Pergamon

Int. J. Heat Mass Transfer. Vol. 38, No. 13, pp. 2507–2515, 1995

0017–9310/95 $9.50+0.00

0017–9310(94)00352–1

Velocity and temperature distributions in compressible turbulent boundary layers with heat and mass transfer

ATILA P. SILVA FREIRE, DANIEL O. A. CRUZ and CLÁUDIO C. PELLEGRINI

Mechanical Engineering Program, Federal University of Rio de Janeiro, C.P. 68503, 21945 Rio de Janeiro, Brazil

(*Received* 4 *January* 1993 *and in final form* 19 *October* 1994)

Abstract—In the present work, asymptotic methods are used to derive new expressions for the law of the wall, for the law of the wake and for the skin-friction and the Stanton number equations, for compressible turbulent boundary layers with heat and mass transfer. The results are compared with previous theories and experiments showing good agreement. The two parameters in the law of the wall, the angular coefficient and the linear coefficient of the straight part of the velocity and the temperature profiles plotted in appropriate logarithmic coordinates, are shown to vary with Mach number. Only the second of these parameters, the linear coefficient of the straight line, is shown to vary with the injection rate. No dependence of these parameters on Eckert number, E could be assessed. Also, it emerges from the present analysis that the dissipation effects become important only when $E = 0(u_\tau^{-1})$, u_τ = non-dimensional friction velocity.

1. INTRODUCTION

The development of high-speed flight vehicles and their components has motivated researchers to advance correlations with which engineers can predict the likely values of important flow parameters such as the skin-friction and the transfer of heat at the wall. These correlations are basically derived through three types of approach: the parametric approach, where non-dimensional groups are used to seek empirical correlations of the experimental data; the transformation approach, where a mathematical transformation is sought which reduces the complex system of partial differential equations which governs the motion of a compressible flow into a simpler system, such as the system of equations for an incompressible flow; and the direct approach, where analytical expressions for the mean flow parameters are obtained after appropriate assumptions are made about the turbulence terms. The parametric and transformation approaches always face the difficult problem of correctly chosing the position in the boundary layer where the characteristic flow parameters should be defined and, in the latter case, what the right transformation parameters should be. The direct approach, on the other hand, presents the difficulty of having to choose a turbulence closure model simple enough to provide analytical solutions, but of wide enough validity to cover the cases of practical interest. Of course, this compromise is hard to be achieved and, so far, only a few analytical solutions are found in literature for selected flow conditions.

The purpose of this work is to extend to compressible flow a recently advanced formulation [1] for the thermal turbulent boundary layer over a porous surface which comprises new expressions for the law of the wall, for the law of the wake and for the Stanton number equation. The present approach uses asymptotic techniques, so that the flow region is divided into distinct parts where dominant effects can be used to derive simplified sets of equations. The resulting equations for the near wall region are readily integrated, yielding analytical solutions for the main flow parameters. From these solutions, the influence of the dissipation terms and of the injection velocity is clearly seen. For the solid surface case, it is shown that the dissipation contributes to the leading order solution with a bilogarithmic term. For flows with transpiration, however, the dissipation also contributes with a trilogarithmic higher order correction. Also, it emerges from the analysis that the dissipation effects become important only when $E = 0(u_\tau^{-1})$, E = Eckert number, u_τ = non-dimensional friction velocity.

The near wall solutions are extended to the defect region by adding Coles' function [2] to their logarithmic term. With arguments similar to those of ref. [1], a Stanton number equation then follows immediately. All predictions are compared with the experimental data of Danberg [3], Squire [4], Jeromin [5] and Mabey *et al.* [6]. Both parameters in the law of the wall are shown to vary with Mach number, whereas only one of them, the linear coefficient of the straight part of the velocity and temperature profiles, is shown to vary with the injection rate. No dependence of these parameters on Eckert number could be determined here due to the scatter in the experimental data.

For a boundary layer with zero pressure gradient along an adiabatic wall, the concepts of eddy

NOMENCLATURE

A, B, D	parameters in laws of the wall
C_f	skin friction coefficient
C_p	specific heat at constant pressure
E	Eckert number $= U_\infty^2/C_p(T_\infty - T_w)$
F	injection rate $= \rho_w v_w/\rho_\infty U_\infty$
h	enthalpy
h_c	$= (h-h_w)/(h_\infty - h_w)$
h_τ	friction enthalpy
k_m	Von Karman's constant in velocity profile
k_t	Von Karman's constant in enthalpy profile
M	Mach number
P	pressure
P_m, P_t	Velocity and enthalpy wake profiles
Pr	Prandtl number
q	heat flux
R	Reynolds number
St	Stanton number
T	Temperature
u, v	velocity components in X and Y direction
u_τ	friction velocity
W	Coles' universal function
X, Y	Cartesian coordinates
y	inner variable $= (Yu_\tau)/\nu$.

Greek symbols

δ	boundary layer thickness
Δ	enthalpy thickness
θ	momentum thickness
λ	bulk viscosity
μ	dynamic viscosity
ν	kinetic viscosity
ρ	density
τ	shear stress
ϕ_m, ϕ_t, ϕ_e	see equations (31)–(33).

Superscript

$'$	turbulent fluctuation.

Subscript

a	parameter in definition of A and of B
m	velocity
t	temperature
w	wall condition
∞	external flow condition
1	correction of order u_τ
2	correction of order v_w.

coefficients of friction and of heat transfer can be used to extend the Crocco relation [7] to turbulent flow and to derive an algebraic relationship between velocity and temperature. Introduction of the mixing-length theory in the flow region where the turbulent stresses are dominant, together with the Crocco equation, results in a logarithmic solution for the velocity profile, the so-called Van Driest transformation [8]. Extension of this solution to the defect region can immediately be obtained by adding Coles' function to the logarithmic term as shown by Maise and MacDonald [9]. The Crocco and the Van Driest equations seem to render the compressible boundary layer problem completely solved. In fact, a skin-friction equation can be derived from the defect region extended Van Driest equation, and Stanton number values can be determined from some Reynolds analogy factor constant. If, however, some simple effects such as transfer of heat at the wall, pressure gradients and transpiration are considered, the degree of complexity increases to a level where the above results do not hold anymore and exact solutions are difficult to obtain.

Most of the studies on compressible turbulent boundary layers with transpiration were made by Squire and his students at Cambridge University [4, 5, 10–18]. These studies favour the direct approach, producing an expression for the law of the wall by straightforward application of the mixing-length theory. Unfortunately, the resulting expression [10] is cast in the form of an elliptic integral which cannot be integrated exactly to yield a skin-friction equation. To circumvent this difficulty, an analysis is carried out in ref. [18] where Van Driest's transformation is applied directly to the expressions derived in ref. [17], so that a skin-friction equation can be derived from a simpler set of equations. The resulting expression provides reasonable overall predictions of the skin-friction but poor predictions of the velocity profile for high injection rates. Another purpose of the present work is to obtain better results than those of ref. [18].

All the above mentioned studies for transpired flows deal with adiabatic conditions and hence no Stanton number equations are derived. Squire [14] extended some of the relationships between temperature and velocity in turbulent boundary layers to flows over porous surfaces. His analysis aims at investigating for which conditions the hypotheses that lead to the Crocco relation break down, so that heat transfer predictions are made through Reynolds analogy constants.

2. EQUATIONS OF MEAN MOTION AND ASYMPTOTIC HYPOTHESES

The set of equations which describe the two-dimensional motion of a fluid along a flat plate is:

(a) continuity equation

$$\frac{\partial}{\partial x_j}(\rho u_j) = 0 \quad j = 1, 2 \tag{1}$$

(b) momentum equation

$$\frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left[-\rho\overline{u'_i u'_j} + \frac{1}{R}\tau_{ij}\right] \quad (2)$$

(c) energy equation

$$\frac{\partial}{\partial x_j}[\rho h u_j] = E u_j \frac{\partial p}{\partial x_j} + E\overline{u'_j\frac{\partial p'}{\partial x_j}}$$

$$+\frac{\partial}{\partial x_j}\left[\frac{1}{RP_r}\frac{\partial h}{\partial x_j} - \rho\overline{u'_j h'_i}\right] + \frac{E}{R}\overline{\tau_{ij}\frac{\partial U_j}{\partial x_j}} \quad (3)$$

where

$$\tau_{ij} = \lambda\delta_{ij}\frac{\partial U_l}{\partial x_L} + \mu\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right]$$

and $U = u + u'$.

The above equations must be complemented by an additional thermodynamic relation, say, the equation of state. The symbols x, u, p, ρ and h have their classical meaning; λ is the bulk viscosity, μ the dynamic viscosity and δ_{ij} the Kronecker delta. All these quantities are non-dimensionalized by the external flow conditions. The bars and the lower cases are used to indicate time-averaged quantities. The dashes indicate turbulent fluctuation. R is the Reynolds number, Pr is the Prandtl number and E is the Eckert number. A summation is understood for repeated indices. The normal velocity u_2, here also denoted v, has been non-dimensionalized according to

$$v = \frac{V - V_w}{U_\infty}. \quad (4)$$

The structure of the boundary layer can be determined from the previous set of equations if the order of magnitude of the turbulent terms is estimated and asymptotic techniques are used. For Eckert number of order unity and flow over a solid surface, the analysis of ref. [19] shows that the classical two-deck structure of the boundary layer holds for compressible flows despite some matching difficulties which occur with the density profiles. In this analysis the order of magnitude of the turbulent terms is estimated based on the measurements of Kistler [20], of Kistler and Chen [21] and of Morkovin [22]. These works show that the turbulent fluctuations can be scaled as

$$0(u') = 0(v') = 0(\rho') = 0(u_\tau)$$

where

$$u_\tau = \frac{1}{U_\infty}\sqrt{\left(\frac{\tau_w}{\rho_w}\right)}.$$

In addition, based on the measurements for the incompressible case (see [1]), we consider here that

$$0(h') = 0(h_\tau)$$

where

$$h_\tau = \frac{q_w}{(\rho c_p(\Delta T)u_\tau U_\infty)} = \frac{St}{u_\tau}.$$

Actually, the relation $0(h') = 0(h_\tau)$ can be derived from the definition of Stanton (St) number and the assumption that there is a region near the wall where $\partial(\overline{v'h'})/\partial y = 0$.

3. ASYMPTOTIC ANALYSIS

The turbulent boundary layer preserves its two-layered structure even when normal injection of fluid at the wall and transfer of heat are considered [17, 19, 23, 24]. These effects, in fact, only contribute with higher order corrections to the classical solutions. Thus, it is licit for us to carry out here an analysis in the same terms as those of refs. [17, 24].

In this section we apply the matched asymptotic expansion method to the problem under study to show how approximate solutions for the defect and wall layers can be obtained. Since most of the analysis concerning the obtaining of the asymptotic expansions is conventional, only a few comments about it will be made here.

3.1. *Wall layer*

To find a solution for the near wall turbulent region, we assume that the asymptotic expansions for the flow parameters can be written as

$$u = u_\tau u_1(x,y) + v_w u_2(x,y) \quad (5)$$

$$v = (u_\tau R)^{-1}[u_\tau v_1(x,y) + v_w v_2(x,y)] \quad (6)$$

$$p = p_1(x,y) + u_\tau p_2(x,y) + v_w p_3(x,y) \quad (7)$$

$$h_c = h_1(x,y) + h_\tau h_2(x,y) + v_w h_3(x,y) \quad (8)$$

$$\rho = \rho_1(x,y) + u_\tau \rho_2(x,y) + v_w \rho_3(x,y) \quad (9)$$

with

$$h_c = \frac{h}{h_\infty - h_w} \quad (10)$$

$$y = \text{inner variable} = Yu_\tau/v.$$

An asymptotic expansion for the viscosity is derived by expanding μ in a Taylor series around h_1, i.e.

$$\mu = \mu(h_1) + h_\tau\frac{\partial\mu(h_1)}{\partial h}h_2 + v_w\frac{\partial\mu(h_1)}{\partial h}h_3. \quad (11)$$

Introduction of expressions (5)–(11) into the equations of mean motion together with an eddy viscosity/mixing-length hypothesis for the turbulent terms, and collection of the coefficients of various powers, gives the equations for the successive approximations. The resulting equations are, except for the differences arising from the transpiration terms, the same as those obtained by Afzal (see ref. [19]). The approximate equations are:

$$\frac{\partial}{\partial x}(\rho_1 u_1) + \frac{\partial}{\partial y}(\rho_1 v_1) = 0 \quad (12)$$

$$\frac{\partial}{\partial y}\left(\rho_1 k_m^2 y^2 \left(\frac{\partial u_1}{\partial y}\right)^2\right)+\frac{\partial}{\partial y}\left(\mu(h_1)\frac{\partial u_1}{\partial y}\right)=0 \quad (13)$$

$$\frac{\partial p_1}{\partial y}=0 \quad (14)$$

$$\frac{\partial^2 h_1}{\partial y^2}=0 \quad (15)$$

$$p_1=\frac{\gamma-1}{\gamma}\frac{1}{E}\rho_1 h_1 \quad (16)$$

$$\frac{\partial}{\partial x}(\rho_1 u_2)+\frac{\partial}{\partial y}(\rho_1 v_2)=0 \quad (17)$$

$$\rho_1\frac{\partial u_2}{\partial y}=\frac{\partial}{\partial y}\left(\rho_1 k_m^2 y^2\left(\frac{\partial u_2}{\partial y}\right)^2\right) \quad (18)$$

$$\frac{\partial p_2}{\partial y}=0 \quad (19)$$

$$\frac{\partial}{\partial y}\left(\rho_1 k_m k_t y^2\frac{\partial u_1}{\partial y}\frac{\partial h_2}{\partial y}\right)+Eu_\tau\rho_1 k_m^2 y^2\left(\frac{\partial u_1}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\frac{\mu(h_1)}{Pr}\frac{\partial h_2}{\partial y}\right)=0 \quad (20)$$

$$p_2=\frac{\gamma-1}{\gamma}\frac{1}{E}(\rho_1 h_2+h_1\rho_2) \quad (21)$$

$$\frac{\partial p_3}{\partial y}=0 \quad (22)$$

$$\rho_1\frac{\partial h_2}{\partial y}=\frac{\partial}{\partial y}\rho_1 k_m k_t y^2\left[\frac{\partial u_1}{\partial y}\frac{\partial h_3}{\partial y}+\frac{\partial u_2}{\partial y}\frac{\partial h_2}{\partial y}\right]+ Eu_\tau k_m^2 y^2\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\mu(h_1)}{Pr}\frac{\partial h_3}{\partial y}\right). \quad (23)$$

Equations (14)–(16) together with the matching conditions for the defect layer (ref. [19]) show that the leading order pressure, temperature and density terms are constant across the wall layer. Hence we have

$$p_1=1 \quad \rho_1=\rho_w/\rho_\infty \quad h_1=h_w/(h_\infty-h_w). \quad (24)\text{–}(26)$$

In order to find a solution for equations (13), (20) and (23) in the fully turbulent region we neglect the influence of the viscous and conduction terms. As a result, the solution of equations (13) and (18) become

$$u_1=\frac{1}{k_m}\ln y+A \quad (27)$$

$$u_2=\frac{1}{4}\left[\frac{1}{k_m}\ln y+A\right]^2 \quad (28)$$

where, in principle, k_m and A are allowed to vary with Eckert number, and the injection rate.

The solution of the second order temperature equation, equation (6i), is

$$h_2=\frac{1}{k_t}\ln Pr\,y+B+\frac{Eu_\tau^3}{St}\frac{1}{k_m k_t}\ln^2 Pr\,y. \quad (29)$$

The third order temperature solution, h_3, is given by

$$h_3=\frac{k_t^2}{4}(k_m A+2)h_B^2+\left[\frac{1}{6}-\frac{Eu_\tau^3}{3St}\frac{1}{k_t^2 k_m}\right]\ln^3 Pr\,y - \frac{Eu_\tau}{12}\frac{k_m}{k_t}u_1^3 \quad (30)$$

where, $h_B=(1/k_t)\ln Pr\,y+B$, and parameters k_t and B are, in general, a function of E and F.

Equations (29) and (30) are derived here for the first time. Their substitution into equation (8) gives the thermal version of the law of the wall for compressible turbulent boundary layers with transfer of heat and of mass. Some authors [25–27] have carried out analyses of the wall region using Taylor series expansions. They, however, arrive at expressions which hold only for the viscous-conductive region and hence are very different from equations (29) and (30).

The asymptotic results of equations (27–30) are leading order results in the sense that they have been obtained with the leading order terms of the density and temperature solutions (equations (25) and (26)). Thus, no influence of ρ_2 and of h_2 on the velocity profiles was derived here. The reason for this is the great analytical difficulty involved in solving the higher order equations. These equations are coupled and the resulting complex system of partial differential equations that has to be solved hampers any attempt at obtaining closed analytical solution. The implication is that, similarly to other analyses that use the direct approach, the present formulation has to resort to experimental data to assess the right dependence on E and F of the parameters in equations (27)–(30). Here, following the recommendation of Fernholtz and Finley [29], we have used the data of Mabey *et al.* [6] to do this. Despite, this apparent limitation of the formulation, we stress this is the only way in which analytical solutions can be obtained which account for such flow effects as compressibility, transpiration and transfer of heat.

3.2. *Defect layer*

To find a solution for the defect region we should use the same procedure as above. Specification of a simple algebraic turbulence model for this part of the flow which describes realistically the physics of the phenomenon and allows analytical solutions to be obtained is, however, very difficult. We then follow Coles' approach and extend equations (5) and (8) to the defect layer simply by considering an additional universal function, the Coles' function. To this end we re-write solutions (5) and (8) as

$$\phi_m=\frac{2u_\tau}{v_w}\left(\sqrt{\left(\frac{uv_w}{u_\tau^2}+1\right)}-1\right)=\frac{1}{k_m}\ln y+A \quad (31)$$

Table 1. Experimental flow conditions

Author	M	$F \times 10^3$	T_w/T_∞	No. of profiles considered
Mabey *et al.* [6]	2.5–4.5	0.0	Adiabatic conditions	28
Squire [4]	1.8–3.6	0.0–3.6	Adiabatic conditions	60
Danberg [3]	6.5	0.0–2.5	3.86–7.65	49

$$\phi_t = \frac{-1+\sqrt{\left(1-2\frac{v_w}{h_\tau}\left[\frac{\phi_e}{h_\tau}-\frac{h}{h_\tau}\right]\right)}}{\frac{u_w}{h_\tau}} = \frac{1}{k_t}\ln Pr\, y + B \qquad (32)$$

with

$$\phi_e = -\frac{Eu_\tau^2}{2k_m k_t}\ln^2 Pr\, y + v_w\left[-\frac{Eu_\tau^3}{6S_t k_m k_t^2}\ln^3 Pr\, y - \left[\frac{k_m}{4k_t}+\frac{A_t}{2}\right]u_1^2 + \frac{Eu_\tau k_m}{6k_t}\left[\frac{u_2^2}{St}-\frac{1}{2}\right]u_1^3\right]. \qquad (33)$$

Equations (31) and (32) are the law of the wall expressions for transpired flows. The laws of the wake can then be cast as

$$\phi_m = \frac{1}{k_m}\ln y + A + P_m W\left(\frac{y}{\delta_m}\right) \qquad (34)$$

and

$$\phi_t = \frac{1}{k_t}\ln Pr\, y + B + P_t W\left(\frac{y}{\delta_t}\right) \qquad (35)$$

where W denotes Coles' universal function and P_m and P_t are in general a function of M and F. By the present approach, the compressibility effects are hoped to be fully accounted for by parameters k_m, k_t, A, B, P_m and P_t. Similar analyses carried out for flows over solid surfaces at adiabatic conditions by Winter and Gaudet [28] and by Mabey *et al.* [6], show that this is indeed a good assumption, leading to very accurate expressions for the velocity profile and for the skin-friction.

Expressions for the skin-friction and the Stanton number are readily obtained if $(Y, u) = (\delta_m, U_\infty)$ and $(Y, t) = (\delta_t, T_\infty)$ are substituted into equations (34) and (35), respectively. This yields a transcedental equation which must be solved for the determination of u_t, and an algebraic expression which determines St.

4. RESULTS

Very few works are available in literature which present the combined effects of transpiration and of transfer of heat in compressible flows. Indeed, most of the experiments on transpired flow have been performed in adiabatic flow conditions. One work that seems to conform to our conditions, and presents velocity and temperature profiles in fair detail, is the work of Danberg [3]. Even so, some severe discrepancies in his data [29] mean that they must be considered with reserve. In particular, the values of C_f and of St must be seen as an approximation since they are evaluated from the limiting gradients of the profiles.

The present formulation is expected to improve the results of ref. [18] for transpired flows along adiabatic walls, if only appropriate functional dependences of k_m and A on Mach number are determined. The analysis of the experiments will then include the data of Squire [4] as well as the basic test data of Mabey *et al.* [6]. The test conditions are shown in Table 1.

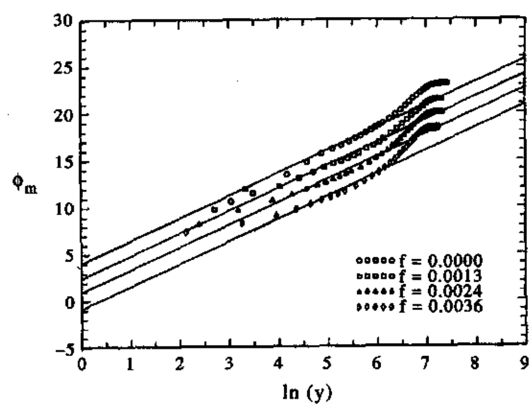

Fig. 1. Typical velocity profiles for transpired flows. Data from Squire [4].

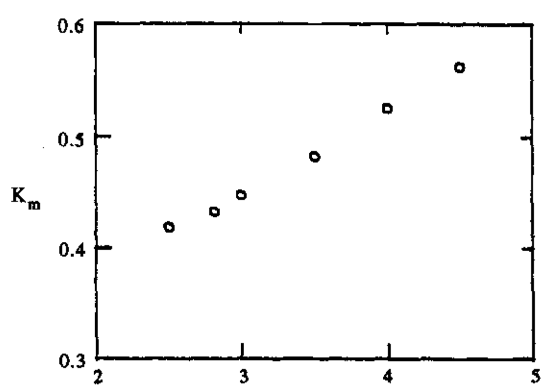

Fig. 2. Variation of k_m with Mach number; 31 profiles of refs. [4,6] considered.

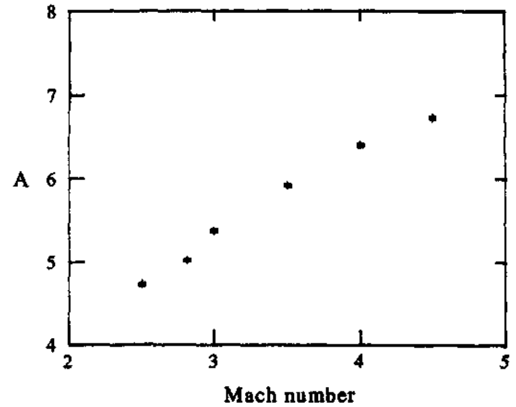

Fig. 3. Variation of A with M; 31 profiles of refs. [4,6] considered.

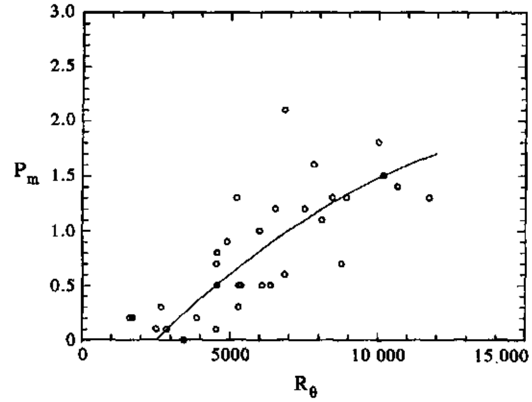

Fig. 4. Variation of P_m with R_θ. Data from Danberg [3].

A comparison between the measurements and equation (5) is made in Fig. 1. Following the trends previously observed by Squire and by Mabey *et al.*, k_m shows an appreciable dependence on Mach number (Fig. 2), being apparently invariant with an injection rate. For the lowest Mach number, k_m assumes the value of 0.43, increasing significantly until reaching the value 0.6 for $M = 6.5$. The value of $k_m = 0.6$ found for the data of Danberg is about 10% lower than the theoretical value given by the correlation of Winter and Gaudet [28]. Contrary to the work of Mabey *et al.*, A is found to vary with M. It is a fact that the slope of the straight-line portion of the logarithmic curve is independent of C_f, whereas A depends critically on C_f. This certainly introduces some uncertainty into the analysis of the data; particularly knowing that the values of C_f obtained by Danberg are only approximate. Figure 3, however, gives clear indication that A increases with M, assuming a value of 9.0 for Danberg's condition. A dependence of A on the injection rate is also apparent from the data. This is in accordance with previous observations for incompressible and compressible flows. It is now generally recognized in literature [12] that A decreases with the increase in the injection rate. Here we found that the procedure introduced by Simpson [30] applied to compressible flow accounts well for this dependence. Thus, we write

$$A = \frac{2u_\tau}{v_w}\left(\sqrt{\left(\frac{u_a v_w}{u_\tau}+1\right)}-1\right)+\frac{1}{k_m}\ln\frac{1}{u_a} \quad (36)$$

where

$$u_a = \frac{1}{k_m}\ln u_a + A_{u0} \quad A_{u0} = 9.0. \quad (37)$$

The behaviour of the wake profile is more difficult to assess due to its sensitivity to the line fitting of the logarithmic part of the profile. Squire [11], however, has shown that for flows along solid surfaces, Maise and McDonald's formulation yields a value of P_m which is virtually independent of M, but varies with R_θ. Our present formulation seems to follow this trend, as indicated by Fig. 4, which was compiled using Danberg's data. This figure yields a value of 1.75 for P_m, which is lower than the value commonly found in literature for incompressible flow, 2.5.

Values of skin-friction obtained with the present formulation are compared in Fig. 5 with the data provided by Fig. 3 of ref. [10] and the results of ref. [18]. The better agreement of the present theory is noticeable.

The analysis of the temperature data was much more difficult to carry out than the previous analysis of the velocity data, since the influence of the Eckert number also had to be accounted for and the scatter in the data of Danberg was appreciable. Typical temperature profiles, plotted under appropriate coordinates defined by equation (31), are shown in Figs. 6 and 7. The linear, logarithmic and wake regions of the flow come out nicely in these figures, giving an indication that the present formulation is consistent. Since the data of Danberg were given for a single value of M, 6.5, no dependence of parameters k_t, B, D and P_t on M could be assessed. Parameter k_t showed no dependence on F, assuming a value of 0.60.

Parameter B was assumed to have the same qualitative behaviour of parameter A, so that we write

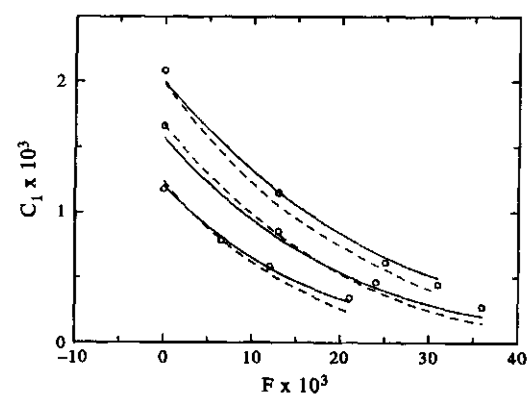

Fig. 5. Skin-friction results for transpired flow. O, experiments; —, present approach; ---, Silva Freire [18].

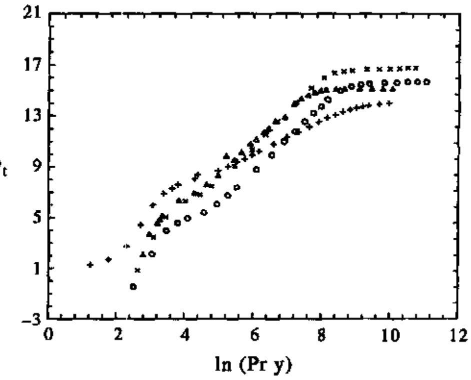

Fig. 6. Typical temperature profiles for unblown flow. Data from Danberg [3].

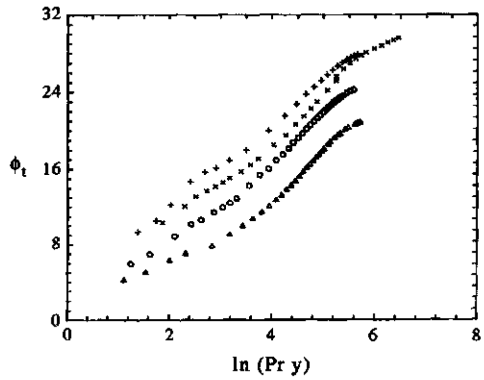

Fig. 7. Typical temperature profiles for transpired flow. Data from Danberg [3].

$$B = \frac{2h_\tau}{v_w}\left(\sqrt{\left(\frac{h_a v_w}{h_\tau}+1\right)}-1\right)+\frac{1}{k_t}\ln\frac{1}{h_a} \qquad (38)$$

where

$$h_a = \frac{1}{k_t}\ln h_a + B_0 \quad B_0 = 11.0. \qquad (39)$$

The implication of expressions (36, 37) to (38, 39) is that both parameters, A and B, were theoretically determined through the linear and logarithmic solutions patching point.

The wake profile, P_t, showed wide variation with F ranging from 0.5 to 6.0. Here we assumed P_t to be constant and equal to 2.

The values of Stanton number obtained from equation (35) are shown in Table 2 for some selected flow conditions. The theoretical values are normally higher than the experimental values by a margin of 10–30%. This is acceptable since the error found in literature for skin friction and for Stanton number data on transpired compressible turbulent boundary layer flows normally range, even for the lowest injection rates, from 20 to 50%. Please note that the present results were obtained using C_f values as quoted by Danberg which, as mentioned before, are only approximate. A complete comparison of the results provided by equation (35) with the data of Danberg is shown in Fig. 8 where 43 profiles are considered.

5. CONCLUSION

The present work has derived new expressions for the law of the wall and for the law of the wake, for both the velocity and the temperature fields in compressible transpired flows. From these, simple algebraic expressions follow for prediction of the skin-friction

Table 2. Stanton number predictions

Run	Injection rate $\times 10^4$	T_w/T_∞	R_θ	$S_t \times 10^4$ Theoretical	$S_t \times 10^4$ Experimental
5	9.2	4.1	3206	21.66	22.94
6	9.1	4.1	4028	18.76	22.29
7	8.9	4.3	4698	18.99	20.05
8	9.2	4.2	5288	19.53	17.38
12	24.9	3.9	6517	9.76	10.21
13	24.5	3.6	8156	8.54	6.63
14	24.9	3.8	8450	9.15	8.45
15	25.8	3.7	10166	9.15	6.49
20	0.0	4.4	3296	48.00	34.96
21	0.0	4.1	3891	34.46	28.81
22	0.0	5.5	1619	65.09	44.52
23	0.0	5.1	1853	46.29	40.82
26	8.9	5.2	2994	27.31	35.13
27	8.9	5.5	3448	28.76	31.33
28	8.4	5.7	4554	30.13	29.61
29	8.8	5.3	5281	26.55	26.11
43	8.7	7.7	4541	52.79	94.25
44	16.2	7.7	6365	44.21	45.58
45	24.4	7.6	8928	26.35	23.74
49	7.2	7.0	8753	31.20	37.71
50	10.1	6.9	11737	31.35	29.26
51	0.0	4.7	3841	22.60	27.37
52	0.0	4.7	4557	25.63	26.83

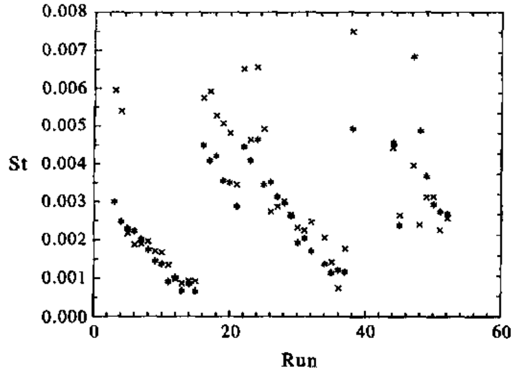

Fig. 8. Stanton number predictions. ×, Present approach; *, Danberg [3].

coefficient and of the Stanton number. Although obtained with a constant density hypothesis for the near wall region (which result from the asymptotic analysis), the above expressions correlate fairly well with the experimental data. Only the results for the temperature field must be seen with caution since they are based on the data of Danberg which present large scatter. This scatter is attributed to the lack of accurate skin-friction data, and to the low Reynolds numbers of the experiments, which cast severe doubts as to whether the flow is fully turbulent. The former difficulty could apparently be circumvented if, for example, skin-friction equations were taken from refs. [18, 28] to produce reliable results for the unblown and blown data, respectively. However, most of the data in literature suggest that the additive constants in the laws of the wall increase with Eckert number, which makes the use of the equations in [18, 28] uncertain since they have been developed for adiabatic flow conditions only. Of course, with the latter difficulty there is not much we can do except consider the higher Reynolds number data. Considering all these aspects, we decided here to make a critical analysis of the data of Danberg, using for the derivation of the above equations only those we judged to be consistent. The present formulation is the only one that can be found in literature which furnishes a complete set of closed analytical solutions for this type of problem. The authors are confident that this formulation retains most of the important features of the problem, and think that a better tuning of the parameters, as more reliable data become available, will greatly improve the results.

Acknowledgements—This work was financially supported partially by the Brazilian National Research Council (CNPq) through grants nos 411131/88-5 and 500620/90-3, and partially by the Rio de Janeiro Research Foundation (FAPERJ) through grant no. E-29/170358/88. The authors are thankful to Mr Andre C. Barros and to Mr Marcelo R. A. S. Santos for their help with the computations.

REFERENCES

1. M. A. F. Medeiros and A. P. Silva Freire, The transfer of heat in turbulent boundary layers with injection or suction: universal laws and Stanton number equations, *Int. J. Heat Mass Transfer* **35**, 991–995 (1992).
2. D. Coles, The law of the wake in turbulent boundary layers, *J. Fluid Mech.* **1**, 191–226 (1956).
3. J. E. Danberg, Characteristics of the turbulent boundary layer with heat and mass transfer: data tabulation, NOLTR 675–6 (1967).
4. L. C. Squire, Further experimental investigations of compressible turbulent boundary layers with air injection, ARC R&M No. 3627 (1970).
5. L. O. F. Jeromin, An experimental investigation of the compressible turbulent boundary layer with air injection, ARC R&M No. 3526 (1968).
6. D. G. Mabey, H. V. Meier and W. G. Sawyer, Experimental and theoretical studies of the boundary layer on a flat plate at Mach numbers from 2.5 to 4.5, RAE TR 74127 (1974).
7. L. Crocco, Transformations of the compressible turbulent boundary layer with heat exchange, *AIAA J.* **1**, 2723–2731 (1963).
8. E. R. Van Driest, Turbulent boundary layer in compressible fluids, *J. Aero. Sci.* **18**, 145–160 (1951).
9. G. Maise and H. McDonald, Mixing length and kinematic eddy viscosity in a compressible boundary layer, *AIAA J.* **6**, 73–80 (1968).
10. L. C. Squire, A law of the wall for compressible turbulent boundary layers with air injection, *J. Fluid Mech.* **37**, 449–456 (1969).
11. L. C. Squire, Eddy viscosity distributions in compressible turbulent boundary layers with injection, *Aero. Q.* **XXIII**, 169–182 (1971).
12. L. C. Squire, Turbulent boundary layers with suction or injection, Conference on data and computation for complex turbulent flows, Report of Evaluator (1980).
13. L. C. Squire and V. K. Verma, The calculation of compressible turbulent boundary layers with fluid injection, ARC CP No. 1265 (1973).
14. L. C. Squire, The relationship between temperature and velocity in turbulent boundary layers with injection. In *Recent Advances in Aerospace Sciences* (Edited by C. Casci), pp. 165–185 (1986).
15. G. D. Thomas, Compressible turbulent boundary layers with combined air injection and pressure gradient, ARC R&M No. 3779 (1976).
16. P. G. Marriot, Compressible turbulent boundary layers with discontinuous air transpiration: an experimental and theoretical investigation, ARC R&M No. 3780 (1977).
17. A. P. Silva Freire, An asymptotic solution for transpired incompressible turbulent boundary layers, *Int. J. Heat Mass Transfer* **31**, 1011–1021 (1988).
18. A. P. Silva Freire, An extension of the transpired skin-friction equation to compressible turbulent boundary layers, *Int. J. Heat Mass Transfer* **31**, 2395–2398 (1988).
19. A. P. Silva Freire, On the matching conditions for a two-deck compressible turbulent boundary layer model, *Z. Angew. Math. Mech.* **69**, 100–104 (1989).
20. A. L. Kistler, Fluctuation measurements in a supersonic turbulent boundary layer, *Phys. Fluids* **2**, 290–296 (1959).
21. A. L. Kistler and W. S. Chen, A fluctuating pressure field in a supersonic turbulent boundary layer, *J. Fluid Mech.* **16**, 41–64 (1963).
22. M. V. Morkovin, Effects of compressibility on turbulent flows, *Proceedings of an International Symposium on Mecanique de la turbulence*, pp. 367–380 (1962).
23. A. P. Silva Freire and M. H. Hirata, Approximate solutions to singular perturbation problems: the intermediate variable technique, *J. Math. Anal. Appl.* **145**, 241–253 (1990).
24. A. P. Silva Freire, Asymptotic structure and characteristics of turbulent boundary layers: short review and recent developments. In *Trends in Heat and Mass Transfer*, Research Trends, Trivandrum, India, in press.

25. C. L. Tien, A note on distributions of temperature and eddy diffusities for heat in turbulent flow near a wall, *ZaMP* **15**, 63–66 (1964).
26. R. N. Meroney, Turbulent sublayer temperature distribution including wall injection and dissipation, *Int. J. Heat Mass Transfer* **11**, 1406–1408 (1968).
27. R. A. Antonia, Behaviour of the turbulent Prandtl number near the wall, *Int. J. Heat Mass Transfer* **23**, 906–908 (1980).
28. K. G. Winter and L. Gaudet, Turbulent boundary layer studies at high Reynolds numbers at Mach numbers between 0.2 and 2.8, ARC R&M No. 3712 (1973).
29. H. H. Fernholtz and P. J. Finley, A critical commentary on mean flow data for two-dimensional compressible turbulent boundary layers, AGARDograph No. 253 (1980).
30. R. L. Simpson, Characteristics of the turbulent boundary layer at low Reynolds number with and without transpiration, *J. Fluid Mech.* **42**, 769–802 (1970).