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MAXIMUM DRAWDOWN: MODELS AND APPLICATIONS

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Abstract

Financial series may possess fractal dimensions which would induce cycles of many different durations. This inherent characteristic would explain the turbulent cascades in stock markets when strong local dependence is observed. A drawdown is defined as the percentual accumulated loss due to a sequence of drops in the price of an investment. It is collected over non-fixed time intervals and its duration is also a random variable. The maximum drawdown occuring during a fixed investment horizon is a flexible measure that may provide a different perception of the risk and price flow of an investment. In this paper we propose statistical models from the extreme value theory for the severity and duration of the maximum drawdown. Our empirical results indicate that there may exist a relation between the pattern of the GARCH volatility of an investment and the fluctuations of the severity of the maximum drawdown and that, typically, extreme (but not outlying) maximum drawdowns occur during stress periods of high volatility. We suggest applications for the maximum drawdown, including the computation of the Maximum Drawdown-at-Risk with exceedance probability α, and the classification of investments according to their performance when controlling losses via the maximum drawdown.

Key Words: Drawdown; Drawup; Maximum Drawdown-at-Risk; Extreme value distributions; GARCH volatility.

Clients of financial institutions usually follow closely the performance of their investments, even when these are not meant for short term objectives. Although high volatility raises tension, a sequence of consecutive drops in the price of a portfolio may take the investor to withdraw from the market. On the other hand, a long lasting sequence of drops may be an extreme test for traders and their trading methodology. Traders typically fix a small proportion $0 < p < 1$ such that, if the capital available is less than p times the initial capital, the investment is withdrawn.

A drawdown is defined as the loss in percentual from the last local maximum to the next local minimum of an investment. Similarly, a drawup is defined as the gain from the last local minimum to the next local maximum of an investment. Both quantities are collected over non-fixed time intervals and their durations are also random variables. Risk measures based on the drawdown may provide a different perception of the risk and price flow of an investment. They may be seen as a complementary approach beyond the standard deviation and Value-at-Risk.

An important characteristic of a drawdown is that it is defined over highly correlated data. Many successive drops of a stock price suggest the existence of a time dependent subordinated process causing local dependence. This topic was first introduced by Mandelbrot [1967] in the context of modeling some aspects of a phenomenon showing an intermittent turbulence. According to Mandelbrot [1972, 1997]), financial time series would possess fractal dimensions which would induce cycles of many different durations. This inherent characteristic would explain the turbulent cascades in stock markets, the fat tails returns distributions, and the presence of long memory in stock returns and squared stock returns. Here we focus in the statistical modeling of these sequences of losses (gains) in stock markets.

Previous literature addressing the problem of obtaining the probability distribution of the drawdown includes Johansen and Sornette [2001], that used the Stretched Exponential to model the drawdown severity from indexes, commodities and currencies. They found that typically this distribution fits well the bulk of the data, but under-estimates one to ten extreme observations. Mendes and Brandi [2003] empirically showed that the Modified Generalized Pareto distribution and its sub-models fit very well the drawdown data, including most of those observations previously found to be atypical.

Other works focused on the usefulness of this measure in financial applications. For example, Chekhlov et al. [2000] introduced the conditional Drawdown-at-Risk (DaR) as an optimality constraint to obtain optimal portfolio allocation (see also Palmquist et al 1999).

In this paper we are interested in charactering the distribution of the maximum drawdown occurring during a fixed time period. We consider two definitions of the maximum drawdown and propose models from the extreme value theory for each one. We discuss which modeling strategy would be more operational to be used on a daily basis. Previous work of Acar and James [1997] also studied the distributional properties of the maximum drawdown. However, they started by assuming a normal distribution for the daily returns.

We provide an illustration using three stock indexes. Our empirical results indicate that there may exist a relation between the pattern of the GARCH volatility of an investment and the fluctuations of the severity of the maximum drawdown. We observed that typically extreme (but not outlying) maximum drawdowns occur during stress periods of high volatility. Our empirical findings also suggest that long lasting maximum drawdowns may occur during periods of low volatility and in this case they might be considered outliers. We recall that in many cases, traders, fund managers, and investors can handle periods of large volatility, but cannot handle a long string of consecutive losses.

Then we estimate the probability distribution of the maximum drawdown for the three indexes. We show that the empirical distribution when compared to the estimated one, underestimates high quantiles of the distribution of the maximum drawdown. The best fitted model is used to compute the risk measure MDaR_α, the Maximum Drawdown-at-Risk with exceedance probability α. We note that the Value-at-Risk with exceedance probability α, the VaR_α, is a downside risk measure concerned with a single loss. The MDaR_α, being a sum of individual correlated losses, is a new type of multi period downside risk measure. It may be seen as an upper bound for the loss resulted by a marked-to-market investment during a certain period and therefore as a useful concept in determining market risk. The knowledge of its statistical model may help to set aside bank capital during a fixed period to absorb losses from that investment.

Another application suggested is the classification of investments according to their performance when controlling losses via maximum drawdowns. Acar and James [1997] used the maximum drawdown to investigate portfolios' and funds' performances. We envision the use of their estimated densities to discriminate the portfolio's performances, to check, for example, the potential benefits from international diversification. In fact, the objective of this paper was rather to open up several suggestions for applications using the proposed models for the maximum drawdown than to carry on extensive investigation on them.

The remaining of this paper is as follows. In Section 2 we give two definitions of the maximum drawdown, propose statistical models for its severity and duration, estimate the models for three stock market indexes, and investigate the relation between the fluctuations of the maximum drawdown and the GARCH volatility of the daily returns. In Section 3 we suggest applications for the maximum drawdowns and compute the MDaR_α risk measure. In Section 4 we summarize our findings and provide directions for further works.

Statistical Models for the Maximum Drawdown

Definitions

Let P_t denote the price of an investment at day t. Let r_t denote the percentual logarithm daily return, i. e., $r_t = (lnP_t - lnP_{t-1}) * 100$, and assume $\{r_t\}_{t\geq 1}$ is a stationary sequence. We will consider two different definitions of the maximum drawdown[3]. Even though they result in similar data sets, they admit different statistical modeling approaches.

Definition 1. Let P_k be a local maximum and P_l be the next local minimum. This means that $P_k > P_{k+1} \geq \cdots \geq P_l$, with $l-k \geq 1$, $k \geq 1$. A drawdown is then defined as $X_i = (lnP_l - lnP_k) * 100$, $i = 1, 2, ...$ Consider a fixed period of N business days, and the collection of drawdowns $X_1, ..., X_T$, $T < N$, occurring during this period. Definition 1 states that the maximum drawdown is $M = min(X_1, ..., X_T)$.

Definition 2. Let P_k, P_l, and N be as in Definition 1. A drawdown is now defined as $X_i = r_{k+1} + r_{k+2} + \cdots + r_l$, $i = 1, 2....$ Consider the collection of T drawdowns occurring during the N business days. Definition 2 states that the maximum drawdown is $M = min(X_1, ..., X_T)$. Note that now we can write the (maximum) drawdown as

[3] A third definition may also be considered. It is relevant when one is considering marked-to-market investments. Let r_t be the daily log-returns and consider a fixed period of N business days. Consider the partial sums of consecutive returns r_1, $r_1 + r_2$, ..., $r_1 + r_2 + ... + r_N$ occurring during this period. This definition states that the maximum drawdown is the smallest partial sum of daily returns, i.e.,

$$M = min(\sum_{t=1}^{j} r_t \ , j = 1, 2, ..., N) \ .$$

Note that this definition involves positive and negative log-returns whose sums represent the marked-to-market investment value.

a (smallest) random sum of daily log-returns, i.e,

$$M = \sum_{j=1}^{D} r_{k+j} \ ,$$

for some $1 \leq k \leq N$, and where D is the length of the sequence of negative returns. It is easy to see that the two definitions are equivalent, resulting in the same data set. However, Definition 2 opens up the possibility of modeling separately the duration D of a drawdown and the severity r_t of losses (mimicking the classical risk model in Actuary, see next section).

The classical actuarial homogeneous risk model

One of the main topics in classical risk theory in actuary is the modeling of the total excess claims amount S occurring during the lifetime of a contract, usually one year. The basic assumption is that $S = \sum_{i=1}^{D} Y_i$, where Y_i are the iid excess losses independent of D, the random number of claims exceeding the retention limit u in one year. It is suggested to obtain the distribution of S by convolutions, to model D using a Negative Binomial (NB(k,p)) or a Poisson distribution (Poisson(λ)), and to model Y using a Gamma or a Pareto distribution.

We apply this modeling structure to the maximum drawdown obtained from Definition 2. Thus, we model separately the number of consecutive losses D (the duration of the maximum drawdown) and the severity of the negative returns r_t, obtaining by convolution the distribution of M:

$$Pr\{M < s\} = Pr\{D = 0\} + \sum_{d=1}^{\infty} Pr\{r_1 + \cdots + r_d < s\} Pr\{D = d\}. \qquad (1)$$

To model D we experiment the NB(k,p), the Poisson(λ), and also a truncated Poisson since the value zero is never observed. We note that the drawdowns observations, having aggregated data possessing short range dependence, may be thought as independent. Estimates are obtained by maximum likelihood and the chi-square test statistic (given in the next subsection) is used to test the good quality of the fits to the observations of D. The r_ts are modelled using a distribution given in the next subsection.

Models from the extreme value theory

Recently we have seen a large amount of research applying techniques from the extreme value theory (EVT) in the computation of risk measures. Examples include Danielsson and de Vries [1997], McNeil [1998], Embrechts, McNeil, and Strauman [1999], McNeil and Frey [2000], Embrechts [2000], etc. For a comprehensive review of methodologies based on EVT models, see Focardi and Fabozzi [2003].Many of these applications rely on theoretical results from a branch of the EVT, a collection of methodologies arranged under the general name of peaks over threshold (POT) methods. The derived techniques usually apply the generalized Pareto distribution (GPD) to model the extreme tails of the underlying distribution.

Let X represent our random variable of interest, the drawdown, let F_X represent its distribution function, and let $X_1, X_2, ...$ be observations of F_X. Mendes and Brandi [2003] empirically showed that drawdowns may be well modelled using the modified generalized Pareto distribution (MGPD), a flexible extension of the GPD. The POT stability property (Embrechts, Klüppelberg, and Mikosch, 1997) implies that the tail of an MGPD should also be an MGPD. By noting that the maximum drawdown is a tail event, we propose the MGPD as the model distribution for the maximum drawdown from Definition 1. The flexibility of this distribution allows us to consider it for modeling the negative returns composing the maximum drawdown from Definition 2.

The MGPD was proposed in Anderson and Dancy (1992) and has distribution function given by

$$G_\xi(m) = \begin{cases} 1 - (1 + \xi \frac{m^\gamma}{\psi})^{-1/\xi}, & \text{if } \xi \neq 0 \\ 1 - e^{-\frac{m^\gamma}{\psi}}, & \text{if } \xi = 0 \ , \end{cases} \tag{2}$$

where $\gamma \in \Re$ is the modifying parameter. When $\gamma < 1$ the corresponding densities are strictly decreasing with heavier tails; the GPD is recovered by setting $\gamma = 1$; $\gamma > 1$ induces concavity.

In the applications that follow, the MGPD is fitted to data by maximum likelihood. We chose to estimate by maximum likelihood because these estimates possess good (asymptotic) properties and allow the use of well known statistical tests. Let $m_1, ..., m_n$ represent the data. The maximum likelihood estimates of the full model $MGPD(\gamma, \xi, \psi)$ are obtained by maximizing the logarithm of the likelihood function $\prod_{i=1}^n g(\gamma, \xi, \psi; m_i)$, where $g(\gamma, \xi, \psi; m_i)$ is given by

$$g(\gamma,\xi,\psi;m_i) = \begin{cases} (1+\xi\psi^{-1}m_i^{\gamma})^{\frac{-1-\xi}{\xi}}\psi^{-1}\gamma m_i^{\gamma-1}, & \text{if } \xi \neq 0 \\ e^{-\psi^{-1}m_i^{\gamma}}\psi^{-1}\gamma m_i^{(\gamma-1)}, & \text{if } \xi = 0\ . \end{cases} \tag{3}$$

Maximum likelihood estimates are also obtained for the three nested constrained models: 1) $MGPD(\gamma,0,\psi)$, the constrained modified generalized Pareto (CMGPD, also known as the Weibull distribution); 2) $MGPD(1,\xi,\psi)$, the GPD; and 3) $MGPD(1,0,\psi)$, the unit exponential distribution.

The likelihood ratio test is used to discriminate between the nested models. Standard errors of the estimates and confidence intervals based on simple Bootstrap techniques may also be easily obtained. Goodness of fits are graphically checked by means of qq-plots.

We formally test the goodness of fit using the Kolmogorov test whose test statistic is $T_n = \max_i \max\{\frac{i}{n} - F_0(m_{(i)}), F_0(m_{(i)}) - \frac{(i-1)}{n}\}$, where $m_{(1)} \leq m_{(2)} \leq ... \leq m_{(n)}$ are the ordered observations of the maximum drawdown M, n is the sample size, and F_0 is the distribution of M under the null hypothesis. Large values of the test statistic leads one to reject the null hypothesis and to conclude that M is not well modeled by F_0. Critical values for small sample sizes are obtained in (Bickel and Doksum, 2001).

Illustration

We now provide an illustration of the modelling strategies proposed for the maximum drawdowns from Definition 1 and from Definition 2. Since the two definitions result in the same data set, it would be interesting to find out which one would provide a more accurate fit to the data. To this end we use daily closing prices of three stock indexes. Two are from developed markets, the US Nasdaq and the British FTSE. The third is the Brazilian IBOVESPA, a volatile market.

Descriptive Analysis

We start by collecting data in blocks of 6 months which correspond to approximately $N = 130$ working days (corrections were made when necessary to assure that the block did not end during a run of consecutive losses). Simple statistics based on the observations of the iid sample $M_1, M_2, ..., M_n$ of the maximum drawdown M for the three indexes are given in Table Exhibit 1. This table provides the sample size

s of closing prices, the period covered (month and year), the number of maximum drawdowns collected (n), the smallest M_s, the median M_m, and the largest M_l maximum drawdown, the median and the longest duration in number of days of the maximum drawdowns (D_{me} and D_{ma}), the mean block size of consecutive daily returns (business days) B_{mean}, and the mean number b_{mean} of drawdowns within blocks.

Exhibit 1: Simple statistics of maximum drawdowns from the three indexes.

Index	s	Period	n	M_s, M_m, M_l (%)	D_{me} - D_{ma}	B_{mean}	b_{mean}
Nasdaq	7959	08/71 - 08/02	61	-29.18, -5.71, -2.07	5 - 16	130.5	26.13
FTSE	4316	06/84 - 08/02	33	-24.51, -4.94, -2.14	5 - 7	130.3	31.21
IBOVESPA	2171	04/94 - 12/02	16	-42.51, -13.89, -4.42	4 - 7	130.3	30.69

Notation in table: s: sample size of closing prices; n: number of maximum drawdowns collected; M_s,M_m, M_l: the smallest, median, and largest maximum drawdown; D_{me} and D_{ma}: median and maximum durations in number of days; B_{mean}: mean block size; b_{mean}: mean number of drawdowns within blocks.

The Nasdaq series yielded a sample of $n = 61$ maximum drawdowns with 23 of them lying between -4% and -6%. They seem to last longer than those from the other two indexes, as indicated by the numbers in Table Exhibit 1. For the FTSE, more than half (17) of the 33 maximum drawdowns lied between -4% and -6%. This concentration of observations is an indication that the concavity of the density of an MGPD with $\gamma > 1$ should provide a good fit. The IBOVESPA provided a quite different picture. The severity of its maximum drawdowns is greater with all of them smaller than -4%.

Models Fitting

We first fit the MGPD models to data of maximum drawdowns from Definition 1. For the Nasdaq, the best fit (and estimates) turned out to be the $MGPD(\gamma = 2.03, \xi = 0.71, \psi = 39.55)$. For the FTSE it is the $MGPD(\gamma = 3.58, \xi = 1.17, \psi = 342.37)$, and for the IBOVESPA we found a $MGPD(\gamma = 1.45, \xi = 0.0, \psi = 72.07)$. The FTSE fit was poor for the smallest observation which showed up as an outlier (the second smallest is -14.45%), as we can see in the qq-plot at the left hand side of Figure Exhibit 3.

Then, we find the distribution of M based on the random sum approach. We collect the whole set of K negative daily returns composing all the n maximum drawdowns. That is, $K = d_1 + d_2 + ... + d_n$, with d_i being the duration of M_i, $i = 1, ..., n$. Usually K is large. For the Nasdaq $K = 316$ and approximately half of the losses are greater than -1%. The FTSE data contain two very large outliers, -13.03% and -11.48%, quite distant from the third largest daily loss which is -5.88%. Data summaries of the three (very left skewed) data sets and the value of K are given in Exhibit 2.

Exhibit 2: *Simple statistics of daily negative returns composing the maximum drawdowns from Definition 2.*

Index	Smallest	Median	Mean	Largest	K
Nasdaq	-12.0500%	-0.9674%	-1.5200%	-0.0104%	316
FTSE	-13.0300%	-0.9451%	-1.3710%	-0.0186%	149
IBOVESPA	-17.2100%	-3.3520%	-4.1950%	-0.0368%	69

The extreme value nested models are fitted to the losses data. The standard log-likelihood ratio test indicated as best fits for the Nasdaq, FTSE and IBOVESPA, respectively, the $MGPD(\gamma = 1.17, \xi = 0.2573, \psi = 1.31)$, the $MGPD(\gamma = 1.37, \xi = 0.4199, \psi = 1.11)$ and the $MGPD(\gamma = 1.20, \xi = 0.0, \psi = 5.99)$.

Figure Exhibit 3 compares the two MGPD fits on data from definitions 1 and 2 in the case of the FTSE index. On the left hand side we show the fit on the 33 observations of maximum drawdown from Definition 1. At the right hand side we can see the good adherence (except for the two extreme observations) of the MGPD fit to the 149 negative daily returns composing the maximum drawdowns from Definition 2.

The Kolmogorov goodness of fit test confirmed the good adherence of all six MGPD fits obtained so far. For example, in the case of the FTSE fits shown in Figure Exhibit 3, the values of the test statistic T_n are, respectively, 0.1115 and 0.0416. These should be compared to the critical values of 0.2400 and 0.1100, at the 5% confidence level. Similar results were found for the other two indexes.

Next, we estimate the distribution of D, the random number of daily losses composing the maximum drawdown. The empirical distribution is given in Table Exhibit 4, where we note that the most probable duration of the maximum drawdown, in number of days, is 3, 5, and 4, respectively for the Nasdaq, FTSE and IBOVESPA. They never lasted only one day.

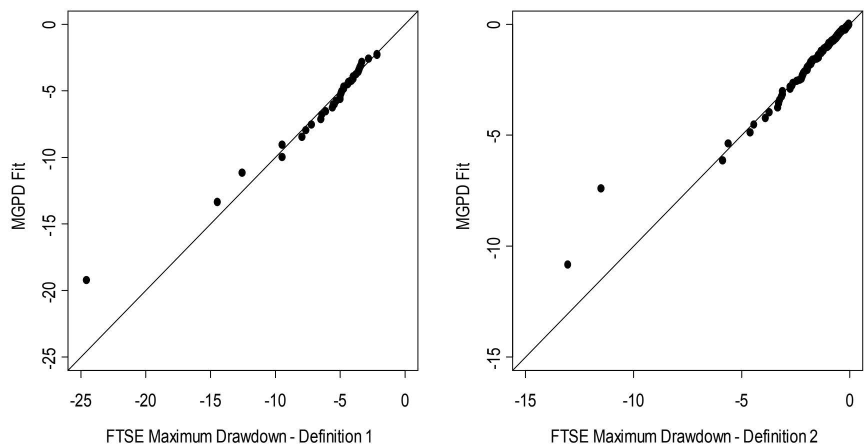

Exhibit 3: QQ-plots of the FTSE MGPD fits on the maximum drawdowns obtained from Definition 1 (at left) and for all losses composing the maximum drawdowns from Definition 2 (at right).

In the three cases we strongly rejected goodness of fit for the Negative Binomial. For the Nasdaq duration data we also rejected the Poisson and the Truncated Poisson distributions. For the FTSE we accepted the Poisson (p-value=0.53 and $\widehat{\lambda} = 4.515$) and the Truncated Poisson (p-value 0.29 and $\widehat{\lambda} = 4.463$) fits. For the IBOVESPA we did not reject goodness of fit for the Poisson (p-value=0.577 and $\widehat{\lambda} = 4.312$) and rejected for the Truncated Poisson (p-value 0.025). So, we model D, the number of consecutive negative returns composing the maximum drawdowns from Definition 2, for the Nasdaq using the empirical distribution, and for the FTSE and the IBOVESPA using the Poisson distribution.

Using expression (1) we obtain by convolutions the distribution of M for the three indexes. They are the lighter lines of Figure Exhibit 5, for the Nasdaq (first row) and IBOVESPA (second row). The entire distribution functions are shown at the right hand side of this figure. At the left hand side we take a zoom on the tail of the distributions, and consider probabilities between 0.90 and 1. Figure Exhibit 5 also shows the distribution function of the maximum drawdown obtained from Definition 1, in darker lines. The distributions obtained under the two methodologies seem to be close to each other. However, for the Nasdaq and for very small probabilities, the quantiles under the simple approach (Definition 1) are much more extreme.

Exhibit 4: Empirical distribution of the durations of maximum drawdowns from definitions 1 and 2 and for the three indexes.

Index	Duration (number of days)															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Nasdaq	0	.08	.20	.15	.20	.13	.11	.06	.02	.03	0	0	0	0	0	.02
FTSE	0	.09	.15	.24	.28	.15	.09	0	0	0	0	0	0	0	0	0
IBOVESPA	0	.12	.12	.38	.13	.19	.06	0	0	0	0	0	0	0	0	0

To decide which modeling strategy may provide the best fit for the maximum drawdown we rely on the Kolmogorov test described in Section 2.3. For each index and for each fitted distribution using (Definition 1, Definition 2) we compute the test statistic T_n. Their values are equal to (0.1278, 0.1299), (0.1115, 0.2216), and (0.1539, 0.1699), respectively for the Nasdaq, FTSE, and IBOVESPA. Based on the critical values (respectively, 0.171, 0.240, 0.350) we conclude that we do not have enough evidence to reject the null hypothesis of good adherence to the data for all six fits. In the case of the FTSE, the values indicate that the simple approach using Definition 1 may be the preferred one. In the case of the Nasdaq, the two methodologies provided close values for the test statistic, but Figure 2 suggests that for extreme quantiles the simple approach may be more conservative. In the case of the IBOVESPA, the approach based on Definition 1 also provided a slightly better fit.

We should note that modeling via the random number of daily losses is more computationally intensive and may be less attractive to be used on a daily basis. However, for a large enough data set, whenever a good fit for the duration D can be found, one should consider the second approach as an interesting and sophisticated modeling alternative. In addition, finding the distribution of the duration of a crisis has much appealing by its own.

From now on we keep the best approach which is the the simple modeling of M from Definition 1, and use the estimated distribution $MGPD(\hat{\gamma}, \hat{\xi}, \hat{\psi})$ of the maximum drawdown in applications.

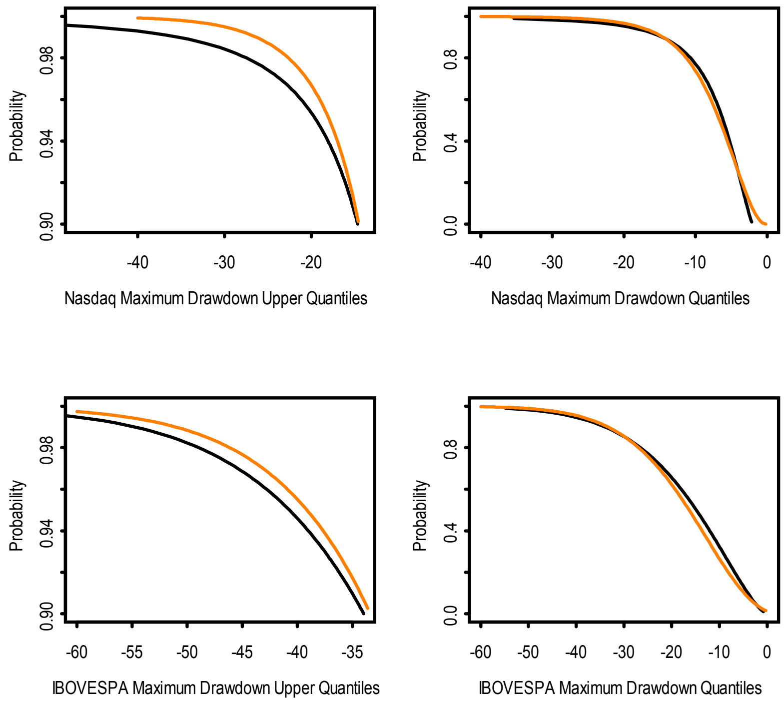

Exhibit 5: The estimated distribution functions (zoom at left) of the 6-months maximum drawdown from definitions 1 (darker) and 2 (lighter), for the Nasdaq and IBOVESPA.

Applications

The MDaR$_\alpha$ risk measure

Like the unconditional Value-at-Risk, the Maximum Drawdown-at-Risk (MDaR$_\alpha$) is just a quantile with exceedance probability α of the distribution of the maximum drawdown. It is a measure of the maximum possible (cumulated) loss that an investment may incur during a fixed horizon. It is thus a downside risk measure that may be used, for example, to discriminate and classify stock markets. To illustrate, we use the three indexes and the six fits of the last subsection and compute the MDaR$_\alpha$. Actually, they are the quantiles shown in the zooms of Figure Exhibit 5. In Table Exhibit 6 we give the MDaR$_\alpha$ for $\alpha = 0.05, 0.01, 0.001$, together with the empirical estimates.

In Table Exhibit 6 we note that the simple approach provided more conservative estimates for the three indexes. For $\alpha = 0.01$, the MDaR_αs for the Nasdaq and IBOVESPA are, respectively, -35.36% and -54.78%. However, for $\alpha = 0.001$, the values -79.87% and -72.45% suggest that there may exist less difference in the extreme tails of these markets.

Exhibit 6: $MDaR_\alpha$ empirical and based on Definition 1 estimates.

	MDaR_α			
	Empirical (n)		Definition 1 - MGPD	
Index (n)	0.01	0.05	0.01	0.05
Nasdaq (61)	—	-17.5	-35.36	-19.42
FTSE (33)	—	-24.18	-22.09	-12.94
IBOVESPA (16)	—	-42.43*	-54.78	-40.72

Notation in table: $-42.43*$ approximated value.

It is worth to investigate if the Maximum Drawdown-at-Risk is a coherent measure of risk in the sense of Artzner et all. [1998]. To see that, let D_1 and D_2 be the maximum drawdowns corresponding to the portfolios (or stocks) X_1 and X_2. Let $\mathrm{MDaR}_\alpha(X_i)$ represent the Maximum Drawdown-at-Risk associated to portfolio X_i, $i = 1, 2$. Assume that the distributions F_1 and F_2 of D_1 and D_2 belong to a Pareto family. Note that $\mathrm{MDaR}_\alpha(X_i) = F_i^{-1}(1-\alpha)$. Under comonotonicity (that is, $\rho(X_1, X_2) = 1$, which implies $\rho(D_1, D_2) = 1$) it is easy to show that

$$F_1^{-1}(1-\alpha) + F_2^{-1}(1-\alpha) = F_{(D_1+D_2)}^{-1}(1-\alpha) \ ,$$

where $F_{(D_1+D_2)}(\cdot)$ represent the distribution of the sum $D_1 + D_2$.

According to results in Embrechts et al. [1999], for $(1-\alpha)$ large enough, whenever F_1 and F_2 are heavy tails distributions we have that

$$F_{(D_1+D_2)}^{-1}(1-\alpha) < F_1^{-1}(1-\alpha) + F_2^{-1}(1-\alpha).$$

Now, let F_D be the distribution of the maximum drawdown obtained from the portfolio $(X_1 + X_2)$. Under comonotonicity we obtain $F_{(D_1+D_2)}^{-1}(1-\alpha) = F_D^{-1}(1-\alpha)$. We wonder that for $\rho(X_1, X_2) < 1$ the distribution F_D possesses lighter tails than $F_{D_1+D_2}$. This would make the MDaR_α a subadditive measure of risk:

$$\mathrm{MDaR}_\alpha(X_1 + X_2) < \mathrm{MDaR}_\alpha(X_1) + \mathrm{MDaR}_\alpha(X_2).$$

It is easy to see that the MDaR_α, being a quantile, also possesses the other properties required by Artzner, Delbaen, Eber, and Heath [1998] for a coherent measure of risk. In fact, it is a location and scale invariant positive measure.

The Maximum Drawdown fluctuations

We wonder if there would exist a relation between the maximum drawdown sizes fluctuations and the corresponding daily returns (fractionally integrated) GARCH volatility. In particular we wonder if extreme maximum drawdowns would occur during periods of large volatility and whether or not the extent of the degree of fractional integration is related to the weight of the tail of the maximum drawdown distribution. This seems reasonable since during crisis extreme returns are observed. We checked this out for the three indexes.

For example, Figure Exhibit 7 shows the Nasdaq series of daily returns superposed by two times its GARCH volatility (fractional parameter $d = 0.46$) and the values of its 6-month maximum drawdowns, plotted at the end of their durations. We observe that the maximum drawdown seems to follow the fluctuations of the volatility of the daily returns with "outlier" maximum drawdowns occurring during stress periods. The extreme drawdowns of severities -28.53% and -29.17%, respectively, occurred during the crisis of October, 1987, and April, 2000. This feature was also observed for the other two indexes.

An interesting finding of this empirical analysis is that among the three data sets, the longest observed duration (16 business days) occurred for the Nasdaq during a period of low volatility (first and second weeks of February, 1984)[4]. Its value is large, -13.20%, but not extreme if compared to the minimum of -29.17%. Even though it is the 8th smaller Nasdaq maximum drawdown, we see this value as an outlier, because it is extreme when compared to those observed during periods of low volatility. It corresponds to position 3284 in Figure Exhibit 7, and can be seen at the upper plot, where we represent the maximum drawdowns durations using vertical bars. We recall that even when the sizes of drops are small, a drawdown duration large enough may induce investors to give up of their investment, or a portfolio manager to change positions.

[4]This may be related to the presence of long memory in the mean and variance of daily returns.

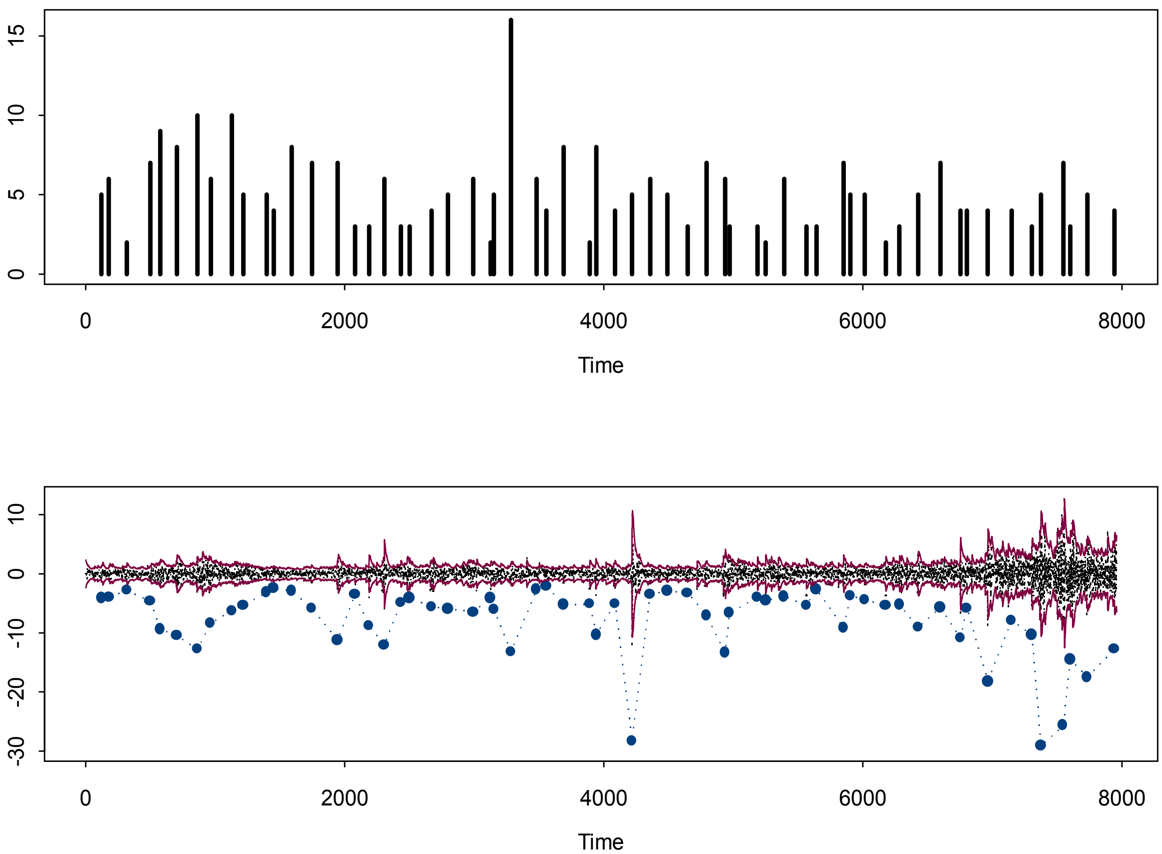

Exhibit 7: At top, the maximum drawdown values plotted at the end of their durations. At the bottom, the Nasdaq series of daily returns superposed by 2 times its GARCH volatility.

We also consider the possibility of the existence a common pattern in the behavior of drawdowns and drawups among markets. Figure Exhibit 8 shows the maximum drawdowns and drawups from the Nasdaq and the FTSE in time. These are observations from January 6, 1984 to August 15, 2002. We note that the maximum drawdowns' and drawups' fluctuations follow a similar pattern, although possessing different magnitudes. This might be an indication of globalization and contagion. Actually, we could identify the extreme maximum drawdowns of -24.51% and -28.25% respectively for the Nasdaq and FTSE, during the global crisis around October, 20, 1987 (approximately position 960 in Figure Exhibit 8).

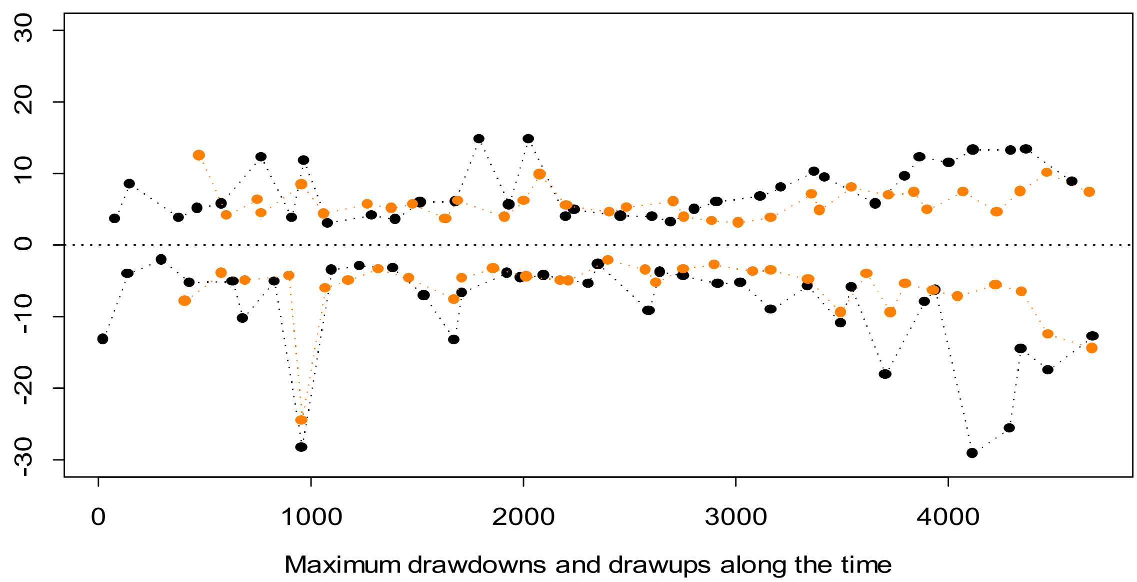

Exhibit 8: Nasdaq (dark) and FTSE (light) maximum drawups and drawdowns plotted at the end of their durations.

However, at the end of the series we observe extreme maximum drawdowns just for the Nasdaq (-29.17% and -25.66%). This also makes sense since we now identify a local crisis.

Conclusions

In this paper we proposed statistical models for two different definitions of the maximum drawdown (drawup). We focused on statistical modeling issues and empirical investigation of the usefulness of derived risk measures. We proposed extreme value distributions for modeling the maximum drawdown (drawup) which were fitted by maximum likelihood. Formal statistical tests were carried on to test goodness of fits.

The proposed modeling strategies were illustrated using three stock market indexes. For these data we found that the fluctuations of the maximum drawdown (drawup) may be related to the daily returns (fractionally integrated) GARCH volatility. The behavior of the maximum drawdown severities may reflect local and global crisis and may therefore be a common pattern among (correlated) markets.

We suggested applications for the maximum drawdown (drawup) estimated distributions. We computed the Maximum Drawdown-at-Risk α, the MDaR_α, and argued that it may be considered a coherent measure of risk. The model based estimates were compared to the empirical ones. The MDaR_α may also be used as constrains in portfolio optimization procedures. The authors have experimented using the maximum-drawdown estimated distributions in this useful application of selecting investments.

We leave for future work a comprehensive empirical investigation of the findings suggested in this paper. There are many open possibilities such as comparisons of the performance of portfolios using the MDaR_α (for example, diversified and non diversified, classical and robust, etc.); the assessment of effect of the maximum drawdown block size on the results obtained; comparisons involving other distributions (for instance, the GEV extreme value distribution) for the maximum drawdown (drawup) data; comparisons involving other estimation methods, including non-parametric and Bayesian methods; the use of alternatives to the log-likelihood ratio test, since at the null $\xi = 0$ the likelihood function is still continuous but non-differentiable; and so on. In fact, the objective of this paper was rather to open up several suggestions for applications using the proposed models for the maximum drawdown than to carry on extensive investigation on them.

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