right]$$

$$= \sum_{i=1}^{j}\left[\left\{\frac{a^{\left(1+\frac{1}{a-1}\right)}X^{a}Y^{b}Y^{\left(\frac{b}{a-1}\right)}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right\}+\left\{\frac{b^{\left(1+\frac{1}{b-1}\right)}X^{a}X^{\left(\frac{a}{b-1}\right)}Y^{b}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right\}\right]$$

$$= \sum_{i=1}^{j}\left[\left\{a^{\left(1+\frac{1}{a-1}-\frac{1}{a-1}\right)}X^{a}Y^{b}Y^{\left(\frac{b}{a-1}-\frac{b}{a-1}\right)}\right\}+\left\{b^{\left(1+\frac{1}{b-1}-\frac{1}{b-1}\right)}X^{a}X^{\left(\frac{a}{b-1}-\frac{a}{b-1}\right)}Y^{b}\right\}\right]$$

$$= \sum_{i=1}^{j}[\{a\,X^{a}Y^{b}\}+\{b\,X^{a}Y^{b}\}] = \sum_{i=1}^{j}(a+b)[X^{a}Y^{b}]$$

It means that if the service provider wants to maximize their profit, they have to minimize P_x and P_y. Since $X \le \bar{X}$ and $Y \le \bar{Y}$, then $X^* = \bar{X}$ and $Y^* = \bar{Y}$. So optimal P_x and P_y will be $P_x = a\bar{X}^{a-1}\bar{Y}^{b}$ and $P_y = b\bar{X}^{a}\bar{Y}^{b-1}$ with maximum profit of $\sum_{i=1}^{j}(a+b)[X^{a}Y^{b}]$; where *i* shows the number of consumers. According to this case, we obtain the Lemma 2a.

Lemma 2a: If the provider would like to apply usage-based pricing scheme then the optimal price will be $P_x = a\bar{X}^{a-1}\bar{Y}^{b}$ and $P_y = b\bar{X}^{a}\bar{Y}^{b-1}$ with maximum profit of $\sum_{i=1}^{j}(a+b)[X^{a}Y^{b}]$; where *i* shows the number of consumers.

Case 3a. The providers apply the two-part tariff price by setting up $P_x > 0, P_y > 0$ and $P > 0$, it means that we have subscription fee if the consumers choose this service and the prices is during the peak and nonpeak hours.

By using the Eq. (8) and (9), substitute those equations into Eq. (4) which is the constraint of consumer optimization problem. So, the constraint will be

$$X^{a}Y^{b} - P_xX - P_yY - PZ \ge 0 \Leftrightarrow X^{a}Y^{b} - (aX^{a-1}Y^{b})X - (b\,X^{a}Y^{b-1})Y - P \ge 0$$

$$\Leftrightarrow X^{a}Y^{b} - aX^{a}Y^{b} - bX^{a}Y^{b} - P \ge 0 \Leftrightarrow P \le X^{a}Y^{b} - aX^{a}Y^{b} - bX^{a}Y^{b}$$

The provider optimization problem will be

$$\max_{P,P_X,P_Y} = \sum_{i=1}^{j}(P_XX^* + P_YY^* + PZ^*)$$

$$= \sum_{i=1}^{j}\left[P_X\left(\frac{P_x^{\left(\frac{1}{a-1}\right)}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right)+P_Y\left(\frac{P_y^{\left(\frac{1}{b-1}\right)}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right)+(X^{a}Y^{b}-a\,X^{a}Y^{b}-bX^{a}Y^{b})\right]$$

$$= \sum_{i=1}^{j}\left[\left(\frac{P_x^{\left(1+\frac{1}{a-1}\right)}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right)+P_Y\left(\frac{P_y^{\left(1+\frac{1}{b-1}\right)}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right)+(X^{a}Y^{b}-a\,X^{a}Y^{b}-bX^{a}Y^{b})\right]$$

$$= \sum_{i=1}^{j}\left[\left\{\frac{(a\,X^{a-1}Y^{b})^{\left(1+\frac{1}{a-1}\right)}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right\}+\left\{\frac{(b\,X^{a}Y^{b-1})^{\left(1+\frac{1}{b-1}\right)}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right\}+(X^{a}Y^{b}-a\,X^{a}Y^{b}-bX^{a}Y^{b})\right]$$

$$= \sum_{i=1}^{j}\left[\left\{\frac{a^{\left(1+\frac{1}{a-1}\right)}X^{a}Y^{\left(b+\frac{b}{a-1}\right)}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right\}+\left\{\frac{b^{\left(1+\frac{1}{b-1}\right)}X^{\left(a+\frac{a}{b-1}\right)}Y^{b}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right\}+(X^{a}Y^{b}-a\,X^{a}Y^{b}-bX^{a}Y^{b})\right]$$

$$= \sum_{i=1}^{j}\left[\left\{\frac{a^{\left(1+\frac{1}{a-1}\right)}X^{a}Y^{b}Y^{\left(\frac{b}{a-1}\right)}}{a^{\left(\frac{1}{a-1}\right)}Y^{\left(\frac{b}{a-1}\right)}}\right\}+\left\{\frac{b^{\left(1+\frac{1}{b-1}\right)}X^{a}X^{\left(\frac{a}{b-1}\right)}Y^{b}}{b^{\left(\frac{1}{b-1}\right)}X^{\left(\frac{a}{b-1}\right)}}\right\}+(X^{a}Y^{b}-a\,X^{a}Y^{b}-bX^{a}Y^{b})\right]$$

$= \sum_{i=1}^{j}[\{a\,X^aY^b\} + \{b\,X^aY^b\} + (\,X^aY^b - a\,X^aY^b - bX^aY^b)] = \sum_{i=1}^{j}[\,X^aY^b]$

Since we know that P_x and P_y decrease, then X* and Y* will increase. However, since $X \leq \bar{X}$ and $Y \leq \bar{Y}$, then $X^* = \bar{X}$ and $Y^* = \bar{Y}$. In other words, optimal P_x and P_y yang will be $P_x = a\,\bar{X}^{a-1}\bar{Y}^b$, $P_y = b\,\bar{X}^a\bar{Y}^{b-1}$ and $P = \bar{X}^a\bar{Y}^b - a\,\bar{X}^a\bar{Y}^b - b\bar{X}^a\bar{Y}^b$. That is why, the maximum profit achievable of service provider is $\sum_{i=1}^{j}[\,X^aY^b]$; where *i* shows the number of consumers. According to this case we obtain Lemma 3a.

If we assume that $a\bar{X}^a\bar{Y}^b > \bar{X}^a\bar{Y}^b$ and $b\bar{X}^a\bar{Y}^b > \bar{X}^a\bar{Y}^b$, then $(a+b)[\bar{X}^a\bar{Y}^b] > [\bar{X}^a\bar{Y}^b]$; $a, b > 0$. So, the maximum profit obtained by ISP will be when they apply usage-based pricing scheme.

Lemma 3a: If the service providers apply two-part tariff scheme, the best P_x and P_y will be $P_x = a\,\bar{X}^{a-1}\bar{Y}^b$, $P_y = b\,\bar{X}^a\bar{Y}^{b-1}$. Maximum fixed value *P* ISPs provide is the differences between consumer maximum that can be obtained, $\bar{X}^a\bar{Y}^b$, and payment for utilizing the service, $(a+$ $bXaYb$. So, the maximum profit for ISP will be $i{=}1j\ XaYb$ where *i* shows the number of consumers.

3.2. High End and Low End Heterogeneous Consumers

Assume that we have *m* high end consumers (*i* = 1) and *n* low end consumers (*i* = 2). To learn how the willingness to pay affects the pricing scheme, we assume that each consumers have the same upper bound in during peak hours and *Y* during nonpeak hours, $a_1 > a_2$ dan $b_1 > b_2$.

Consumer optimization problem will be:

$$\text{Max}\ (X_i, Y_i, Z_i)\ X_i^{\,a_i}Y_i^{\,b_i} - P_xX_i - P_yY_i - PZ_i \qquad (9)$$

Such that

$$X_i \leq \bar{X}\,Z_i \qquad (10)$$

$$Y_i \leq \bar{Y}\,Z_i \qquad (11)$$

$$X_i^{\,a_i}Y_i^{\,b_i} - P_xX_i - P_yY_i - PZ_i \geq 0 \qquad (12)$$

$$Z_i = 0\ atau\ 1 \qquad (13)$$

Provider optimization problem will be:

$$\text{Max}_{\,P_x,P_y,P}\ m\left(P_x\text{X}_1{}^* + P_y\text{Y}_1{}^* + PZ_1{}^*\right) + n\left(P_x\text{X}_2{}^* + P_y\text{Y}_2{}^* + PZ_2{}^*\right) \quad (4.14)$$

with $(\text{X}_\text{i}{}^*,\ \text{Y}_\text{i}{}^*, \text{Z}_\text{i}{}^*) = \text{argmax}\ X_i^{\,a_i}Y_i^{\,b_i} - P_xX_i - P_yY_i - PZ_i$

such that

$$X_i \leq \bar{X}\,Z_i$$

$$Y_i \leq \bar{Y}\,Z_i$$

$$X_i^{\,a_i}Y_i^{\,b_i} - P_xX_i - P_yY_i - PZ_i \geq 0$$

$Z_i = 0$ or 1

We discuss the way to determine the maximum profit for each pricing scheme provided by service provider.

Case 4a. If the providers use the pure flat fee price by setting up $P_x = 0, P_y = 0$ and $P > 0$, it means that the price that price that providers use will not affect the consumption time (peak and off peak hours), then the consumers can select maximum level of consumption $X_1 = \bar{X}$, $X_2 = \bar{X}$, $Y_1 = \bar{Y}$, and $Y_2 = \bar{Y}$. So, each high level consumer will be charged $P \leq \bar{X}^{a_1}\bar{Y}^{b_1}$ and low level consumer is of $P \leq \bar{X}^{a_2}\bar{Y}^{b_2}$. The case 4a is flat fee scheme so *P* or its price is referred to both heterogeneous consumer types. If we set up $a_1 > a_2$ then the prices for high level consumers will follow the prices for low level consumers $a_1 > a_2 \Leftrightarrow \bar{X}^{a_1} > \bar{X}^{a_2}, b_1 > b_2 \Leftrightarrow \bar{X}^{b_1} > \bar{X}^{b_2}$

Assume that $(m)\bar{X}^{a_1}\bar{Y}^{b_1} < (m+n)\bar{X}^{a_2}\bar{Y}^{b_2}$. it means, if $P = \bar{X}^{a_1}\bar{Y}^{b_1}$, than only the high level consumers can adopt this service. If $P = \bar{X}^{a_2}\bar{Y}^{b_2}$, then both consumers can adopt this service. So, to maximize the profit, the providers will charge$P = \bar{X}^{a_2}\bar{Y}^{b_2}$.

So, the providers optimization problem will be:

$\text{Maks}_{\ P}\ m(PZ_1^*) + n\,(PZ_2^*) = m(\bar{X}^{a_2}\bar{Y}^{b_2}) + n\,(\bar{X}^{a_2}\bar{Y}^{b_2}) = (m+n)\,(\bar{X}^{a_2}\bar{Y}^{b_2})$

So the attainable maximum profit of providers will be $(m+n)(\bar{X}^{a_2}\bar{Y}^{b_2})$; *m* is the number of high level consumers and *n* is low level consumers. According to this case, we obtain Lemma 4a.

Lemma 4a: If the service providers apply flat fee scheme, then the price that providers can charge is $\bar{X}^{a_2}\bar{Y}^{b_2}$ with attainable maximum profit of $(m+n)\,[\bar{X}^{a_2}\bar{Y}^{b_2}]$.

Case 5a. if the providers use usage-based scheme by setting up $P_x > 0, P_y > 0$ and $P = 0$, it means that the providers use differentiation prices, peak price and nonpeak price. Then:
The optimization problem of high end heterogeneous consumers will be:

$$\text{Maks}_{\ X,Y,Z} = X_1^{a_1} Y_1^{b_1} - P_x X_1 - P_y Y_1 \tag{15}$$

To optimize the price we use the necessary and sufficient conditions:

(i) For necessary condition $\frac{\partial(X_1^{a_1} Y_1^{b_1} - P_x X_1 - P_y Y_1)}{\partial X_1} = 0$, then from the differential process we obtain:

$$a_1\ X_1^{a_1-1} Y_1^{b_1} = P \ \Leftrightarrow\ X_1^{a_1-1} = \frac{P_x}{a_1 Y_1^{b_1}} \Leftrightarrow X_1^* = \left(\frac{P_x}{a_1 Y_1^{b_1}}\right)^{\frac{1}{a_1-1}}{}_{x} \tag{16}$$

(ii) Sufficient condition. $\frac{\partial^2(X_1^{a_1} Y_1^{b_1} - P_x X_1 - P_y Y_1)}{\partial X_1^2} > 0$

$$\Leftrightarrow \frac{\partial(a_1\ X_1^{a_1-1} Y_1^{b_1} - P_x)}{\partial X_1} = a_1\,(a_1-1) X_1^{a_1-2} Y_1^{b_1} > 0;\ a_1, b_1 > 0$$

It means that $P_x = a_1\ X_1^{a_1-1} Y_1^{b_1}$ is minimum price and

(i) Necessary condition. $\frac{\partial(X_1^{a_1} Y_1^{b_1} - P_x X_1 - P_y Y_1)}{\partial Y_1} = 0$, from the differential result we have :

$$b_1 X_1^{a_1}\ Y_1^{b_1-1} = P_y \ \Leftrightarrow\ Y_1^{b_1-1} = \frac{P_y}{b_1 X_1^{a_1}} \Leftrightarrow Y_1^* = \left(\frac{P_y}{b_1 X_1^{a_1}}\right)^{\frac{1}{b_1-1}} \tag{17}$$

(ii) Sufficient condition

$$\frac{\partial^2(X_1^{a_1} Y_1^{b_1} - P_x X_1 - P_y Y_1)}{\partial Y_1^2} > 0 \Leftrightarrow \frac{\partial(b_1 X_1^{a_1}\ Y_1^{b_1-1} - P_y)}{\partial Y_1} = b_1\,(b_1-1) X_1^{a_1}\ Y_1^{b_1-2} > 0;$$

$$a_1, b_1 > 0$$

It means that $P_y = b_1 X_1^{a_1}\ Y_1^{b_1-1}$ is minimum price.

The optimization problem of low end heterogeneous consumers will be:

$$\text{Maks}_{X,Y,Z} = X_2^{a_2} + Y_2^{b_2} - P_x X_2 - P_y Y_2.$$

To optimize the prices we use the necessary and sufficient condition as follows.

(i) The necessary condition

$\frac{\partial (X_2^{a_2} Y_2^{b_2} - P_x X_2 - P_y Y_2)}{\partial X_2} = 0$, from the differential result we have:

$$a_2 X_2^{a_2 -1} Y_2^{b_2} = P_x \Leftrightarrow X_2^{a_2 -1} = \frac{P_x}{a_2 Y_2^{b_2}} \Leftrightarrow X_2^* = \left(\frac{P_x}{a_2 Y_2^{b_2}}\right)^{\frac{1}{a_2 - 1}} \quad (18)$$

(ii) The sufficient condition

$$\frac{\partial^2 (X_2^{a_2} Y_2^{b_2} - P_x X_2 - P_y Y_2)}{\partial X_2^2} > 0 \Leftrightarrow \frac{\partial (a_2 X_2^{a_2 -1} Y_2^{b_2} - P_x)}{\partial X_2} = a_2 (a_2 - 1) X_2^{a_2 -2} Y_2^{b_2} > 0;$$

$$a_2, b_2 > 0$$

It means that $P_x = a_2 X_2^{a_2 -1} Y_2^{b_2}$ is minimum price and

(i) The necessary condition

$\frac{\partial (X_2^{a_2} Y_2^{b_2} - P_x X_2 - P_y Y_2)}{\partial Y_2} = 0$, from the differential result we have:

$$b_2 X_2^{a_2} Y_2^{b_2 -1} = P_y \Leftrightarrow Y_2^{b_2 -1} = \frac{P_y}{b_2 X_2^{a_2}} \Leftrightarrow Y_2^* = \left(\frac{P_y}{b_2 X_2^{a_2}}\right)^{\frac{1}{b_2 - 1}} \quad (19)$$

(ii) The sufficient condition

$$\frac{\partial^2 (X_2^{a_2} Y_2^{b_2} - P_x X_2 - P_y Y_2)}{\partial Y_2^2} > 0 \Leftrightarrow \frac{\partial (b_2 X_2^{a_2} Y_2^{b_2 -1} - P_y)}{\partial Y_2} = b_2 (b_2 - 1) X_2^{a_2} Y_2^{b_2 -2} > 0$$

$$a_2, b_2 > 0$$

It means that $P_y = b_2 X_2^{a_2} Y_2^{b_2 -1}$ is minimum price.

This analysis can be applied to peak hour and off peak problems.

(i) For problem during peak hours :
The providers should minimize P_x; $P_x \leq a_1 X_1^{a_1 -1} Y_1^{b_1}$ to maximize objective function (15). On the other, if the providers already set up the price $P_x \leq a_2 X_2^{a_2 -1} Y_2^{b_2}$, then the profit is not optimal if $X_1^* \leq \bar{X}$ or $X_2^* \leq \bar{X}$. So, the best price P_x will be $a_2 X_2^{a_2 -1} Y_2^{b_2} \leq P_x \leq a_1 X_1^{a_1 -1} Y_1^{b_1}$.

(ii) For problem during off peak hours:
The providers should minimize P_y; $P_y \leq b_1 Y_1^{b_1 -1} X_1^{a_1}$, to maximize objective function (15). On the other hand, if the providers set up $P_y \leq b_2 X_2^{a_2} Y_2^{b_2 -1}$, then there is no optimal profit if $Y_1^* \leq \bar{Y}$ atau $Y_2^* \leq \bar{Y}$. So, the best P_y price is $b_2 X_2^{a_2} Y_2^{b_2 -1} \leq P_y \leq b_1 Y_1^{b_1 -1} X_1^{a_1}$.

When the price is in interval $a_2 X_2^{a_2 -1} Y_2^{b_2} \leq P_x \leq a_1 X_1^{a_1 -1} Y_1^{b_1}$ and $b_2 X_2^{a_2} Y_2^{b_2 -1} \leq P_y \leq b_1 Y_1^{b_1 -1} X_1^{a_1}$, the demand of high end consumers will still remain at a $\bar{X}$ and $\bar{Y}$, meanwhile the demand of low end consumers will increase gradually since the price goes down. So, both consumers (high end and low end) can apply this service with optimal price for peak hour is $P_x = a_2 X_2^{a_2 -1} Y_2^{b_2}$ and the off peak hour optimal price will be $P_y = b_2 X_2^{a_2} Y_2^{b_2 -1}$.

The provider optimization problem will be:

$$\text{Maks}_{P_x, P_y} \; m\left(P_x \mathrm{X}_1^* + P_y \mathrm{Y}_1^*\right) + n\left(P_x \mathrm{X}_2^* + P_y \mathrm{Y}_2^*\right)$$

$$= (m+n)\left[P_x\left(\frac{P_x^{\left(\frac{1}{a_2-1}\right)}}{a_2^{\left(\frac{1}{a_2-1}\right)}Y_2^{\left(\frac{b_2}{a_2-1}\right)}}\right)+P_y\left(\frac{P_y^{\left(\frac{1}{b_2-1}\right)}}{b_2^{\left(\frac{1}{b_2-1}\right)}X_2^{\left(\frac{a_2}{b_2-1}\right)}}\right)\right]$$

$$= (m+n)\left[\left(\frac{P_x^{\left(1+\frac{1}{a_2-1}\right)}}{a_2^{\left(\frac{1}{a_2-1}\right)}Y_2^{\left(\frac{b_2}{a_2-1}\right)}}\right)+\left(\frac{P_y^{\left(1+\frac{1}{b_2-1}\right)}}{b_2^{\left(\frac{1}{b_2-1}\right)}X_2^{\left(\frac{a_2}{b_2-1}\right)}}\right)\right]$$

$$= (m+n)\left[\left(\frac{P_x^{\left(\frac{a_2}{a_2-1}\right)}}{a_2^{\left(\frac{1}{a_2-1}\right)}Y_2^{\left(\frac{b_2}{a_2-1}\right)}}\right)+\left(\frac{P_y^{\left(\frac{b_2}{b_2-1}\right)}}{b_2^{\left(\frac{1}{b_2-1}\right)}X_2^{\left(\frac{a_2}{b_2-1}\right)}}\right)\right]$$

$$= (m+n)\left[\left(\frac{\left(a_2X_2^{a_2-1}Y_2^{b_2}\right)^{\left(\frac{a_2}{a_2-1}\right)}}{a_2^{\left(\frac{1}{a_2-1}\right)}Y_2^{\left(\frac{b_2}{a_2-1}\right)}}\right)+\left(\frac{\left(b_2X_2^{a_2}Y_2^{b_2-1}\right)^{\left(\frac{b_2}{b_2-1}\right)}}{b_2^{\left(\frac{1}{b_2-1}\right)}X_2^{\left(\frac{a_2}{b_2-1}\right)}}\right)\right]$$

$$= (m+n)\left[\left(\frac{a_2^{\left(\frac{a_2}{a_2-1}\right)}X_2^{(a_2-1)\left(\frac{a_2}{a_2-1}\right)}Y_2^{b_2\left(\frac{a_2}{a_2-1}\right)}}{a_2^{\left(\frac{1}{a_2-1}\right)}Y_2^{\left(\frac{b_2}{a_2-1}\right)}}\right)+\left(\frac{b_2^{\left(\frac{b_2}{b_2-1}\right)}X_2^{a_2\left(\frac{b_2}{b_2-1}\right)}Y_2^{(b_2-1)\left(\frac{b_2}{b_2-1}\right)}}{b_2^{\left(\frac{1}{b_2-1}\right)}X_2^{\left(\frac{a_2}{b_2-1}\right)}}\right)\right]$$

$$= (m+n)\left[a_2X_2^{a_2}Y_2^{b_2}+b_2X_2^{a_2}Y_2^{b_2}\right] = (m+n)(a_2+b_2)\left[X_2^{a_2}Y_2^{b_2}\right]$$

The optimal price for peak hour will be $P_x = a_2\bar{X}^{(a_2-1)}\bar{Y}^{b_2}$ and off peak hour optimal price will be $P_y = b_2\bar{Y}^{(b_2-1)}\bar{X}^{a_2}$ with maximum profit of $(m+n)(a_2+b_2)(\bar{X}^{a_2}\bar{Y}^{b_2})$. According to this case, we obtain Lemma 5a.

Lemma 5a: If the provider use usage-based price, then the optimal price for peak hour will be $P_x = a_2\bar{X}^{(a_2-1)}\bar{Y}^{b_2}$ and off peak hour optimal price will be $P_y = b_2\bar{Y}^{(b_2-1)}\bar{X}^{a_2}$ with maximum profit of $(m+n)(a_2+b_2)(\bar{X}^{a_2}\bar{Y}^{b_2})$.

Case 6a. If the providers use two-part tariff then $P_x > 0$, $P_y > 0$, and $P > 0$. First order condition to optimization problem of high end/low end consumers is by using Eq.(16)-(19). Eq.(16) and (18) are high end consumer demand curve and low consumer in peak hour. Eq. (17) and (19) are the demand curves of high end and low end consumers during off peak hour. If $a_1 > a_2$ then high end price will follow the low end price, then $a_1(m) < a_2(m+n) \Leftrightarrow a_1 < \frac{a_2(m+n)}{m}$

It means that the consumers will be charge of $P_x = a_1X_1^{(a_1-1)}, P_y = b_1Y_1^{(b_1-1)}X_1^{a_1}$ and $P = X_1^{a_1}Y_1^{b_1} - (a_1+b_1)\left(X_1^{a_1}Y_1^{b_1}\right)$, then high end consumers will adopt this service. If the consumers price are $P_x = a_2X_2^{a_2-1}Y_2^{b_2}$, $P_y = b_2Y_2^{b_2-1}X_2^{a_2}$ and $P = X_2^{a_2}Y_2^{b_2} - (a_2 +$ $b2\ X2a2\ Y2b2$, then both consumers can adopt this scheme. This is due to many consumers see that subscription fee as barrier entry, then the providers can select to decrease this barrier entry to attract consumer's attention. Then to maximize the profit, the providers charge

$P_x = a_2X_2^{a_2-1}Y_2^{b_2}$, $P_y = b_2Y_2^{b_2-1}X_2^{a_2}$ and $P = X_2^{a_2}Y_2^{b_2} - (a_2+b_2)\left(X_2^{a_2}Y_2^{b_2}\right)$.

The provider optimization problem will be:

$$\text{Maks}_{P_x,P_y}\ m\left(P_xX_1^* + P_yY_1^* + PZ_1^*\right) + n\left(P_xX_2^* + P_yY_2^* + PZ_2^*\right)$$

$$= m\left[a_2\bar{X}^{a_2}\bar{Y}^{b_2} + b_2\bar{X}^{a_2}\bar{Y}^{b_2} + \left\{X_2^{a_2}Y_2^{b_2} - (a_2+b_2)\left(X_2^{a_2}Y_2^{b_2}\right)\right\}\right] + n\left[a_2\bar{X}^{a_2}\bar{Y}^{b_2} + b_2\bar{X}^{a_2}\bar{Y}^{b_2} + \left\{X_2^{a_2}Y_2^{b_2} - (a_2+b_2)\left(X_2^{a_2}Y_2^{b_2}\right)\right\}\right]$$

$$= m(\bar{X}^{a_2}\bar{Y}^{b_2}) + n(\bar{X}^{a_2}\bar{Y}^{b_2}) = (m+n)(\bar{X}^{a_2}\bar{Y}^{b_2})$$

Lemma 6a: If the providers use two-part tariff scheme, optimal P_x and P_y will be $a_2\,\bar{X}^{a_2-1}\bar{Y}^{b_2}$, $b_2\,\bar{X}^{a_2}\,\bar{Y}^{b_2-1}$, and $P = X_2{}^{a_2}\,Y_2{}^{b_2} - (a_2+b_2)\left(X_2{}^{a_2}\,Y_2{}^{b_2}\right)$, with attainable maximum profit of $(m+n)[\bar{X}^{a_2}\,\bar{Y}^{b_2}]$.If $a_2\,\bar{X}^{a_2} > \bar{X}^{a_2}$ and $b_2\,\bar{Y}^{b_2} > \bar{Y}^{b_2}$ then$(m+n)\,(a_2+b_2)(\bar{X}^{a_2}\,\bar{Y}^{b_2}) >$ $(m+n)\,[\bar{X}^{a_2}\,\bar{Y}^{b_2}]$; $a_2, b_2 > 0$. That is why, if there exist two type of consumers based on willingness to pay then the usage-based price is better than flat fee and two-part tariff schemes. So, the usage based price always benefits and dominates the pure flat-fee and two-part tariff schemes. For high end and low end consumers based on willingness to pay, the providers should set up prices that encourage the consumers to consume much. If the set up price follows for high end consumers then the providers will lose the low end consumers. If the set up price follows the low end consumers, then both consumers can adopt the scheme.

3.3. High Demand and Low Demand Heterogeneous Consumers

For example, we assume that there exist two types of consumers which are high demand consumers (type 1) with maximum consumption level of $\bar{X}_1$ and $\bar{Y}_1$ and low demand consumers (type 2) with maximum level of consumption $\bar{X}_2$ and $\bar{Y}_2$. There are m consumer of type 1 dan *n* consumer of type 2 with $a_1 = a_2 = a$ and $b_1 = b_2 = b$.

The discussion about to obtain the maximum profit for each pricing scheme applied by providers is as follows (follows from [1]).

Case 7a. If the providers would like to apply flat-fee scheme then providers will set up $P_x = 0$, $P_y = 0$ and $P > 0$. it means that this scheme will have price if high and low demand consumers choose to join the scheme then the consumers will fully utilize the scheme by choosing the consumption level of $X_1 = \bar{X}_1,\ Y_1 = \bar{Y}_1$ or $X_2 = \bar{X}_2,\ Y_2 = \bar{Y}_2$ with maximum utility $\bar{X}_1{}^a\bar{Y}_1{}^b$ or $\bar{X}_2{}^a\bar{Y}_2{}^b$ (high and low level consumers, respectively). Then, the providers will give a price to each high consumption level consumers of $P \le \bar{X}_1{}^a\bar{Y}_1{}^b$ and each low level consumer which is not more than $P \le \bar{X}_2{}^a\bar{Y}_2{}^b$as flat fee service. Since we assume that the providers cannot differentiate the high and low level demand consumers and have to charge the same price for both consumers then the providers should set $P = \bar{X}_1{}^a\bar{Y}_1{}^b$ by only serving the high level demand consumers or fixing the price $P = \bar{X}_2{}^a\bar{Y}_2{}^b$to serve two type of consumers. If we assume that $m\left(\bar{X}_1{}^a\bar{Y}_1{}^b\right) < (m+n)\left(\bar{X}_2{}^a\bar{Y}_2{}^b\right)$, then the providers can set up $P = \bar{X}_2{}^a\bar{Y}_2{}^b$ and serve both consumers with attainable maximum profit of $(m+n)\left(\bar{X}_2{}^a\bar{Y}_2{}^b\right)$. According to this case we proceed to Lemma 7a.

Lemma 7a : if the providers use the flat fee scheme, then the fixed price will be $P = \bar{X}_2{}^a\bar{Y}_2{}^b$ with the attainable maximum profit of $\left(\bar{X}_2{}^a\bar{Y}_2{}^b\right)$.

Case 8a. if the providers use usage based scheme by setting up $P_x > 0$, $P_y > 0$, and $P = 0$, then the providers provide the differentiation prices which are peak and off peak hours. First order conditions for optimality of consumers of high and low demand level will be

For high demand level consumers:

$$\text{Maks}_{\,X,Y,Z} = X_1{}^aY_1{}^b - P_xX_1 - P_yY_1$$

To optimize the prices we use the necessary and sufficient conditions as follows.

(i) Necessary condition. $\frac{\partial(X_1{}^a\,Y_1{}^b - P_xX_1 - P_yY_1)}{\partial X_1} = 0$. From that differensial, we have

$$a\,X_1{}^{a-1}Y_1{}^b = P_x \Leftrightarrow X_1{}^* = \left(\frac{P_x}{aY_1{}^b}\right)^{\frac{1}{a-1}} \quad (20)$$

(ii) Sufficient condition

$$\frac{\partial^2(X_1{}^a\,Y_1{}^b - P_xX_1 - P_yY_1)}{\partial X_1{}^2} > 0 \Leftrightarrow \frac{\partial(a\,X_1{}^{a-1}Y_1{}^b - P_x)}{\partial X_1} = a\,(a-1)X_1{}^{a-2}Y_1{}^b > 0; a, b > 0$$

and

(i) Necessary condition

$\frac{\partial(X_1{}^aY_1{}^b - P_xX_1 - P_yY_1)}{\partial Y_1} = 0$. From that differensial, we have

$$b\,Y_1{}^{b-1}X_1{}^a = P_y \Leftrightarrow Y_1{}^* = \left(\frac{P_y}{bX_1{}^a}\right)^{\frac{1}{b-1}} \quad (21)$$

(iii) Sufficient condition

$$\frac{\partial^2(X_1{}^aY_1{}^b - P_xX_1 - P_yY_1)}{\partial Y_1{}^2} > 0 \Leftrightarrow \frac{\partial(b\,Y_1{}^{b-1}X_1{}^a - P_y)}{\partial Y_1} = b(b-1)Y_1{}^{b-2}X_1{}^a > 0; a, b > 0$$

For low level heterogeneous consumers:

$$\text{Max}_{X,Y,Z} = X_2{}^aY_2{}^b - P_xX_2 - P_yY_2$$

To optimize the prices we use the necessary and sufficient conditions as follows.

(i) Necessary condition

$\frac{\partial(X_2{}^aY_2{}^b - P_xX_2 - P_yY_2)}{\partial X_2} = 0$. From that differential, we have

$$a\,X_2{}^{a-1}Y_2{}^b = P_x \Leftrightarrow X_2{}^* = \left(\frac{P_x}{aY_2{}^b}\right)^{\frac{1}{a-1}} \quad (22)$$

(ii) Sufficient condition

$\frac{\partial^2(X_2{}^aY_2{}^b - P_xX_2 - P_yY_2)}{\partial X_2{}^2} > 0 \Leftrightarrow \frac{\partial(a\,X_2{}^{a-1}Y_2{}^b - P_x)}{\partial X_2} = a\,(a-1)X_2{}^{a-2}Y_2{}^b > 0; a, b > 0$ and

(i) Necessary condition

$\frac{\partial(X_2{}^aY_2{}^b - P_xX_2 - P_yY_2)}{\partial Y_2} = 0$. From that differential, we have

$$b\,Y_2{}^{b-1}X_2{}^a = P_y \Leftrightarrow Y_2{}^* = \left(\frac{P_y}{bX_2{}^a}\right)^{\frac{1}{b-1}} \quad (23)$$

If we assume that $m(\bar{X}_1) < (m+n)(\bar{X}_2)$, then the providers can fix $P_x = a\,X_2{}^{a-1}Y_2{}^b$ and $P_y = b\,Y_2{}^{b-1}X_2{}^a$ which serve high and low consumption level consumers.
The provider optimization problem will be

$$\text{Max}_{P_x,P_y}\ m\left(P_x\mathrm{X}_1{}^* + P_y\mathrm{Y}_1{}^*\right) + n\left(P_x\mathrm{X}_2{}^* + P_y\mathrm{Y}_2{}^*\right) = (m+n)\left[P_x\left(\frac{P_x}{aY_2{}^b}\right)^{\frac{1}{a-1}} + P_y\left(\frac{P_y}{bX_2{}^a}\right)^{\frac{1}{b-1}}\right]$$

$$= (m+n)\left[\frac{P_x{}^{\frac{a}{a-1}}}{a^{\frac{1}{a-1}}Y_2{}^{\frac{b}{a-1}}} + \frac{P_y{}^{\frac{b}{b-1}}}{b^{\frac{1}{b-1}}X_2{}^{\frac{a}{b-1}}}\right]$$

It is given that as long as P_x and P_y decrease, then $X_1{}^*$, $X_2{}^*$, $Y_1{}^*$ and $\mathrm{Y}_2{}^*$ will increase. However, sinceX_1, X_2, Y_1 and Y_2 are subjected to $\bar{X}_1$, $\bar{X}_2$, $\bar{Y}_1$ and $\bar{Y}_2$ then the best P_x and P_y will be $a\,\bar{X}_2{}^{a-1}\,\bar{Y}_2{}^b$ dan $b\bar{X}_2{}^a\,\bar{Y}_2{}^{b-1}$with attainable maximum profit

$$(m+n)\left[\frac{P_x^{\frac{a}{a-1}}}{a^{\frac{1}{a-1}}\bar{Y}_2^{\frac{b}{a-1}}}+\frac{P_y^{\frac{b}{b-1}}}{b^{\frac{1}{b-1}}\bar{X}_2^{\frac{a}{b-1}}}\right]=(m+n)\left[\frac{\left(a\,\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}\right)^{\frac{a}{a-1}}}{a^{\frac{1}{a-1}}\bar{Y}_2^{\frac{b}{a-1}}}+\frac{\left(b\,\bar{X}_2^{\,b-1}X_2^{\,a}\right)^{\frac{b}{b-1}}}{b^{\frac{1}{b-1}}\bar{X}_2^{\frac{a}{b-1}}}\right]=$$

$$(m+n)\left[\frac{a^{\frac{a}{a-1}}.\ \bar{X}_2^{\,(a-1)\left(\frac{a}{a-1}\right)}}{a^{\frac{1}{a-1}}\bar{Y}_2^{\frac{b}{a-1}}}+\frac{b^{\frac{b}{b-1}}.\ \bar{Y}_2^{\,(b-1)\left(\frac{b}{b-1}\right)}}{b^{\frac{1}{b-1}}\bar{X}_2^{\frac{a}{b-1}}}\right]=(m+n)(a+b)\left[\bar{X}_2^{\,a}\bar{Y}_2^{\,b}\right]$$

So, if the providers use the pure usage-based scheme, then the peak hour optimal prices will be $P_x = a\bar{X}_2^{\,a-1}$ and in off peak hours will be $P_y = b\bar{Y}_2^{\,b-1}$ with attainable maximum profit of $(m+n)(a+b)\left[\bar{X}_2^{\,a}\bar{Y}_2^{\,b}\right]$. According to this case we obtain Lemma 8a.

Lemma 8a : If the providers apply usage based price, then the peak hour optimal price will be $P_x = a\,\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}$ and the off peak hour optimal price will be $P_y = b\,\bar{Y}_2^{\,b-1}\bar{X}_2^{\,a}$. The maximum profit is $(m+n)(a+b)\left[\bar{X}_2^{\,a}\bar{Y}_2^{\,b}\right]$.

Kasus 9a. if the providers use Jika two-part tariff, $P_x > 0$, $P_y > 0$, and $P > 0$, the first order condition for optimization of high end/low end consumers is by using Eq. (20)-(23).

It is known that $X_1 > X_2 \Leftrightarrow aX_1^{\,a-1}Y_1^{\,b} > b\,X_2^{\,b-1}Y_2^{\,b}$, the best $P_x = a\,X_2^{\,a-1}Y_2^{\,b}$ fixed by the providers. Then the high and low level consumption consumers can adopt. If they use $P_x = aX_1^{\,a-1}Y_1^{\,b}$, then the providers can only attract the high level demand consumers only. It occurs also for off peak hour price, $P_y = b\,X_2^{\,a}\,Y_2^{\,b-1}$ is the best off peak hour price.

By using Eq.(15) , we have

$$X_2^{\,a}Y_2^{\,b} - P_xX_2 - P_yY_2 - PZ_2 \geq 0 \Leftrightarrow X_2^{\,a}Y_2^{\,b} - \left(a\,X_2^{\,a-1}Y_2^{\,b}\right)X_2 - \left(b\,X_2^{\,a}\,Y_2^{\,b-1}\right)Y_2 - P \geq 0$$

$$\Leftrightarrow X_2^{\,a}Y_2^{\,b} - aX_2^{\,a}Y_2^{\,b} - bX_2^{\,a}\,Y_2^{\,b} - P \geq 0 \ \Leftrightarrow P \leq X_2^{\,a}Y_2^{\,b} - aX_2^{\,a}Y_2^{\,b} - bX_2^{\,a}Y_2^{\,b}$$

karena $X_1^* \leq \bar{X}_1$ dan $X_2^* \leq \bar{X}_2$, maka $P \leq \bar{X}_2^{\,a}\bar{Y}_2^{\,b} - a\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - b\bar{X}_2^{\,a}\bar{Y}_2^{\,b}$

The optimization problem of the provider will be

$$\text{Max}_{\,P_x,P_y,P} = m\left(P_x\text{X}_1^* + P_y\text{Y}_1^* + P\text{Z}_1^*\right) + n\left(P_x\text{X}_2^* + P_y\text{Y}_2^* + P\text{Z}_2^*\right)$$

$$= m\left[\left(a\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}\right)\bar{X}_1 + \left(b\bar{X}_2^{\,a}\bar{Y}_2^{\,b-1}\right)\bar{Y}_1 + \left\{\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - a\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - b\bar{X}_2^{\,a}\bar{Y}_2^{\,b}\right\}\right]$$
$$+\,n\left[\left(a\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}\right)\bar{X}_2 + \left(b\bar{X}_2^{\,a}\bar{Y}_2^{\,b-1}\right)\bar{Y}_2 + \left\{\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - a\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - b\bar{X}_2^{\,a}\bar{Y}_2^{\,b}\right\}\right]$$

$$= m\left[\left(a\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}\right)\bar{X}_1 + \left(b\bar{X}_2^{\,a}\bar{Y}_2^{\,b-1}\right)\bar{Y}_1 + \left\{\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - a\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - b\bar{X}_2^{\,a}\bar{Y}_2^{\,b}\right\}\right]$$
$$+\,n\left[(a+b)\bar{X}_2^{\,a}\bar{Y}_2^{\,b} + \left\{\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - a\bar{X}_2^{\,a} - b\bar{Y}_2^{\,b}\right\}\right]$$

$$= m\left[\left(a\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}\right)\bar{X}_1 + \left(b\bar{X}_2^{\,a}\bar{Y}_2^{\,b-1}\right)\bar{Y}_1 + \left\{\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - a\bar{X}_2^{\,a}\bar{Y}_2^{\,b} - b\bar{X}_2^{\,a}\bar{Y}_2^{\,b}\right\}\right] + n\left(\bar{X}_2^{\,a}\bar{Y}_2^{\,b}\right) =$$
$$m\left[\left(a\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}\right)(\bar{X}_1 - \bar{X}_2) + \left(b\bar{X}_2^{\,a}\bar{Y}_2^{\,b-1}\right)(\bar{Y}_1 - \bar{Y}_2)\right] + (m+n)\left(\bar{X}_2^{\,a}\,\bar{Y}_2^{\,b}\right)$$

It means that the providers use two-part tariff scheme by setting up $P_x > 0$, $P_y > 0$, and $P > 0$, then the providers can determine the optimal price $P_x = a\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}$, $P_y = b\bar{X}_2^{\,a}\bar{Y}_2^{\,b-1}$, and subscription fee *P* which equals to low level consumers surplus, so we can obtain maximum profit of $m\left[\left(a\bar{X}_2^{\,a-1}\bar{Y}_2^{\,b}\right)(\bar{X}_1 - \bar{X}_2) + \left(b\bar{X}_2^{\,a}\bar{Y}_2^{\,b-1}\right)(\bar{Y}_1 - \bar{Y}_2)\right] + (m+n)\left(\bar{X}_2^{\,a}\,\bar{Y}_2^{\,b}\right)$. According to this case, we obtain Lemma 9a.

If $X \geq 0$ and $Y \geq 0$, then $m\left[\left(a\ \bar{X}_2^{\ a-1}\ \bar{Y}_2^{\ b}\right)(\bar{X}_1 - \bar{X}_2) + \left(b\ \bar{X}_2^{\ a}\ \bar{Y}_2^{\ b-1}\right)(\bar{Y}_1 - \bar{Y}_2)\right] + (m+n)\left(\bar{X}_2^{\ a}\ \bar{Y}_2^{\ b}\right) > (m+n)(a+b)\left[\bar{X}_2^{\ a}\bar{Y}_2^{\ b}\right] > (m+n)\left(\bar{X}_2^{\ a}\bar{Y}_2^{\ b}\right)$. So, *two-part tariff* scheme offers better price compared to pure *flat fee* and pure *usage-based* scheme for high demand and low demand heterogeneous consumers.

Lemma 9a: If the providers use two part tariff scheme, optimal P_x and P_y will be $a\bar{X}_2^{\ a-1}\bar{Y}_2^{\ b}$ dan $b\bar{X}_2^{\ a}\bar{Y}_2^{\ b-1}$, respectively and $P = \bar{X}_2^{\ a}\bar{Y}_2^{\ b} - a\bar{X}_2^{\ a}\bar{Y}_2^{\ b} - b\bar{X}_2^{\ a}\bar{Y}_2^{\ b}$. So, the attainable profit of providers will be $m\left[\left(a\bar{X}_2^{\ a-1}\bar{Y}_2^{\ b}\right)(\bar{X}_1 - \bar{X}_2) + \left(b\bar{X}_2^{\ a}\bar{Y}_2^{\ b-1}\right)(\bar{Y}_1 - \bar{Y}_2)\right] + (m+n)\left(\bar{X}_2^{\ a}\ \bar{Y}_2^{\ b}\right)$ that is larger price compared to attainable price by flat-fee or usage-based scheme.

The comparison between original Cobb-Douglass with modified Cobb Douglass proposed in [1] is presented in Table 1 as follows. The analysis to seek for the advantage of using original Cobb-Douglass is explained as follows.

1. **For Homogenous Consumers**, if we assume that
   a. $a\bar{X}^a > a\log(\bar{X}+1)$; $a > 0$ then $a\bar{X}^a > a\ log\ (\bar{X}+1) \Leftrightarrow \log e^{(a\bar{X}^a)} > log\ (\bar{X}+1)^a$.
   b. $b\bar{Y}^b > b\log(\bar{Y}+1)$; $b > 0$ then $b\bar{Y}^b > b\ log\ (\bar{Y}+1) \Leftrightarrow \log e^{(b\bar{Y}^b)} > log\ (\bar{Y}+1)^b$.
   So,$a\bar{X}^a + b\bar{Y}^b > a\ log\ (\bar{X}+1) + b\ log\ (\bar{Y}+1)$
   We obtain maximum profit by adopting original Cobb-Douglass.

2. **For High-End dan Low-End Heterogeneous Consumers**
   If we assume that
   a. $a_2\bar{X}^{a_2} > a_2\log(\bar{X}+1)$;$a_2 > 0, a_2\bar{X}^{a_2} > a_2\log(\bar{X}+1) \Leftrightarrow \log e^{(a_2\bar{X}^{a_2})} > log\ (\bar{X}+1)^{a_2}$
   b. $b_2\bar{Y}^{b_2} > b_2\log(\bar{Y}+1)$; $b_2 > 0$ then $b_2\bar{Y}^{b_2} > b_2\log(\bar{Y}+1) \Leftrightarrow \log e^{(b_2\bar{Y}^{b_2})} > log\ (\bar{Y}+1)^{b_2}$

   So we obtain $a_2\bar{X}^{a_2} + b_2\bar{Y}^{b_2} > a_2\log(\bar{X}+1) + b_2\log(\bar{Y}+1)$ and obtain maximum profit by adopting original Cobb-Douglass.

3. **For High-Demand dan Low-Demand Heterogeneous Consumers**
   I we assume that
   a. $\left(a\bar{X}_2^{\ a-1}\right)(\bar{X}_1 - \bar{X}_2) > \frac{a}{\bar{X}_1+1}(\bar{X}_1 - \bar{X}_2)$; $a > 0$ then $\bar{X}_2^{\ a} > a\log(\bar{X}_2+1)$; $a > 0$

$$\bar{X}_2 < \bar{X}_1 \Leftrightarrow \bar{X}_2 \geq \frac{1}{\bar{X}_1} \Leftrightarrow \bar{X}_2 > \frac{1}{\bar{X}_1+1} \Leftrightarrow a\bar{X}_2 > \frac{a}{\bar{X}_1+1}$$

$\Leftrightarrow a\bar{X}_2^{\ a-1} > \frac{a}{\bar{X}_1+1} \Leftrightarrow \left(a\bar{X}_2^{\ a-1}\right)(\bar{X}_1 - \bar{X}_2) > \frac{a}{\bar{X}_1+1}(\bar{X}_1 - \bar{X}_2)$. Next,

$$\bar{X}_2^{\ a} > a\log(\bar{X}_2+1) \Leftrightarrow \log e^{(\bar{X}_2^{\ a})} > \log\ (\bar{X}_2+1)^a$$

   b. $\left(b\bar{Y}_2^{\ b-1}\right)(\bar{Y}_1 - \bar{Y}_2) > \frac{b}{\bar{Y}_1+1}\left((\bar{Y}_1 - \bar{Y}_2)\right)$; $b > 0$ then $\bar{Y}_2^{\ b} > b\log(\bar{Y}_2+1)$; $b > 0$

$$\bar{Y}_2 < \bar{Y}_1 \Leftrightarrow \bar{Y}_2 \geq \frac{1}{\bar{Y}_1} \Leftrightarrow \bar{Y}_2 > \frac{1}{\bar{Y}_1+1} \Leftrightarrow b\bar{Y}_2 > \frac{b}{\bar{Y}_1+1} \Leftrightarrow b\bar{Y}_2^{\ b-1} > \frac{b}{\bar{Y}_1+1}$$

$$\Leftrightarrow \left(b\bar{Y}_2^{\ b-1}\right)(\bar{Y}_1 - \bar{Y}_2) > \frac{b}{\bar{Y}_1+1}\left((\bar{Y}_1 - \bar{Y}_2)\right)$$

$$\bar{Y}_2^{\ b} > b\log(\bar{Y}_2+1) \Leftrightarrow \log e^{(\bar{Y}_2^{\ b})} > \log\ (\bar{Y}_2+1)^b$$

We have $m\left[\left(a\bar{X}_2{}^{a-1}\right)(\bar{X}_1-\bar{X}_2)+\left(b\bar{Y}_2{}^{b-1}\right)(\bar{Y}_1-\bar{Y}_2)\right]+(m+n)\ \left(\bar{X}_2{}^{a}+\ \bar{Y}_2{}^{b}\right)>$ $m\left[\frac{a}{\bar{X}_1+1}(\bar{X}_1-\bar{X}_2)+\frac{b}{\bar{Y}_1+1}\left((\bar{Y}_1-\bar{Y}_2)\right)\right]+\ (m+n)[a\log(\bar{X}_2+1)+\ b\log(\bar{Y}_2+1)]$

Again, we obtain maximum profit by using Cobb-Douglass utility function.

Table 1. Comparison between Original Cobb-Douglass and Modified Cobb-Douglass

Consumer type		Original Cobb-Douglass	Modified Cobb-Douglass
Homogen	Standard form	X^a+Y^b	$a\log(\bar{X}+1)$ $+b\log(\bar{Y}+1)$
	Pricing Scheme	*Usage-Based*	Flat-Fee atau Two-Part Tariff
	Consumer Price	$P_x=a\bar{X}^{a-1}$ and $P_y=b\bar{Y}^{b-1}$	$P=a\log(\bar{X}+1)+\ b\log(\bar{Y}+1)$
	Maximum profit for provider	$\sum_i[a\bar{X}^a+b\bar{Y}^b]$	$\sum_i[a\log(\bar{X}+1)+\ b\log(\bar{Y}+1)]$
Heterogen *High-End* dan *Low-End*	Pricing Scheme	Usage-Based	Flat-Fee atau Two-Part Tariff
	Consumer Price	$P_x=\ a_2\,\bar{X}^{a_2-1}$ and $P_y=\ b_2\,\bar{Y}^{b_2-1}$	$P=a_2\ \log(\bar{X}+1)$ $+\,b_2\ \log(\bar{Y}+1)$
	Maximum profit for provider	$(m+n)$ $(a_2\,\bar{X}^{a_2}+b_2\,\bar{Y}^{b_2})$	$(m+n)$ $[a_2\ \log(\bar{X}+1)+\ b_2\ \log(\bar{Y}+1)]$
Heterogen *High-Demand* dan *Low-Demand*	Pricing Scheme	Two-part tariff	Two-part tariff
	Consumer Price	$P_x=a\,\bar{X}_2{}^{a-1}$ and $P_y=\ b\,\bar{Y}_2{}^{b-1}$	$P_x=\frac{a}{\bar{X}_1+1}$ and $P_y=\frac{b}{\bar{Y}_1+1}$
	Maximum profit for provider	$m\left[\left(a\bar{X}_2{}^{a-1}\right)(\bar{X}_1-\bar{X}_2)+\right.$ *bY2b−1Y1−Y2+m+n X2a+ Y2b*	$m\left[\frac{a}{\bar{X}_1+1}(\bar{X}_1-\bar{X}_2)\right.$ $\left.+\frac{b}{\bar{Y}_1+1}\left((\bar{Y}_1-\bar{Y}_2)\right)\right]$ $+\ (m+n)[a\log(\bar{X}_2+1)$ $+\ b\log(\bar{Y}_2+1)]$

4. Conclusion

The main purpose of this paper is to aid ISP to determine the best strategy to be offered to consumers with optimal prices. The previous work done by researchers does not solve the problem of pricing strategy when ISP adopt flat fee and usage based scheme with or without subscription fee.

We found that the marginal and monitoring prices can be neglected for monopoly supplier with homogen consumers, pure usage based price is better than fat fee and two-part tariff since supplier gets maximum profit.

In case of heterogeneous marginal supply based on the willingness to pay, the usage based price can extract all low end consumer surpluses and leave the surplus only for high-end consumer if it is in company's benefit to serve the consumer. However, the flat-fee and two-part tariff prices can only extract all surpluses from low-end consumers.

If we compare to modified Cobb-Douglass proposed by [1] by using original Cobb-Douglass, we obtain maximum profit using the original Cobb-Douglass in some cases including pricing strategy, consumer price and maximum profit obtained by providers.

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