MODELING AND ANALYSIS OF MARKET SHARE DYNAMICS IN A DUOPOLY SUBJECT TO AFFINE FEEDBACK ADVERTISING POLICIES AND DELAYS

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“Todo lo ha hecho ella”
Para Norma y Rayda.

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MODELAGEM E ANÁLISE DA DINÂMICA DE FATIA DE MERCADO EM UM DUOPÓLIO SUJEITO A PUBLICIDADE AFIM REALIMENTADA E ATRASOS

Walter Aliaga Aliaga

Setembro/2017

Orientador: Amit Bhaya

Programa: Engenharia Elétrica

Esta tese apresenta extensões aos modelos de Vidale-Wolfe e Lanchester para a dinâmica de duopólios. As novidades nas extensões propostas são a introdução explícita de um conjunto de clientes indecisos nos modelos existentes, os quais consideram apenas os conjuntos de clientes das duas empresas concorrentes, e o uso de políticas de publicidade afins com realimentação. Demonstra-se que sob a classe proposta de políticas de publicidade, os modelos estendidos de Vidale-Wolfe e Lanchester, apesar de terem dinâmicas diferentes , apresentam pontos de equilíbrio idênticos com as mesmas propriedades de estabilidade. A introdução de um terceiro conjunto de clientes indecisos também motiva a introdução de um modelo mais elaborado da dinâmica de mercado baseado no modelo Replicador-Mutador da teoria dos jogos evolucionários. A proposta do modelo é realizada identificando estratégias com elementos de uma matriz de preferência consistindo nas preferências de escolha das empresas pelos clientes e a matriz de mutação representa probabilidades de transição de um conjunto de clientes para outro. O modelo proposto é analisado em relação aos pontos de equilíbrios e suas propriedades de estabilidade, bem como a sensibilidade paramétrica sob as políticas de publicidade propostas. Todos os modelos propostos são analisados quanto à estabilidade, a presença de oscilações, a existência de bifurcações de Hopf quando atrasos de implementação ou de adoção são introduzidos. Os modelos estendidos de Vidale-Wolfe e Lanchester são robustos para atrasos em implementacão enquanto que para o atraso de adocão apresentam a existência de bifurcações. O modelo Replicador-Mutador proposto é robusto para ambos tipos de atrasos.

Abstract of Thesis presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Doctor of Science (D.Sc.)

MODELING AND ANALYSIS OF MARKET SHARE DYNAMICS IN A DUOPOLY SUBJECT TO AFFINE FEEDBACK ADVERTISING POLICIES AND DELAYS

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This thesis presents extensions of the Vidale-Wolfe and Lanchester models for market share duopoly dynamics. The novelties in the proposed extensions are the explicit introduction of a set of undecided clients into existing models, which consider only the sets of clients of the two competing firms, as well as the use of decentralized affine feedback advertising policies. It is shown that, under the proposed class of advertising policies, the extended Vidale-Wolfe and Lanchester models, despite having different dynamics, have equilibria in identical locations, with the same stability properties. The introduction of a third set of undecided clients also motivates the introduction of a more elaborate model of market share dynamics based on the replicator-mutator model from evolutionary game theory. This is done by identifying strategies with the entries of a preference matrix consisting of the choice preferences of firms by clients and the mutation matrix representing transition probabilities from one set of clients to another. The proposed model is analysed with respect to equilibria and their stability properties, as well as parametric sensitivity, under the proposed advertising policies. All proposed models are analysed for stability, the presence of oscillations, existence of Hopf bifurcations when implementation or adoption delays are introduced. The extended models of Vidale-Wolfe and Lanchester are robust to implementation delays, while for adoption delays, bifurcations can occur. The proposed replicator-mutator model is robust for both types of delays.

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x_i	Market share of the firm i
λ_i	Decaying rate of the firm i
A	Preference matrix of the clients with respect to the firms.
Q	Mutation matrix of the clients in relation to the firms
a_{ij}	Element of the preference matrix representing the clients' preference for firm i in relation to firm j
q_{ij}	Element of the mutation matrix that represents the probability of change of choice of the clients of firm i for firm j
μ	Fidelity parameter of the clients
Δa_{ij}	Variation of element a_{ij} of preference matrix A
Δq_{ij}	Variation of element q_{ij} of mutation matrix Q
$\Delta\mu$	Variation of fidelity parameter μ
x_τ	Market share with delay value τ
τ	Delay value
τ_c	Critical delay value

Chapter 1

Introduction

A market is a set of agreements by which buyers and sellers exchange goods and services [1]. The number of participating firms, the uniformity of the product, and the form of competition among firms define the structure of a market [2]. A predominant form of market structure is the oligopoly, which represents an interdependent market [2] with few supply-side firms and a large number of buyers on the demand side [3]. The simplest type of oligopoly is the duopoly, where the market consists of two companies offering similar or identical products [4]. This thesis will deal exclusively with the case of duopolies which, although studied for a long time, remains a subject of intense research.

Markets are not static; they vary in time; therefore the competition strategies which exist in the firms which make up the market such as price, quality, advertising, etc. are also influenced by changes in the market [5]. On the other hand, the consumer buying process is strongly influenced by cultural, social, personal, and psychological factors. Thus, characteristics such as behavior, value systems, social hierarchy, age, occupation, personality, motivation, and perception have significant influence in consumer decision-making [6].

Given these conditions and with the purpose of influencing the consumer buying process, firms use advertising to promote the sale of their products. Thus, advertising can be understood as a form of communication used to induce consumers to take a particular action concerning goods or services [7]. Advertising involves communicating the “value proposition” of the company or brand using the paid media to inform, persuade, and remind consumers of the product offered [6]. The accounting of advertising and the return on its investment becomes an important parameter to be considered by firms, thus, quantifying the effects of advertising becomes a significant task.

Advertising results can be assessed by communication impacts and by the effects on sales and profits [6]. Several models have been developed using different approaches [7], with the most popular continuos-time models being those of Nerlove and Arrow [8], Vidale and Wolfe [9], and Lanchester [10].

The models proposed by Vidale and Wolfe and Lanchester are sales-advertising models characterized by a direct relationship between the rate of change in sales and advertising [11]. The Vidale-Wolfe model represents competitive interaction indirectly through its influence on the unsaturated market, while the Lanchester model describes direct competition for market share assuming that the market is saturated (namely, that the market shares of the firms add up to one) [12]. Over time, extensions and diverse approaches to the original models have been presented [13], [14]. In recent years, new modeling has been proposed under new approaches such as population dynamics and evolutionary game theory.

Evolutionary Game Theory studies the behavior of large populations of agents which interact strategically repeatedly subject to frequency-dependent selection processes [15]. In the context of socioeconomic models, it is assumed that agents have the capacity to adapt their behavior, thus changing their strategy in response to return, which in turn, is determined by the behavior of the population as a whole [16]. In this way, the Evolutionary Game Theory is useful in the socioeconomic context because it allows studying the diversity, interaction, and evolution of social systems [17].

Finally, the presence of delays can be found in several systems in which, in many cases, the time lag between the actual system information and the time when it becomes available is significant [18]. In the specific case of models of economic phenomema, it is natural to suppose that there is a delay between the time when the economic decision is made and the time when the decision produces results. Classes of models in which some model variable depends on past as well as current values also imply the existence of delays [19].

1.1 Motivation

Market dynamics and consumer behavior constitute complex systems characterized by processes of interaction between the different agents in the market. On the other hand, the information flow and processes which determine the dynamics of these systems are influenced by the presence of delays. In this context, the extension or generalization of existing models in the literature and the formulation of new models contempalting delays offer the possibility of a better understanding of these systems.

1.2 Objectives

The objectives of this thesis are:

- To study the Vidale-Wolfe and Lanchester models introducing a third population of undecided users and decentralized affine feedback advertising policies.
- To propose a model based on the evolutionary dynamics that considers the interaction between the various agents that integrate a duopolistic market under decentralized affine feedback advertising policies.
- To analyze the existence of delays in the proposed duopoly models subject to decentralized affine feedback advertising policies.

1.3 Structure of the thesis

This thesis is organized as follows. Chapter 1 starts by presenting a general introduction to the subject of the thesis. The motivation for the proposed models and advertising policies, as well as the objectives of the thesis are also given in this chapter.

In Chapter 2 a review of the main features of market dynamics and the consumer behavior processes is presented. The chapter also offers an overview of duopoly models subject to advertising policies and discusses the existence of different types of delays in markets and consumer behavior.

Chapter 3 examines the Vidale-Wolfe and Lanchester models in which a third population of undecided users has been introduced and also proposes the use of a decentralized affine feedback advertising policy. The proposed models are analysed with regard to the existence and stability properties of their equilibria. Chapter 4 analyses the models studied in Chapter 3 subject to the presence of delays in the available information and in the response of the users to advertising of the firms.

Chapter 5 presents a new model of duopolistic dynamics based on the evolutionary dynamics. The chapter studies the main characteristics of the formulated model as well as the existence, the stability and the parametric sensitivity of its solutions. In Chapter 6 the evolutionary model proposed in the previous chapter is studied assuming the presence of delays.

Finally, Chapter 7 exposes the conclusions of the thesis and indicates possibilities for future research.

Chapter 2

Duopolistic markets, advertising, consumer behavior and models: a brief review

In this chapter, we first present in section 2.1 an overview of the dynamics of the duopolistic market with competitive advertising. Subsequently, in section 2.2 we examine the main characteristics of consumer behavior describing the interaction between clients and firms in a market. The next section 2.3 presents a review of the duopoly models under competition in advertising. Section 2.4 discusses different types of delays that arise in advertising processes. Finally, in section 2.5 we present and explain the advertising policies that will be used throughout this thesis.

2.1 Duopolistic market and advertising

A market is a set or group of buyers and sellers interacting, resulting in the possibility of exchanging between them [20]. A predominant form of market structure is the oligopoly which represents a market with few enterprises and a large number of buyers from the demand side [3] and characterized because the firms are aware of each other [21]. The simplest type of oligopoly is the duopoly, where the market consists of two companies offering similar or identical products [4].

Markets are dynamic, therefore, the strategies of competition existing in the firms which conform the market such as price, quality, publicity, etc. are also influenced by market changes [5]. On the other hand, the consumer buying process is strongly influenced by cultural, social, personal and psychological factors [6]. Therefore, given these conditions and in order to influence the behavior of people [22] and in this way to impact in the process of consumer's purchase, firms use advertising to promote the sale of their products [23]. Thus, advertising can be understood as a form of communication used to induce consumers to take a particular action with respect to products or services [7].

Advertising involves communicating the “value proposition” of the company or brand using the paid media to inform, persuade, and remind consumers of the product offered [6]. In this sense, advertising results can be assessed by communication impacts and the effects on sales and profits [6]. Thus, the accounting of advertising and the return on its investment becomes an important parameter to be considered by firms.

2.2 Consumer behavior and interaction between clients and firms in a duopoly

Consumers demand goods and services produced by firms [2]. In this context, the consumer behavior is a dynamic process involving the action of a group of individuals. It is also a process that includes many decision making and involves states of emotion and coping strategies [24].

The decision making of people is characterized by particular conditions of the individual such as education, experience, emotion, stress [25] and by social conditions as for example family relationships, labor relations, friendships, markets, etc [26].

In the traditional economic sphere, it is assumed that individuals, in making choices, have a condition of rationality that leads them to select what they perceive to be in their best interests [2]. Thus, in economics terms, rationality can be interpreted as a behavior to maximize profit [27].

However, in many cases, the actual behavior in the client decision-making process is different from the rational consumption model because the behavior of the people may vary depending on the type of situation in which they are found, as well as the particular characteristics of the decision to consider or choose [28]. Thus, in real situations, agents use inductive rules of thumb to make decisions instead of absolutely rational reasons [29]. Additonally people’s choices may change due to social influence. Thus, the behavior of others, for example, can give relevant information about the products they will purchase, so that people can learn from their own experiences and their environment [30]. The behavioral approach considers the environmental setting in which the decision making happens [31]. Behavioral economics uses concepts from psychology and anthropology together with economic principles allowing a better understanding of real markets [30].

In competitive duopolies, there is a permanent process of clients migrating from a firm or service provider to their competitor. This consumer behavior occurs either for intrinsic (e.g. desire to try a new brand) or for extrinsic motives (such as price, coupon) [32], [33]. This migration between firms is denominated in some areas as churn, turnover rate or client evasion. The *churn rate* is a key metric for some business segments, especially for Software and Service (SaaS) companies to track the percentage of clients that have canceled the service [34, 35]. From the modeling point of view, it is important to include the churn phenomenon in the dynamics [36]. Moreover, studies show that advertising affects all clients in the market [7], and may affect their preferences regarding the firms, resulting in churn, as well as the effect of decay or forgetfulness.

The main contributions of this thesis are in this context: specifically, it is proposed to model the advertising process as affecting three populations. The first two are the clients of the two competing firms in the duopoly, while the third is that of undecided clients. The latter population is not modeled explicitly in existing advertising models, and it will be shown that its consideration allows for an extension as well as a unification of the Vidale-Wolfe and Lanchester models. Continuing with the idea of three populations, a new model is proposed. It considers client preferences in more detail and is based on the well known replicator-mutator dynamics model from evolutionary game theory. Finally, the impact of different types of delays on these models is also analysed.

Figure 2.1 summarizes the above discussion, in the form of a graph representing the interactions between sets of clients and firms. Nodes of the graph represent sets of clients or firms. Edges represented as solid arrows represent possible transitions between the sets of clients. These transitions occur as a result of advertising by the firms, which are assumed to affect all three sets of clients, as well as due to interactions between clients belonging to the different sets. Existing models in the literature do not consider the set of undecided clients.

Due to the existence of interactions between the various market agents (clients-clients, clients-firms, firms-clients, firms-firms) and assuming that agents are not blind followers of habitual behavior and, unlike, they are able to modify their behavior according to changes in the environment and the environment, in turn, is also affected by the behavior of agents [17], an analysis of the resulting dynamics among the agents involved contemplates the study of social and economic systems before to processes of opinion formation, decision-making, emergence of behavior, norms, or conduct.

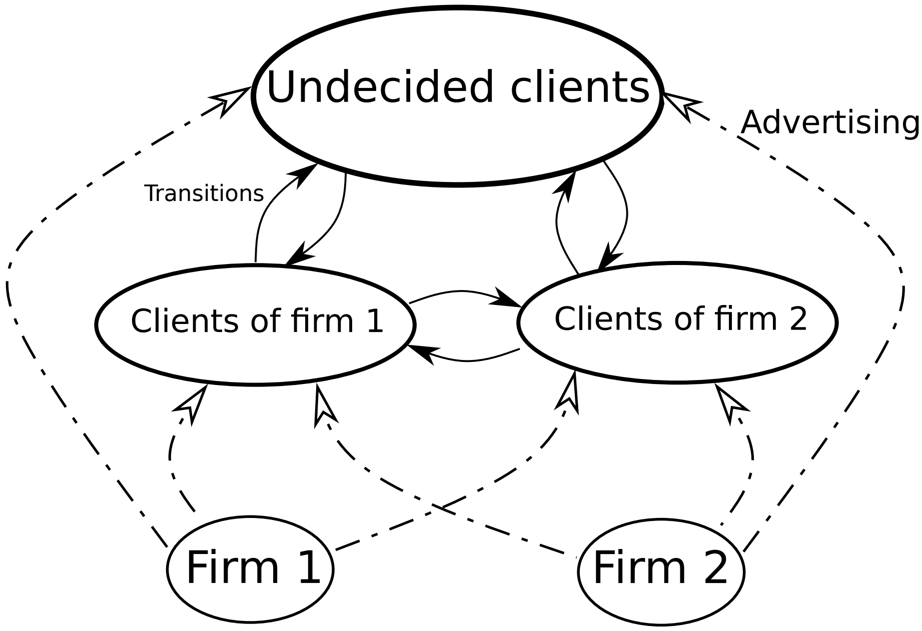


Figure 2.1: Representation of the interactions between clients and firms in a duopoly. The continuous arrows represent the transitions between clients of firm 1, firm 2 and the undecided users. The dashed arrows indicate that advertising by both companies affects all three types of clients.

2.3 Review of existing models of market share dynamics under advertising

The following is a brief summary of the various continous-time models of the market share dynamics under advertising.

2.3.1 Dynamics of monopoly models with advertising

Advertising dynamics models describe the dynamics of the market share under the influence of a control action, called advertising. These models date back to pioneering works by Nerlove and Arrow [8] and Vidale and Wolfe [9] and have since been developed in a number of ways. The most basic model of a firm advertising a single product is called a monopoly model. Denoting the market share (fraction of the population that buys the product) by x, the most general model of its evolution over time can be written as the following ordinary differential equation (ODE):

$$\dot{x} = C(x, u) - D(x), \tag{2.1}$$

where $C(x, u)$ is the term representing the growth of the market share as a function of the current share x and advertising action u , and $D(x)$ is the term representing decrease or loss of the market share. In this terminology, the model proposed by

Nerlove and Arrow [8] can be described as follows. Instead of modeling the share x directly, Nerlove and Arrow propose to use an abstract quantity called goodwill for the firm's product and choice $C(x,u) = u$, $D(x) = \lambda x$. Thus, from an initial value x_0 of goodwill, its evolution is described by ODE:

$$\dot{x} = u - \lambda x(t), \qquad x(0) = x_0 \tag{2.2}$$

Vidale and Wolfe [9] proposed to use a total number of consumers, denoted M, and the fraction of this total conquered by the firm as market share, denoted x. With these definitions, they proposed the choice resulting in the following ODE:

$$\dot{x} = \frac{b}{M}u(1-x) - \lambda x, \qquad x(0) = x_0, \tag{2.3}$$

where $\tilde{u} = \frac{bu}{M}$ is the advertising expenditure and λ the consumer loss ratio attributed to consumers desisting from the product of the firm. Comparing Nerlove-Arrow (2.2) and Vidale-Wolfe models (2.3), it is observed, under constant advertising effort, that, in order to reach equilibrium in the market share denominated x_{eq}, the corresponding constant advertising efforts are:

$$u_{eq}^{NA} = \delta x_{eq} \tag{2.4}$$

$$u_{eq}^{VW} = \frac{\lambda x_{eq}}{1 - x_{eq}} \tag{2.5}$$

Another notable aspect of the Vidale-Wolfe model is the saturation in the growth of the market share as it approaches its maximum value (which is 1, imposed by normalization). As can be clearly seen from equation (2.5) this means that advertising effort tends to infinity as the desired equilibrium share x_{eq} approaches 1.

2.3.2 Models of duopoly advertising dynamics

In this section, the extensions of monopoly models to the case of duopolies will be presented briefly, aiming to motivate the models that will be analyzed and proposed as objects of study in this work.

2.3.2.1 Vidale-Wolfe model

Deal [37] proposed an extension of the Vidale-Wolfe model [9] in the case of a duopoly, expressing the variation of the market share of participating firms in the following way:

$$\dot{x}_i = k_i u_i (1 - x_1 - x_2) - \lambda_i x_i, \qquad i = 1, 2 \tag{2.6}$$

where $x_i = \frac{S_i}{M}$ is the current market share of firm i, u_i is the advertising expenditure of firm i, b_i is the advertising response rate of firm i, λ_i is the decay constant of firm i, and $k_i = \frac{b_i}{M}$, considering M as the potential market of firms, and S_i the firm's sales rate i.

The model proposed by Deal [37] considers that the effects of advertising act only on the unconquered part of the market, thus discarding the influence of advertising on the market shares conquered by competing firms. This hypothesis in Deal's model been questioned by empirical studies that evidence the influence of advertising on the market as a whole [7].

Several extensions and variations to the Vidale-Wolfe model have also been developed in the literature. Many of these studies use the approach of differential games and optimal control formulating advertising strategies based on the Nash equilibrium. For more details of these model extensions and variations, see [13, 14, 38].

2.3.2.2 Lanchester model

The Lanchester model [10] was originally formulated for combat problems and was later discussed by Kimball [39] as a model for the analysis of competition advertising. The Lanchester model can be understood as an extension of the Vidale-Wolfe model [9] within a duopoly with competition in advertising. The model of Lanchester [10] represents the dispute of the market shares between the two participating firms as follows:

$$\begin{aligned} \dot{x}_1 &= k_1 u_1 (1 - x_1) - k_2 u_2 x_1 \\ x_2 &= 1 - x_1 \end{aligned} \tag{2.7}$$

where x_i is the market share relative to firm i $(i = 1, 2)$, k_i is the advertising response rate of firm i, and u_i is the advertising expenditure of firm i.

The Lanchester model [10], different from the Deal model [37], represents the dynamics of competition in advertising, modeling advertising as the only cause of variation in the market share of firms. On the other hand, the Lanchester model does not consider the decay term contemplated in the Vidale-Wolfe model, which is used to represent the effects produced by factors such as the quality of the product or service, as well as competition in advertising with other firms not modeled in the duopoly [7].

Similar to the Vidale-Wolfe model, several extensions to the Lanchester model have been developed in the literature. Again, the most frequent approach in these studies is differential games, in which open-loop and closed-loop strategies are formulated in the Nash equilibrium sense. For reviews of model extensions and variations, see [13, 14, 38].

2.3.2.3 Duopoly model using population dynamics

Consumers interact with each other, so that their opinions or behaviors change over time, resulting in behavior similar to the population dynamics between different species in biological systems. Thus, biological population dynamics can be used to study consumer behavior [40],[41]. Wang *et al.* [40] analyze the effect of advertising on sales of similar products considering the Lotka-Volterra population model [42], [43]. The Wang *et al.* model [40] representing the competition of two firms can be expressed as:

$$\begin{aligned} \dot{x}_1 &= x_1 \left(b_1 - a_{11} x_1 - a_{12} x_2 \right) \\ \dot{x}_2 &= x_2 \left(b_2 - a_{21} x_1 - a_{22} x_2 \right) \end{aligned} \tag{2.8}$$

where x_1 and x_2 are the sales of firms, b_1 and b_2 are the intrinsic growth coefficients of firms, a_{11} and a_{22} are the growth restriction coefficients in firms' own products, a_{12} is the competition coefficient (or predation) of firm 2 in relation to firm 1, and a_{21} the competition coefficient (or predation) of firm 1 in relation to firm 2. The intrinsic growth of sales can be affected by price, quality, promotion and advertising. Wang *et al.* [40] also assume a functional relation $f(\cdot)$ between advertising level q_i of firm i and the intrinsic growth rate coefficient b_i:

$$b_i = f(q_i), \qquad i = 1, 2 \tag{2.9}$$

(the reader is referred to [40] for more details on the function f).

In the next section, we argue that a generalization of model (2.8) can be arrived at using models from evolutionary game theory.

2.3.2.4 Duopoly model using evolutionary games

Evolutionary Game Theory studies the behavior of large populations of agents which interact strategically repeatedly. Changes in the behavior of these populations are driven by natural selection processes through differences in birth and death rates or by "myopic" decision rules applied by individual agents [15].
Evolutionary Game Theory originated as an application of the classical Game Theory formulated by Neumann and Morgenstern [44] to biological contexts, arising from the perception that frequency-dependent aptitude introduces a strategic aspect to evolution [45]. In classical Game Theory, the interactions between rational agents are modeled as games of two or more players who can choose from a set of strategies. Thus, Game Theory is the mathematical study of interactive decision-making in the sense that decision-makers take their own choices and those of others into account [46].

Evolutionary Game Theory was first developed by Fisher [47] in an attempt to explain the approximate equality of sexual proportion in mammals. Later Lewontin [48] made the first explicit application of the Game Theory to evolutionary biology. In 1972 Maynard Smith defined the concept of Evolutionary Stable Strategy. In the year 1973, Maynard Smith and Price in their paper "The Logic of Animal Conflict" [49] generalized the concept of an evolutionarily stable strategy.
Maynard Smith's seminal text "Evolution and the Theory of Games" [50] appears in 1982, followed in 1984 by Robert Axelrod's famous work "The Evolution of Cooperation" [51]. Since then, there has been a constant interest by economists and social scientists in the Evolutionary Game Theory.
The interest of applying the Theory of Evolutionary Games in social and economic sciences is based on three aspects. The first aspect is that the concept of biological evolution can be understood as cultural evolution, referring to changes in beliefs and customs over time. The second aspect is that the premises of limited rationality in the approach to evolutionary game theory [52] are in many cases more appropriate for modeling social systems. The third aspect is that the Evolutionary Game Theory explicitly represents a dynamic theory.
Deterministic evolutionary games of large populations can be described by the equations of the replicator developed by Taylor and Jonker [53] and Zeeman [54] that propose to represent the selection process as follows:

$$\begin{aligned} \dot{x}_i &= x_i \left[f_i(\mathbf{x}) - \phi \right] \\ \mathbf{x} &= (x_1, \dots, x_n) \\ \mathbf{f} &= (f_1, \dots, f_n) \end{aligned} \tag{2.10}$$

where:

- x_i is the fraction of the population with strategy i and it is assumed that $\sum\limits_{i=1}^{n} x_i = 1$,
- $f_i = \sum\limits_{j=1}^{n} a_{ij} x_j$ is the aggregate return with strategy i (also referred to as fitness of strategy i),
- a_{ij} is the payoff of agents with strategy i in interaction with agents with strategy j, and
- $\phi = \mathbf{f}^T \mathbf{x}$ is the mean population return [55].

At this point, it is worth mentioning the equivalence between the replicator equation and the general Lotka-Volterra equation which describes the interaction of

n species, since this can be represented as:

$$\dot{y}_i = y_i \left(r_i + \sum_{j=1}^{n} b_{ij} y_j \right) \qquad i = 1, \ldots, n \tag{2.11}$$

where y_i is the abundance of species i, r_i the rate of growth of species i, and b_{ij} the interaction between species i and j.

In [56] it is argued that the replicator equation with n strategies can be transformed into a Lotka-Volterra equation of $n-1$ species. Thus, expressing equation (2.10) in summation form the replicator equation can be represented as:

$$\dot{x}_i = x_i \left(\sum_{j=1}^{n} a_{ij} x_j - \phi \right) \qquad i = 1, \ldots, n \tag{2.12}$$

Rewriting equation (2.11) for the case of $n-1$ species and considering b_{ij} as the interaction between species, the Lotka-Volterra equation can be expressed as:

$$\dot{y}_i = y_i \left(r_i + \sum_{j=1}^{n-1} b_{ij} y_j \right) \qquad i = 1, \ldots, n-1 \tag{2.13}$$

Thus, equations (2.12) and (2.13) are equivalent for $r_i = a_{in} - a_{nn}$ and $b_{ij} = a_{ij} - a_{nj}$.

The correspondence between Lotka-Volterra equations and the replicator equation represents a link between ecological theory and the theory of evolutionary games [56], [57]. Moreover, given the use of the Lotka-Volterra dynamics in the duopoly modeling, presented in subsubsection 2.3.2.3, the use of the replicator dynamics is suggested in this context as well. Before formalizing this idea, a generalization of the replicator dynamics is presented.

The more general description of evolutionary dynamics includes frequency-dependent selection processes and mutation processes and can be represented by the equations called Replicator-Mutator:

$$\dot{x}_i = \sum_{j=1}^{n} x_j f_j q_{ji}(\mu) - x_i \phi \tag{2.14}$$

where,

- x_i is the fraction of the population with strategy i with $\sum_{i=1}^{n} x_i = 1$,
- a_{ji} is the payoff of the agents with strategy j in interaction with agents with strategy i,
- $f_j = \sum_{i=1}^{n} a_{ji} x_i$ is the fitness strategy j,

- q_{ij} is the mutation rate of strategy i for the strategy j with $\sum_{j=1}^{n} q_{ij} = 1$,
- μ is the mutation parameter, and
- $\phi = \mathbf{f}^T\mathbf{x}$ is the average fitness of the population.

Thus, it follows that the replicator equation is a special case of the Replicator-Mutator equation in the absence of mutation processes [57].

Although it was originated in a biological context, the theory of evolutionary games provides analysis tools for several areas, including in particular the social and economic sciences [55, 57–60].

In the specific context of markets and advertising, we highlight the work developed by Wang [61], which presents an evolutionary model for an online advertising ecosystem, also covering some aspects of advertising strategies. Details of this model are given in the appendix B.

2.4 Delays in markets and consumer behavior

The presence of delays can be found in several systems of diverse areas as mathematics, biology, economics, physics and social sciences [19].

Delays in the economy may arise in many ways; one manner is a delay between the time the economic decision is made and the time when the decision produces the results. Another form is the estimate of the expected values when the function to determine the result is dependent on the current and past values [19].

In the particular case of a duopolistic market, it is possible to indicate that clients and firms can require different sources of information past and present in the decision-making process [18]. So, in the consumer decision process, an existence of a time gap between the recognition of the necessity of a product and the purchase of the same is perceived. This time can be generated as a result of factors internal or external to the consumer such as age, social level, availability of time, information search, product prices or product quality [62].

On the other hand, in the decision-making process of the firms it is observed that the acquisition and processing of data besides being difficult are costly in resources and time, therefore, in many cases, the information of the system is available after the strategic decision is implemented [18]. Thus time delay between the actual information of the system and the moment when the information is available or known is significant [18].

Within this framework, two types of delays in the duopolistic market approach are proposed in the following sections. In order to represent the duopoly models with delay we first write a general market share model in a manner similar to (2.1), that is:

$$\dot{x} = f(x, u) - g(x) \tag{2.15}$$

where $f(x, u)$ is the term representing the growth of the market share as a function of the market share x and advertising action u , and $g(x)$ is the term representing decrease or loss of the market share.

2.4.1 Implementation delay in advertising policy

We define implementation delay following [63]:

Definition 1. *Implementation delay is said to occur when the market share information utilized to define advertising policy is lagged or delayed with respect to the instant when the latter is applied.*

From the definition, denoting $u_\tau = u(t - \tau)$ it follows that the following modification to (2.15):

$$\dot{x} = f(x, u_\tau) - g(x) \tag{2.16}$$

represents a market share dynamics model with implementation delay.

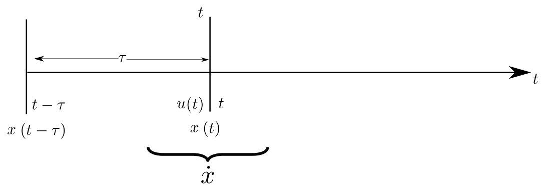


Figure 2.2: A timeline illustrating implementation delay: current time is t, at which (feedback) advertising effort $u(\cdot)$ will be applied. If the only market share information available is that of past instant $(t-\tau)$, this means that the advertising effort applied at time t can be expressed as $u(x(t-\tau))$.

2.4.2 Adoption delay

The effect of the company's advertising policy on clients is not immediate, thus in this section we define the idea of adoption delay following [64]:

Definition 2. *Adoption delay is said to occur when advertising policy put into effect at time t, acting on the market share variables at time t, will only affect the dynamics at time $t + \tau$.*

Supposing time $T = t + \tau$, to be the current time instant, the market share dynamics in general form can be written as:

$$\dot{x}(T) = f\left(x(t), u(t)\right) - g(T) \tag{2.17}$$

Since $x(t) = x(T - \tau) = x_\tau$ and $u(t) = u(T - \tau) = u_\tau$, equation (2.17) can be rewritten as:

$$\dot{x}(T) = f\left(x_\tau, u_\tau\right) - g(x(T)) \tag{2.18}$$

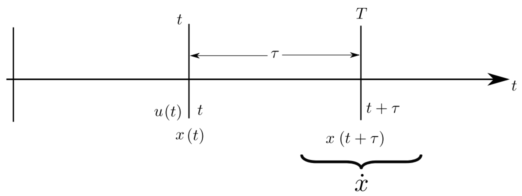


Figure 2.3: A timeline illustrating adoption delay: current time is $T = t + \tau$, but (feedback) advertising effort $u(x(t))$ is applied at time $t = T - \tau$, based on the known market share $x(t)$ at time t.

2.5 Decentralized affine feedback advertising policy

The decentralized affine feedback advertising policy used in this thesis is formulated as:

$$u_1 = k_1 x_1 + c_1 \tag{2.19}$$
$$u_2 = k_2 x_2 + c_2 \tag{2.20}$$

where:
u_i is the advertising effort of firm i
x_i is the market share of firm i.
k_i is the proportional effort of firm i
c_i is the constant effort of firm i.

Note that, the advertising policy of the firms, expressed by two terms, the first being proportional to market share ($k_i x_i$) and the second having a constant value (c_i), is also decentralized in the sense that firm i bases its policy based only on information about its own market share x_i.

Affine control has been proposed, in the context of predator-prey models and using full state feedback, in the textbook [65]. Prior to this, decentralized affine control, also in the context of predator-prey models, was used in [66], [67] and subsequently in [68] . Taking inspiration from these applications, it was then proposed to use decentralized affine advertising (DAA) policies in models of duopolies in [69], [70] . The main motivations for the use of DAA policies in this thesis are summarized below:

- DAA policies have a natural interpretation as proportional plus constant control, and are easy to implement, in contrast to optimal controls, which are usually very hard to calculate and also to implement.
- The simple mathematical form of DAA policies also permits analytical derivations of stability and bifurcation results for the models proposed in this thesis.

Given the fact that, in this thesis, all advertising policies or controls are affine and decentralized, and determined by the choice of the parameters $(k_i, c_i, i = 1, 2)$, the results presented are analytical, allowing a policy designer to predict what happens under different scenarios, for different choices of the parameter values. This is in contrast with the approach of optimal control, which determines a (usually open loop and not necessarily decentralized) policy that takes the system state from an initial set of market shares to a desired final set of market shares, minimizing some cost function. In the proposed approach, costs can be evaluated by substituting the proposed controls into a specified cost function and using the results in a “flight simulator” mode [71]. It is also possible, for example, to draw isocost contours that connect reachable states with the same terminal cost.

Chapter 3

Vidale-Wolfe model and extended Lanchester model under affine advertising control policy

In this chapter, the Vidale-Wolfe and Lanchester models are reexamined from the perspective of the existence of a population of undecided clients in a duopolistic market. The chapter begins by examining Deal's extension of the Vidale-Wolfe model and concludes that it is essentially equivalent to a model with an undecided set of clients, in addition to the two usual sets of clients of the competing firms. In section 3.2 an extension of the Lanchester model considering the existence of a third set of undecided clients showing that this model is a genuine extension and is not subsumed by the earlier models is formulated. Section 3.3 presents an equilibrium and stability analysis of the models introduced in the previous sections, subjected to affine advertising control policies. Subsequently, numerical simulations that verify the analytical results are given in section 3.4. Finally, section 3.5 summarizes the conclusions of the chapter.

3.1 Vidale-Wolfe model considering undecided users

Deal [37] proposed an extension of the Vidale-Wolfe model to the case of a duopoly considering that the effects of advertising act only on the unconquered part of the market, thus discarding the influence of advertising on the market shares conquered by the competing firms. Thus denoting the unconquered part of the market by x_3, which corresponds to the population of undecided individuals, the Vidale-Wolfe model can be expressed by the following equations:

$$\begin{aligned}\dot{x}_1 &= x_3 u_1 - \lambda_1 x_1 \\ \dot{x}_2 &= x_3 u_2 - \lambda_2 x_2 \\ \dot{x}_3 &= -x_3 u_1 - x_3 u_2 + \lambda_1 x_1 + \lambda_2 x_2\end{aligned} \tag{3.1}$$

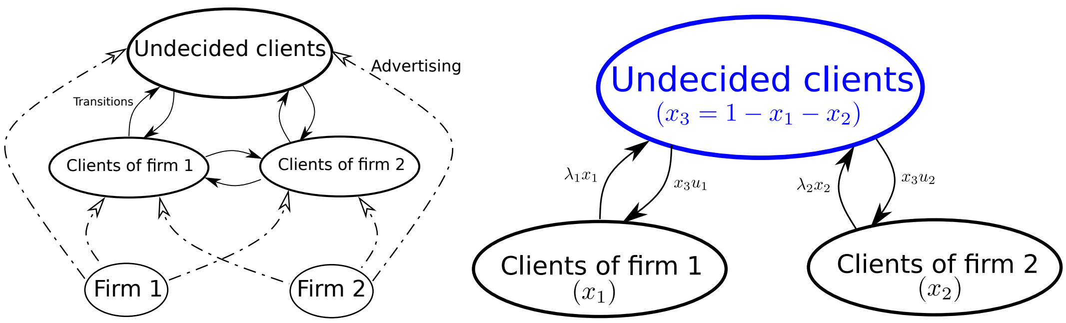
Assuming that the total population size is constant and normalized to 1 [22], i.e., $x_1 + x_2 + x_3 = 1$, the model (3.1) can be expressed as follows:

$$\begin{aligned}\dot{x}_1 &= u_1 - u_1 x_1 - u_1 x_2 - \lambda_1 x_1 \\ \dot{x}_2 &= u_2 - u_2 x_1 - u_2 x_2 - \lambda_2 x_2\end{aligned} \tag{3.2}$$

Under constant controls, the equilibrium point of the system (3.2) is calculated to be:

$$(x_1^*, x_2^*) = \left(\frac{\lambda_2 u_1}{\lambda_1 \lambda_2 + \lambda_1 u_2 + \lambda_2 u_1}, \frac{\lambda_1 u_2}{\lambda_1 \lambda_2 + \lambda_1 u_2 + \lambda_2 u_1} \right) \tag{3.3}$$

where x_1, x_2 and x_3 are the state variables representing the market shares of firm 1, firm 2 and undecided users respectively, u_1 and u_2 are the actions, assumed constant, of firm 1 and firm 2 representing positive advertising, λ_1 and λ_2 are the decay terms of firm 1 and firm 2. Figure 3.1 shows the relation between the interaction between clients in the Vidale-Wolfe model.



(a) (b)

Figure 3.1: A graph representation of client and firm interactions in a duopoly with advertising. (a) The nodes represent clients and firms and the edges represent interactions (advertising or transitions) between them, (b) Vidale-Wolfe model (3.10) showing only transitions between client sets of firms 1 and 2 and the set of undecided clients, but no transitions amongst themselves.

3.2 Extended Lanchester model considering undecided users

The Lanchester model [10] can be understood as an extension of the Vidale-Wolfe model [9] within a duopoly with competition in advertising [72]. The Lanchester model in contrast with the Deal model [37] considers advertising to be the sole cause of variation of the market share of firms. For this reason, the Lanchester model does not consider the decay term contemplated in the Vidale-Wolfe model, which is used to represent the loss of market share produced by factors such as quality of the product or service, as well as competition in advertising with other firms not modeled in the duopoly. In addition, the Lanchester model assumes that the market is saturated (i.e., the sum of market shares of the two firms is unity), so that competition takes place in a situation where the market share gained by one firm is equal to that lost by the other. In other words, there are no undecided clients at all. With these assumptions, the classical Lanchester model is given by the following equations:

$$\begin{aligned} \dot{x}_1 &= x_2 u_1 - x_1 u_2 \\ \dot{x}_2 &= x_1 u_2 - x_2 u_1 \end{aligned} \tag{3.4}$$

In order to extend both the Lanchester as well as the Deal-Vidale-Wolfe models, we argue that the firm i's advertising acts on the undecided consumers in a positive sense (i.e., to increase x_i), while it acts negatively on firm j (its competitor). In addition, we assume that the decay terms that occur in the Deal-Vidale-Wolfe model represent migration of clients of firms 1 and 2 to the set of undecided clients. In mathematical terms, the proposed extension of the Lanchester model is as follows:

$$\begin{aligned} \dot{x}_1 &= -x_1 u_2 + (x_3 + x_2) u_1 - \lambda_1 x_1 \\ \dot{x}_2 &= -x_2 u_1 + (x_3 + x_1) u_2 - \lambda_2 x_2 \\ \dot{x}_3 &= -x_3 u_1 - x_3 u_2 + \lambda_1 x_1 + \lambda_2 x_2 \end{aligned} \tag{3.5}$$

Note that x_3 has the same dynamics as in the Vidale-Wolfe model, because of the fact that the Lanchester model just adds and subtracts the terms $x_i u_j$ from the corresponding equations, so that the sum of the first two equations in the Vidale-Wolfe model is the same as the corresponding sum in the Lanchester model. Since the total population size is constant and normalized to 1, the model (3.5) can be expressed as follows:

$$\begin{aligned}\dot{x}_1 &= u_1 - x_1(u_1 + u_2 + \lambda_1)\\ \dot{x}_2 &= u_2 - x_2(u_1 + u_2 + \lambda_2)\end{aligned} \tag{3.6}$$

Thus the fixed points of the system (3.6) are determined as:

$$(x_1^*, x_2^*) = \left(\frac{u_1}{\lambda_1 + u_1 + u_2}, \frac{u_2}{\lambda_2 + u_1 + u_2}\right) \tag{3.7}$$

where x_1, x_2 and x_3 are the state variables representing the market shares of firm 1, firm 2 and undecided users respectively, u_1 and u_2 are the actions of firm 1 and firm 2 representing positive advertising, λ_1 and λ_2 are the decay terms of firm 1 and firm 2, respectively. Figure 3.2 illustrates the relation between the interaction between clients in the extended Lanchester model.

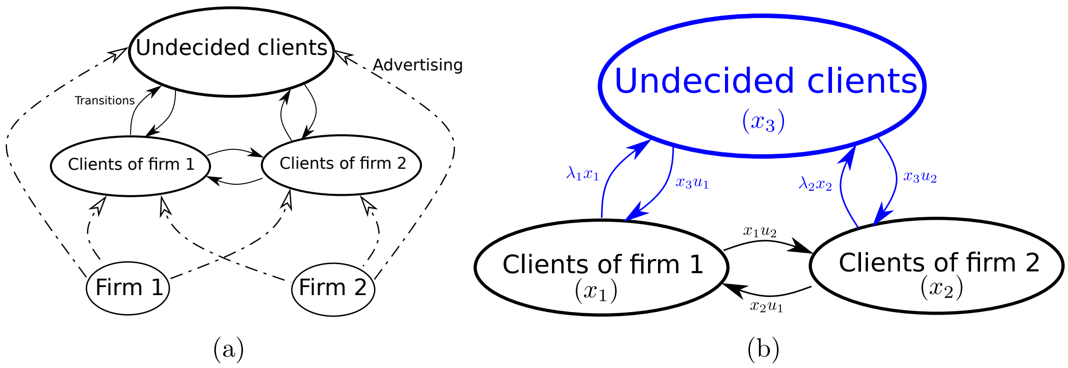


Figure 3.2: A graph representation of client and firm interactions in a duopoly with advertising. (a) The nodes represent clients and firms and the edges represent interactions (advertising or transitions) between them, (b) extended Lanchester model (3.17) showing transitions between all client sets.

3.3 Equilibria and stability analysis of duopoly models considering undecided users under an affine advertising control policy

In this section, we analyze the existence and stability of the equilibrium points of the duopoly models proposed in the previous sections. For this purpose, we first make the following standard assumptions:

Assumption 1. *The market shares of firm* 1 (x_1) *and firm* 2 (x_2) *are considered to be nonnegative values in the interval* $[0, 1]$.

Assumption 2. *The coefficients of models and control policies :* λ_1*,* λ_2*,* u_1*,* u_2*,* k_1*,* k_2*,* c_1 *and* c_2 *are assumed to be positive values.*

Assumption 3. *The decay rates are assumed to be equal for both firms,* $\lambda_1 = \lambda_2 = \lambda$

3.3.1 Affine control in Vidale-Wolfe model

The advertising efforts of the firms are assumed to be affine and decentralized as explained in section 2.5 above:

$$u_1 = k_1 x_1 + c_1 \tag{3.8}$$

$$u_2 = k_2 x_2 + c_2 \tag{3.9}$$

Then, substituting the affine advertising efforts (controls) in the model (3.2) yields:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1^2 - k_1 x_1 x_2 + k_1 x_1 - \lambda x_1 + c_1 - c_1 x_1 - c_1 x_2 \\ \dot{x}_2 &= -k_2 x_2^2 - k_2 x_1 x_2 + k_2 x_2 - \lambda x_2 + c_2 - c_2 x_1 - c_2 x_2 \end{aligned} \tag{3.10}$$

Reordering terms we have:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1^2 - k_1 x_1 x_2 - a x_1 - c_1 x_2 + c_1 \\ \dot{x}_2 &= -k_2 x_2^2 - k_2 x_1 x_2 - b x_2 - c_2 x_1 + c_2 \end{aligned} \tag{3.11}$$

where:
$a = (c_1 + \lambda - k_1)$
$b = (c_2 + \lambda - k_2)$
The dynamics and corresponding equilibrium points for the model (3.11) are shown in table 3.1. Note that for general cases of affine control, particular conditions are considered for control parameters in order to establish analytical solutions for equilibrium points. Thus for policy 4, $k_1 = c_1 = c_2 = c$. For policy 5, $k_1 = c_1 = k$ and for policy 6, $k_1 = k_2 = k$ and $c_1 = c_2 = c$. In addition, in table 3.1, we have that for policy 4, $p = 3c + \lambda + g$ and $q = 3c + \lambda - g$ where $g = (5c^2 + 2c\lambda + \lambda^2)^{\frac{1}{2}}$, for policy 5, $f = \sqrt{\lambda^2 + 4k^2}$ and finally for policy 6, $f = \sqrt{\frac{e^2+2ec+c^2+8ck}{4}}$ where $e = c + \lambda - k$. Summary of these considerations are shown in Appendix A (Table A.4)

Now, the stability is determined by the signs of the determinant and trace of the Jacobian matrix evaluated at the corresponding equilibrium points. The Jacobian matrix for model (3.11) with respect to x_1 and x_2 is given by:

$$J_{VW} = \begin{bmatrix} -2k_1 x_1 - k_1 x_2 - a & -k_1 x_1 - c_1 \\ -k_2 x_2 - c_2 & -2k_2 x_2 - k_2 x_1 - b \end{bmatrix} \tag{3.12}$$

Hence, the determinant and trace for the Jacobian matrix are:

$$\begin{aligned} \det = {} & 2k_1k_2x_1^2 + 4k_1k_2x_1x_2 + 2k_1k_2x_2^2 + ak_2x_1 + 2ak_2x_2 + 2bk1x_1 \\ & + bk_1x_1 - c_1k_2x_2 - c_2k_1x_1 + ab - c_1c_2 \end{aligned} \tag{3.13}$$

$$\mathrm{tr} = -2k_1x_1 - k_1x_2 - k_2x_1 - 2k_2x_2 - a - b \tag{3.14}$$

The conditions that ensure the stability of the equilibrium points of the Vidale-Wolfe model are shown in table 3.3. Note, the expressions for the determinants and traces of the Vidale-Wolfe for special cases of affine control, are displayed in Appendix A (Table A.2).

Policy	Control Parameters	Dynamic Eq. for Vidale-Wolfe model	Equilibrium Points
P_1	$u_1 = c_1$ $u_2 = c_2$	$\dot{x}_1 = -c_1x_1 - c_1x_2 - \lambda x_1 + c_1$ $\dot{x}_2 = -c_2x_1 - c_2x_2 - \lambda x_2 + c_2$	$\left(\frac{c_1}{c_1+c_2+\lambda}, \frac{c_2}{c_1+c_2+\lambda}\right)$
P_2	$u_1 = k_1x_1$ $u_2 = c_2$	$\dot{x}_1 = -k_1x_1^2 - k_1x_1x_2 + k_1x_1 - \lambda x_1$ $\dot{x}_2 = -c_2x_1 - c_2x_2 - \lambda x_2 + c_2$	$\left(0, \frac{c_2}{c_2+\lambda}\right)$ $\left(\frac{k_1-c_2-\lambda}{k_1}, \frac{c_2}{k_1}\right)$
P_3	$u_1 = k_1x_1$ $u_2 = k_2x_2$	$\dot{x}_1 = -k_1x_1^2 - k_1x_1x_2 + k_1x_1 - \lambda x_1$ $\dot{x}_2 = -k_2x_2^2 - k_2x_1x_2 + k_2x_2 - \lambda x_2$	$(0, 0)$ $\left(0, \frac{k_2-\lambda}{k_2}\right)$ $\left(\frac{k_1-\lambda}{k_1}, 0\right)$
P_3^*	$u_1 = kx_1$ $u_2 = kx_2$	$\dot{x}_1 = -kx_1^2 - kx_1x_2 + kx_1 - \lambda x_1$ $\dot{x}_2 = -kx_2^2 - kx_1x_2 + kx_2 - \lambda x_2$	$\left(-\frac{kx_2-k+\lambda}{k}, x_2\right)$ $(0, 0)$
P_4	$u_1 = k_1x_1 + c_1$ $u_2 = c_2$	$\dot{x}_1 = -k_1x_1^2 - k_1x_1x_2 + k_1x_1 - \lambda x_1$ $\dot{x}_2 = -c_2x_1 - c_2x_2 - \lambda x_2 + c_2$	$\left(\frac{2c-p}{2c}, \frac{p}{2(c+\lambda)}\right)$ $\left(\frac{2c-q}{2c}, \frac{q}{2(c+\lambda)}\right)$
P_5	$u_1 = k_1x_1 + c_1$ $u_2 = k_2x_2$	$\dot{x}_1 = -k_1x_1^2 - k_2x_1x_2 - c_1x_1 + k_1x_1 - \lambda x_1 + c_1$ $\dot{x}_2 = -k_1x_1x_2 - k_2x_2^2 - c_1x_2 + k_2x_2 - \lambda x_2$	$\left(\frac{-\lambda-f}{2k}, 0\right)$ $\left(\frac{-\lambda+f}{2k}, 0\right)$ $\left(\frac{k}{k_2-k}, \frac{2kk_2-k\lambda+k_2\lambda-k_2^2}{kk_2-k_2^2}\right)$
P_6	$u_1 = k_1x_1 + c_1$ $u_2 = k_2x_2 + c_2$	$\dot{x}_1 = -k_1x_1^2 - k_1x_1x_2 + k_1x_1 - \lambda x_1 + c_1 - c_1x_1 - c_1x_2$ $\dot{x}_2 = -k_2x_2^2 - k_2x_1x_2 + k_2x_2 - \lambda x_2 + c_2 - c_2x_1 - c_2x_2$	$\left(\frac{-e-c-2f}{4k}, \frac{-e-c-2f}{4k}\right)$ $\left(\frac{-e-c+2f}{4k}, \frac{-e-c+2f}{4k}\right)$

Table 3.1: Dynamic equations and equilibrium points for the Vidale-Wolfe model (3.10) under different control policies

3.3.2 Affine control in the extended Lanchester model

Similar to Vidale-Wolfe model, the affine advertising control of the firms for extended Lanchester model is defined as:

$$u_1 = k_1x_1 + c_1 \tag{3.15}$$

$$u_2 = k_2x_2 + c_2 \tag{3.16}$$

Then, substituting the controls in the model (3.6) yields:

$$\begin{aligned} \dot{x}_1 &= -k_1x_1^2 - k_2x_1x_2 - c_1x_1 - c_2x_1 + k_1x_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_1x_1x_2 - k_2x_2^2 - c_1x_2 - c_2x_2 + k_2x_2 - \lambda x_2 + c_2 \end{aligned} \tag{3.17}$$

Rearranging terms:

$$\begin{aligned}\dot{x}_1 &= -k_1 x_1^2 - k_2 x_1 x_2 - w x_1 + c_1 \\ \dot{x}_2 &= -k_2 x_2^2 - k_1 x_1 x_2 - z x_2 + c_2\end{aligned} \tag{3.18}$$

where:
$w = (c_1 + c_2 + \lambda - k_1)$
$z = (c_1 + c_2 + \lambda - k_2)$
Table 3.2 shows the equations of dynamics and the equilibrium points of the extended Lanchester model for particular cases of the affine control. Note that, similar to the Vidale-Wolfe model, particular cases of control parameters are considered to establish analytical solutions for equilibrium points. For policy 4, $k_1 = c_1 = c_2 = c$. For policy 5, $k_1 = c_1 = k$ and for policy 6, $k_1 = k_2 = k$ and $c_1 = c_2 = c$. In addition, in table 3.2, for policy 4, $p = 3c+\lambda+g$ and $q = 3c+\lambda-g$ where $g = (5c^2 + 2c\lambda + \lambda^2)^{\frac{1}{2}}$, for policy 5, $f = \sqrt{\lambda^2 + 4k^2}$ and finally for policy 6, $f = \sqrt{\frac{e^2+2ec+c^2+8ck}{4}}$ where $e = c + \lambda - k$. Summary of these considerations are shown in Appendix A (Table A.4)

The Jacobian matrix for model (3.18) with respect to x_1 and x_2 is given by:

$$J_{Le} = \begin{bmatrix} -2k_1 x_1 - k_2 x_2 - w & -k_2 x_1 \\ -k_1 x_2 & -2k_2 x_2 - k_1 x_1 - z \end{bmatrix} \tag{3.19}$$

whence, the determinant and trace for the Jacobian matrix are:

$$\begin{aligned}\det = {} & 2k_1^2 x_1^2 + 4k_1 k_2 x_1 x_2 + 2k_2^2 x_2^2 + k_1 w x_1 \\ & + 2k_1 z x_1 + 2k_2 w x_2 + k_2 z x_2 + zw\end{aligned} \tag{3.20}$$

$$\mathrm{tr} = -3k_1 x_1 - 3k_2 x_2 - z - w \tag{3.21}$$

The conditions that ensure stability of the equilibrium points of the extended Lanchester model are shown in table 3.3. The expressions for the determinants and traces of the extended Lanchester model for special cases of affine control are displayed in Appendix A (Table A.3). From the results presented in tables 3.1, 3.2 and 3.3 we can formulate the main result:

Theorem 1. *The extended Lanchester model (3.17) and the Vidale-Wolfe model (3.10) both under affine advertising policy $u_i = k_i x_i + c_i, i = 1, 2$ are equivalent for the same choices of affine policy parameters (k_i, c_i) in the sense that both models have the same equilibrium points with the same stability properties (even though stability conditions, eigenvalues and dynamics differ).*

Proof. Follows from the composition of results given in Table 3.1, Table 3.2 and Table 3.3 □

Policy	Control Parameters	Dynamic equations for extended Lanchester model	Equilibrium Points
P_1	$u_1 = c_1$ $u_2 = c_2$	$\dot{x}_1 = -c_1x_1 - c_2x_1 - \lambda x_1 + c_1$ $\dot{x}_2 = -c_1x_2 - c_2x_2 - \lambda x_2 + c_2$	$\left(\frac{c_1}{c_1+c_2+\lambda}, \frac{c_2}{c_1+c_2+\lambda}\right)$
P_2	$u_1 = k_1x_1$ $u_2 = c_2$	$\dot{x}_1 = -k_1x_1^2 - c_2x_1 + k_1x_1 - \lambda x_1$ $\dot{x}_2 = -k_1x_1x_2 - c_2x_2 - \lambda x_2 + c_2$	$\left(0, \frac{c_2}{c_2+\lambda}\right)$ $\left(\frac{k_1-c_2-\lambda}{k_1}, \frac{c_2}{k_1}\right)$
P_3	$u_1 = k_1x_1$ $u_2 = k_2x_2$	$\dot{x}_1 = -k_1x_1^2 - k_2x_1x_2 + k_1x_1 - \lambda x_1$ $\dot{x}_2 = -k_2x_2^2 - k_1x_1x_2 + k_2x_2 - \lambda x_2$	$(0, 0)$ $\left(0, \frac{k_2-\lambda}{k_2}\right)$ $\left(\frac{k_1-\lambda}{k_1}, 0\right)$
P_3^*	$u_1 = kx_1$ $u_2 = kx_2$	$\dot{x}_1 = -kx_1^2 - kx_1x_2 + kx_1 - \lambda x_1$ $\dot{x}_2 = -kx_2^2 - kx_1x_2 + kx_2 - \lambda x_2$	$\left(-\frac{kx_2-k+\lambda}{k}, x_2\right)$ $(0, 0)$
P_4	$u_1 = k_1x_1 + c_1$ $u_2 = c_2$	$\dot{x}_1 = -k_1x_1^2 - c_1x_1 - c_2x_1 + k_1x_1 - \lambda x_1 + c_1$ $\dot{x}_2 = -k_1x_1x_2 - c_1x_2 - c_2x_2 - \lambda$	$\left(\frac{2c-p}{2c}, \frac{p}{2(c+\lambda)}\right)$ $\left(\frac{2c-q}{2c}, \frac{q}{2(c+\lambda)}\right)$
P_5	$u_1 = k_1x_1 + c_1$ $u_2 = k_2x_2$	$\dot{x}_1 = -k_1x_1^2 - k_2x_1x_2 - c_1x_1 + k_1x_1 - \lambda x_1 + c_1$ $\dot{x}_2 = -k_1x_1x_2 - k_2x_2^2 - c_1x_2 + k_2x_2 - \lambda x_2$	$\left(\frac{-\lambda-f}{2k}, 0\right)$ $\left(\frac{-\lambda+f}{2k}, 0\right)$ $\left(\frac{k}{k_2-k}, \frac{2kk_2-k\lambda+k_2\lambda-k_2^2}{kk_2-k_2^2}\right)$
P_6	$u_1 = k_1x_1 + c_1$ $u_2 = k_2x_2 + c_2$	$\dot{x}_1 = -k_1x_1^2 - k_2x_1x_2 - c_1x_1 - c_2x_1 + k_1x_1 - \lambda x_1 + c_1$ $\dot{x}_2 = -k_1x_1x_2 - k_2x_2^2 - c_1x_2 - c_2x_2 + k_2x_2 - \lambda x_2 + c_2$	$\left(\frac{-e-c-2f}{4k}, \frac{-e-c-2f}{4k}\right)$ $\left(\frac{-e-c+2f}{4k}, \frac{-e-c+2f}{4k}\right)$

Table 3.2: Dynamic equations and equilibrim points for the extended Lanchester model (3.17) under different control policies

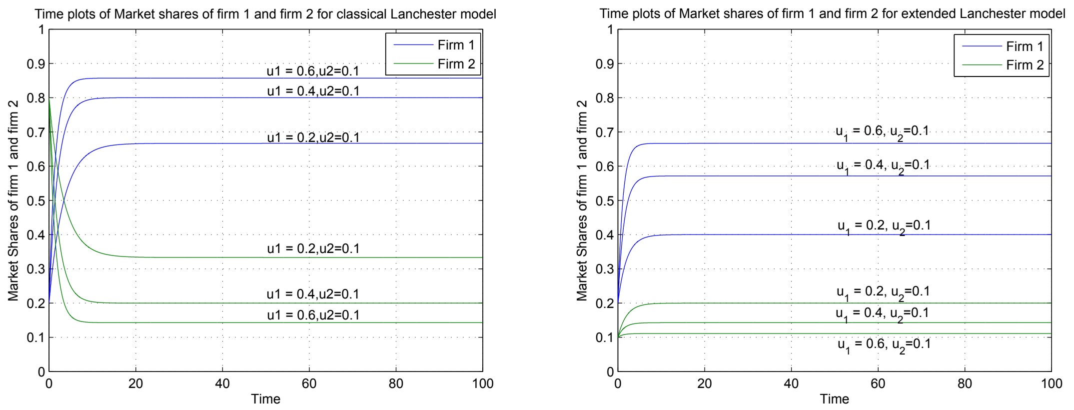
Policy	Equilibrium Points	Stability conditions for VW	Stability conditions for extended Lanchester
P_1	$\left(\frac{c_1}{c_1+c_2+\lambda}, \frac{c_2}{c_1+c_2+\lambda}\right)$	None	None
P_2	$\left(0, \frac{c_2}{c_2+\lambda}\right)$ $\left(\frac{k_1-c_2-\lambda}{k_1}, \frac{c_2}{k_1}\right)$	$k_1 < c_2 + \lambda$ $k_1 > c_2 + \lambda$	$k_1 < c_2 + \lambda$ $k_1 > c_2 + \lambda$
P_3	$\left(0, \frac{k2-\lambda}{k_2}\right)$ $\left(\frac{k_1-\lambda}{k_1}, 0\right)$	$k_1 < k_2$ $k_2 < k_1$	$k_1 < k_2$ $k_2 < k_1$
P_3^*	$\left(-\frac{kx_2-k+\lambda}{k}, x_2\right)$	$\lambda < k$	$\lambda < k$
P_4	$\left(\frac{2c-q}{2c}, \frac{q}{2(c+\lambda)}\right)$	None	None
P_5	$\left(\frac{-\lambda+f}{2k}, 0\right)$ $\left(\frac{k}{k_2-k}, \frac{2kk_2-k\lambda+k_2\lambda-k_2^2}{kk_2-k_2^2}\right)$	$k > \frac{k_2(\lambda-f)}{2(\lambda+f-k_2)}$ $\frac{k_2(\lambda-k_2)}{\lambda-2k_2} < k < \frac{k_2^2}{\lambda}$	$k > \frac{2k_2-f-\lambda}{2}$ $\frac{k_2(\lambda-k_2)}{\lambda-2k_2} < k < \frac{2k_2-\lambda}{2}$
P_6	$\left(\frac{-e-c+2f}{4k}, \frac{-e-c+2f}{4k}\right)$	$k < 2f + \lambda - 2c$	$k < 2f + \lambda - 2c$

Table 3.3: Conditions of the control parameters to guarantee stability of the equilibrium points.

Remark: The extended Lanchester model (3.17) and the Vidale-Wolfe model (3.10) become identical under the control $u_i = k_i x_i$.

3.4 Numerical Results

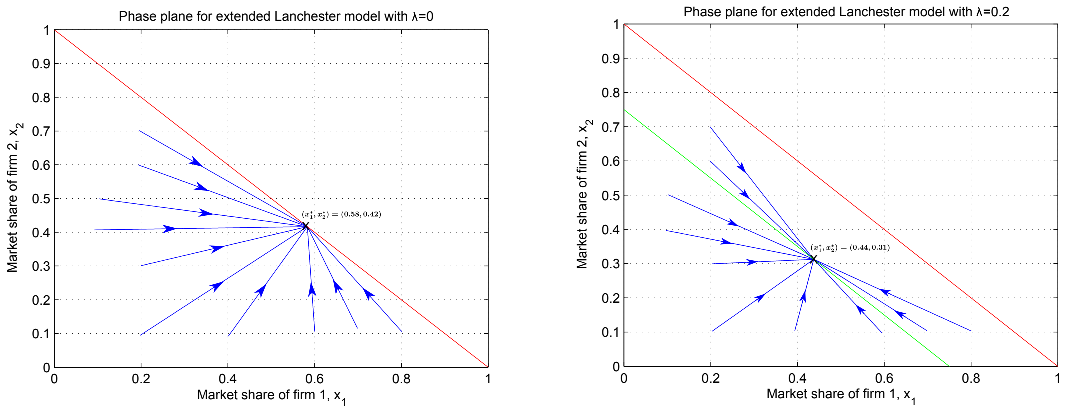
This section presents some numerical simulations to verify the analytical results found in the previous sections. Figure 3.3 allows the comparison of the classical Lanchester model with the extended Lanchester model, thus figure 3.3(a) shows that the equilibrium points are always on the line $x_1+x_2 = 1$ while figure 3.3(b) indicates the influence of considering an undecided population, that is, the equilibrium points lie below the line $x_1 + x_2 = 1$.



(a) (b)

Figure 3.3: Evolution of market shares of firms x_1 and x_2 under affine control policies with parameters $k_1 = 0, c_1 = 0.2, 0.4, 0.6, k_2 = 0, c_2 = 0.1$ for: (a) classical Lanchester model for $x_1(0) = 0.2$, (b) extended Lanchester model with undecided users for $x_1(0) = 0.2$, $x_2(0) = 0.1$ and $\lambda = 0.2$. Note that in classical Lanchester model the market shares are always on the line $x_1 + x_2 = 1$.

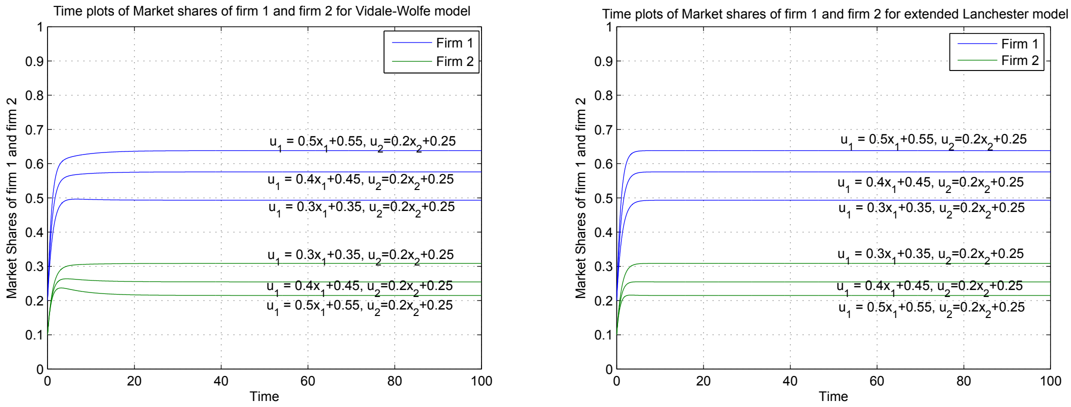
Next, figure 3.4 allows the comparison of the extended Lanchester model for different values of λ. Figure 3.4(a) shows extended Lanchester model for $\lambda = 0$ where the equilibrium point lies on the red line $x_1 + x_2 = 1$, on the other hand, figure 3.4(b) shows extended Lanchester model for $\lambda = 0.2$, then the equilibrium point lies on the green line $x_1 + x_2 = 0.75$. The shift in the line of equilibrium points is a consequence of changing the value of λ from zero to a positive value.



(a) (b)

Figure 3.4: Phase plane of the market shares of firms x_1 and x_2 under affine control policies with parameters $k_1 = 0, c_1 = 0.35, k_2 = 0, c_2 = 0.25$ for: (a) extended Lanchester model with $\lambda = 0$, (b) extended Lanchester model with $\lambda = 0.2$. Note the change in the line containing the equilibrium points. In this case, the new equation of line is $x_1 + x_2 = 0.75$.

Figure 3.5 shows the evolution of market shares of brands under increase in advertising u_1 and u_2 constant. Thus, figure 3.5(a) shows the evolution of market shares in Vidale-Wolfe model and figure 3.5(b) shows the evolution of market shares in extended Lanchester model. In both models three increases in advertising u_1 are considered. Note that the equilibrium points in both models under the same control policy are equal and that the difference in the dynamics of models can be seen in the evolution of market share of firm 2 when the increment in advertising u_1 is greater.



(a) (b)

Figure 3.5: Evolution of market shares of firms x_1 and x_2 under affine control policies for: (a) Vidale-Wolfe model expressed in equation (3.11) , (b) extended Lanchester model expressed in equation (3.18).

Next, figures 3.6(a) and 3.6(b) show the phase planes of the models (3.11) and (3.18) for the following control parameters $k_1 = 0.3, c_1 = 0.35, k_2 = 0.2, c_2 = 0.25$. Thus it is observed that the models of Vidale-Wolfe and extended Lanchester have the same equilibrium points ($x_1^* = 0.49, x_2^* = 0.31$) although they have different trajectories. Therefore numerical results presented in figures 3.6(a) and 3.6(b) are in accordance with Theorem 1.

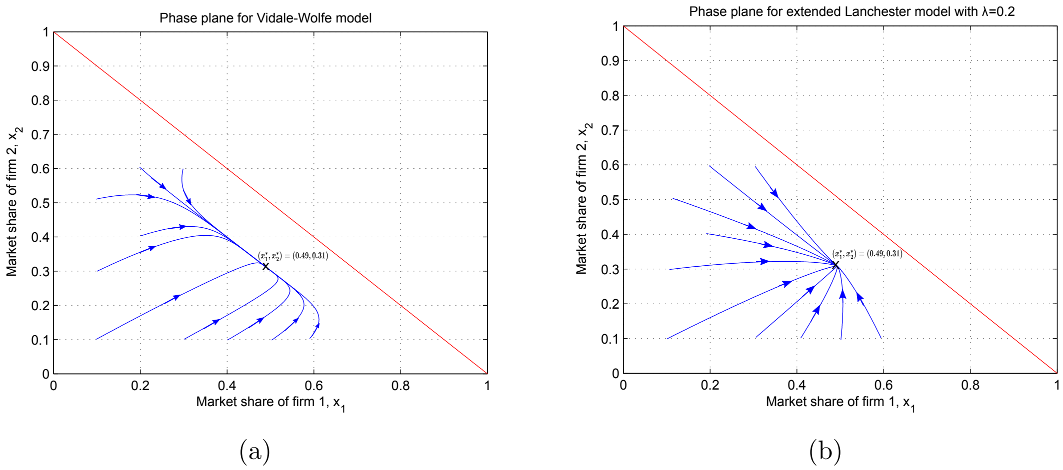


Figure 3.6: Phase plane of the market shares of firms x_1 and x_2 under affine control policies with parameters $k_1 = 0.3, c_1 = 0.35, k_2 = 0.2, c_2 = 0.25$ for: (a) Vidale-Wolfe model expressed in equation (3.11) , (b) extended Lanchester model expressed in equation (3.18). It is observed that the models have the same equilibrium point but different dynamics.

Finally, figure 3.7 shows the phase plane for special case when $u_1 = k_1 x_1$ and $u_2 = k_2 x_2$ with $k_1 = k_2$. Note that in this special case the Vidale-Wolfe and extended Lanchester models have the same dynamic and the fixed points are a set of points.

Remark: The values of λ found in the literature related to empirical data [5, 7, 73, 74] are used for numerical simulations.

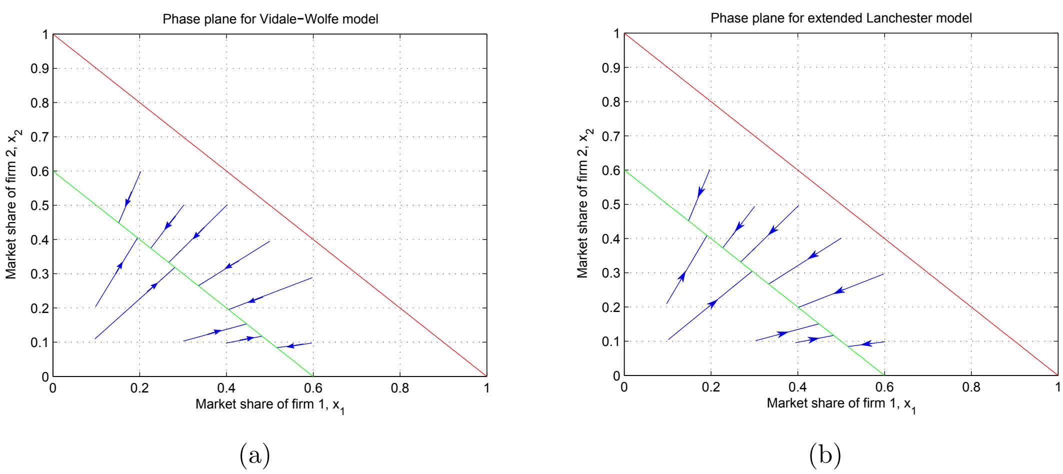


Figure 3.7: Phase plane of the market shares of firms x_1 and x_2 under affine control policies with parameters $k_1 = 0.5, c_1 = 0, k_2 = 0.5, c_2 = 0, \lambda = 0.2$ for: (a) Vidale-Wolfe model expressed in equation (3.11) , (b) extended Lanchester model expressed in equation (3.18).

3.5 Chapter conclusions

This chapter argued for the explicit introduction of a third class of undecided clients into Deal's version of the classical Vidale-Wolfe model, and also into an extension of Lanchester's model, which includes both decay terms and the "spillover" effect of advertising on clients of the rival firm as well as on the undecided clients. The proposed modification of the Lanchester dynamics extends the classical model from the saturated market to the unsaturated market. A complete analysis of the location and stability properties of the equilibria of these two models under a general class of decentralized affine feedback advertising policies leads to the surprising conclusion that, despite differences in the trajectories, under identical advertising policies, the final outcome in terms of equilibrium market share is the same for both models. This is an indication of the fact that, even though Little's algebraic manipulation showed that the Lanchester model for two firms in a saturated market subsumes the single firm Vidale-Wolfe model, and this is no longer true in the duopoly case, there is still a deep similarity between the two models (same equilibrium outcome for same advertising policy).

Chapter 4

Vidale-Wolfe model and extended Lanchester model with delays under affine advertising control policy

In this chapter, the models studied in chapter 3 are described and analyzed considering the presence of delays. Section 4.1 presents an initial overview of the Vidale-Wolfe and Lanchester models considering delays. In section 4.2 and section 4.3 the two types of delays discussed in section 2.4 are incorporated into the Vidale-Wolfe and extended Lanchester models in order to perform stability and bifurcation analyses. Section 4.4 carries out a numercial study of the effect of different values of delay for each firm. Finally, section 4.5 presents the conclusions of the chapter.

4.1 Delays in Vidale-Wolfe and Lanchester models

The Vidale-Wolfe [9] and Lanchester [10] models have been widely studied in various approaches. However, to the best of our knowledge, all published models consider instant access to market information as well as the immediate effect of advertising on market shares. This assumption is not a realistic one, and consideration of delays in market behavior is necessary for a better understanding of market dynamics [75]. However, introduction of delays can induce important changes in the dynamics of systems, including oscillatory, unstable and chaotic behaviors [76]. The introduction of delays has been considered for some classes of duopoly models [63, 64, 77–81]. However, the class of market share models under advertising control policies has not been studied to date. The remainder of this chapter will motivate and describe the introduction of two types of delays into the Vidale-Wolfe and extended Lanchester models. Stability and bifurcation analyses of the proposed delay models finalize the chapter.

4.2 Vidale-Wolfe model with delays under affine advertising control policy

4.2.1 Vidale-Wolfe model with implementation delay

Recalling the definition in subsection 2.4.1 the Vidale-Wolfe model with implementation delay can be defined as follows:

$$\begin{aligned} \dot{x}_1 &= u_{1\tau}\left(1 - x_1 - x_2\right) - \lambda_1 x_1 \\ \dot{x}_2 &= u_{2\tau}\left(1 - x_1 - x_2\right) - \lambda_2 x_2 \end{aligned} \tag{4.1}$$

where the implementation delays affect the advertising policies (now denoted $u_{i\tau}, i = 1, 2$) as follows:
$u_{1\tau} = k_1 x_{1\tau} + c_1 = k_1 x_1\left(t - \tau_1\right) + c_1$
$u_{1\tau} = k_2 x_{2\tau} + c_2 = k_2 x_2\left(t - \tau_2\right) + c_2$
Substituting the above expressions into (4.1), the Vidale-Wolfe model with implementation delay can be expressed as:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 x_{1\tau} - k_1 x_2 x_{1\tau} - k_1 x_{1\tau} - c_1 x_1 - c_1 x_2 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_2 x_1 x_{2\tau} - k_2 x_2 x_{2\tau} - k_2 x_{2\tau} - c_2 x_1 - c_2 x_2 - \lambda x_2 + c_2 \end{aligned} \tag{4.2}$$

In this chapter, it will henceforth be assumed that implementation delays are equal (i.e., $\tau_1 = \tau_2 = \tau$). This assumption is mainly to make stability and bifurcation analysis possible. Section 4.4 relaxes this assumption and carries out a numerical study of the case of unequal implementation delays.
For model (4.2) when $\tau_1 = \tau_2 = \tau$, the Jacobian matrix [82] with respect to equilibrium point is given by:

$$J_{vw} = \begin{bmatrix} A_{vw} & B_{vw} \\ C_{vw} & D_{vw} \end{bmatrix} \tag{4.3}$$

where:
$A_{vw} = -k_1 x_{1\tau} - c_1 - \lambda + e^{-\psi\tau}\left(-k_1 x_1 - k_1 x_2 + k_1\right)$
$B_{vw} = -k_1 x_{1\tau} - c_1$
$C_{vw} = -k_2 x_{2\tau} - c_2$
$D_{vw} = -k_2 x_{2\tau} - c_2 - \lambda + e^{-\psi\tau}\left(-k_2 x_1 - k_2 x_2 + k_2\right)$

At equilibrium it must hold that:

$$(x_1^*(t), x_2^*(t)) = (x_1^*(t-\tau), x_2^*(t-\tau)) \tag{4.4}$$

Hence, the stability of equilibrium points is determined by the following characteristic equation:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{4.5}$$

where:
$P_{1vw} = k_1x_1 + k_2x_2 + c_1 + c_2 + 2\lambda$
$P_{2vw} = k_1\lambda x_1 + k_2\lambda x_2 + c_1\lambda + c_2\lambda + \lambda^2$
$P_{3vw} = k_1k_2x_1^2 + 2k_1k_2x_1x_2 + k_1k_2x_2^2 - 2k_1k_2x_1 - 2k_1k_2x_2 + k_1k_2$
$P_{4vw} = k_1x_1 + k_1x_2 + k2x1 + k2x2 - k1 - k2$
$P_{5vw} = k_1k_2x_1^2 + 2k_1k_2x_1x_2 + k_1k_2x_2^2 + c_1k_2x_1 + c_1k_2x_2 + c_2k_1x_1 + c_2k_1x_2 - k_1k_2x_1 - k_1k_2x_2 + k_1\lambda x_1 + k_1\lambda x_2 + k_2\lambda x_1 + k_2\lambda x_2 - c_1k_2 - c_2k_1 - k_1\lambda - k_2\lambda$
Considering the special case when: $k_1 = k_2 = k$ and $c_1 = c_2 = c$ we have:

$$x_1^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{c^2 + 2c\,(c+\lambda-k) + 8ck + (c+\lambda-k)^2}}{k} \tag{4.6}$$

$$x_2^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{c^2 + 2c\,(c+\lambda-k) + 8ck + (c+\lambda-k)^2}}{k} \tag{4.7}$$

Thus, in this case, the characteristic equation is given by:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{4.8}$$

where:
$P_{1vw} = c + \frac{3}{2}\lambda + \frac{1}{2}\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2} + \frac{k}{2}$
$P_{2vw} = \frac{1}{2}k\lambda + c\lambda + \frac{1}{2}\lambda^2 + \frac{1}{2}\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}$
$P_{3vw} = 2c^2 + 2c\lambda - \left(c - \frac{1}{2}\lambda - \frac{1}{2}k\right)\left(\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}\right) + \frac{1}{2}\lambda^2 + \frac{1}{2}k^2 + 2ck$
$P_{4vw} = \sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2} - 2c - k - \lambda$
$P_{5vw} = -c\lambda - \frac{3}{2}k\lambda - \frac{1}{2}\lambda^2 + \frac{1}{2}\lambda\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}$

4.2.1.1 Stability analysis of the characteristic equation for $\tau = 0$

In this case, the characteristic equation is given by:

$$P_{vw}(\psi, \tau) = \Psi^2 + P_{6vw}\Psi + P_{7vw} \tag{4.9}$$

where:
$P_{6vw} = P_{1vw} + P_{4vw}$
$P_{7vw} = P_{2vw} + P_{3vw} + P_{5vw}$
Then, the stability criteria [83] are satisfied whenever

$$\begin{aligned} P_{6vw} &> 0 \\ P_{7vw} &> 0 \end{aligned} \tag{4.10}$$

Therefore, we may formulate the following proposition:

Proposition 1. *The equilibrium point* (x_1^*, x_2^*) *of model* (4.2) *is a stable equilibrium point for* $\tau = 0$ *whenever the conditions in* (4.10) *hold.*

4.2.1.2 Stability analysis of the characteristic equation for $\tau > 0$

For convenience, we present the statement of the Hopf bifurcation theorem below, taken from [84], since this will be the main tool used for bifurcation analysis throughout this thesis.

Hopf bifurcation theorem[1]

Let $\dot{\mathbf{x}} = A(\lambda)\mathbf{x} + \mathbf{F}(\lambda, \mathbf{x})$ be a C^k, with $k \geqslant 3$, be a planar vector field depending on a scalar parameter λ such that $\mathbf{F}(\lambda, \mathbf{0}) = \mathbf{0}$ and $D_{\mathbf{x}}\mathbf{F}(\lambda, \mathbf{0}) = \mathbf{0}$ for all sufficiently small $|\lambda|$. Assume that the linear part $A(\lambda)$ at the origin has the eigenvalues $\alpha(\lambda) \pm i\beta(\lambda)$ with $\alpha(0) = 0$ and $\beta(0) \neq 0$. Furthermore, suppose that the eigenvalues cross the imaginary axis with nonzero speed, that is:

$$\frac{d\alpha}{d\lambda}(0) \neq 0 \tag{4.11}$$

Then, in any neighborhood U of the origin in $\mathbb{R}^2$ and any given $\lambda_0 > 0$ there is a $\overline{\lambda}$ with $|\overline{\lambda}| < \lambda_0$ such that the differential equation $\dot{\mathbf{x}} = A\left(\overline{\lambda}\right)\mathbf{x} + \mathbf{F}\left(\overline{\lambda}, \mathbf{x}\right)$ has a nontrivial periodic orbit in U.

[1] Also atributed to Poincaré and Andronov and called the Poincaré-Andronov-Hopf theorem.

Returning to the analysis when $\tau > 0$, in this case the characteristic equation is given by equation (4.5). Next, considering $\Psi = iw$ and substituting in (4.5), yields:

$$P_{vw}(i\omega,\tau) = (i\omega)^2 + P_{1vw}\,(i\omega) + P_{2vw} + P_{3vw}e^{-2(i\omega)\tau} + P_{4vw}(i\omega)e^{-(i\omega)\tau} + P_{5vw}e^{-(i\omega)\tau} \quad (4.12)$$

Then, separating the real and imaginary parts, we have:

$$P_{3vw}\cos(2\omega\tau) + \omega P_{4vw}\sin(\omega\tau) = \omega^2 - P_{2vw} - P_{5vw}\cos(\omega\tau) \quad (4.13)$$

$$-P_{3vw}\sin(2\omega\tau) + \omega P_{4vw}\cos(\omega\tau) = -P_{1vw}\omega + P_{5vw}\sin(\omega\tau) \quad (4.14)$$

Solving equations (4.13) and (4.14), we obtain:

$$\begin{aligned} 0 = &- P_{4vw}\omega^3 + P_{4vw}^2\sin(\omega\tau)\,\omega^2 + P_{4vw}P_{2vw}w + P_{4vw}P_{3vw}\cos(2\omega\tau)\,\omega \\ &- P_{1vw}P_{5vw}\omega + P_{5vw}^2\sin(\omega\tau) + P_{3vw}P_{5vw}\sin(2\omega\tau) \end{aligned} \quad (4.15)$$

Now, after rearrangement the characteristic equation becomes:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + e^{-\psi\tau}\left(P_{4vw}\Psi + P_{3vw}e^{-\psi\tau} + P_{5vw}\right) \quad (4.16)$$

Then, the second necessary condition for the existence of a Hopf Bifurcation [85] is formulated as:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \quad (4.17)$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (4.16) we get:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{E_{vw} + F_{vw}i}{G_{vw} + H_{vw}i} \quad (4.18)$$

where:
$E_{vw} = -w\left(4P_{3vw}\cos^2(\omega\tau) + P_{4vw}\omega\sin(\omega\tau) + P_{5vw}\cos(\omega\tau) - 2P_{3vw}\right)$
$F_{vw} = -w\left(4P_{3vw}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)\,P_{4vw}\omega + P_{5vw}\sin(\omega\tau)\right)$
$G_{vw} = -4P_{3vw}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4vw}\omega\cos(\omega\tau) - \tau P_{5vw}\sin(\omega\tau) + P_{4vw}\sin(\omega\tau) - 2w$
$H_{vw} = 4P_{3vw}\tau\cos^2(\omega\tau) + P_{4vw}\tau\omega\sin(\omega\tau) + P_{5vw}\tau\cos(\omega\tau) - P_{4vw}\cos(\omega\tau) - 2P_{3vw}\tau - P_{1vw}$
Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{E_{vw}G_{vw} + F_{vw}H_{vw}}{G_{vw}^2 + H_{vw}^2} \neq 0 \quad (4.19)$$

Hence, from the previous analysis, the following proposition can be formulated:

Proposition 2. *The model (4.2) has Hopf bifurcation for delay value $\tau > 0$ when the equation (4.15) has a positive solution and condition (4.19) holds.*

4.2.2 Numerical results for Vidale-Wolfe model with implementation delay

In this section some numerical results for the Vidale-Wolfe model with implementation delay are presented. The following parameter values are considered: $x_1(0) = 0.2$, $x_2(0) = 0.1$, $\lambda = 0.2$, $k_1 = 0.4$, $c_1 = 0.35$, $k_2 = 0.17$, $c_2 = 0.4$. For these parameters, the equilibrium point is given by: $x_1^* = 0.44$, $x_2^* = 0.39$

First, analyzing for $\tau = 0$, the parameter values in equation (4.9) are substituted. Thus,

$$P_{vw}(\psi, \tau) = \Psi^2 + 1.2954\Psi + 0.174 \tag{4.20}$$

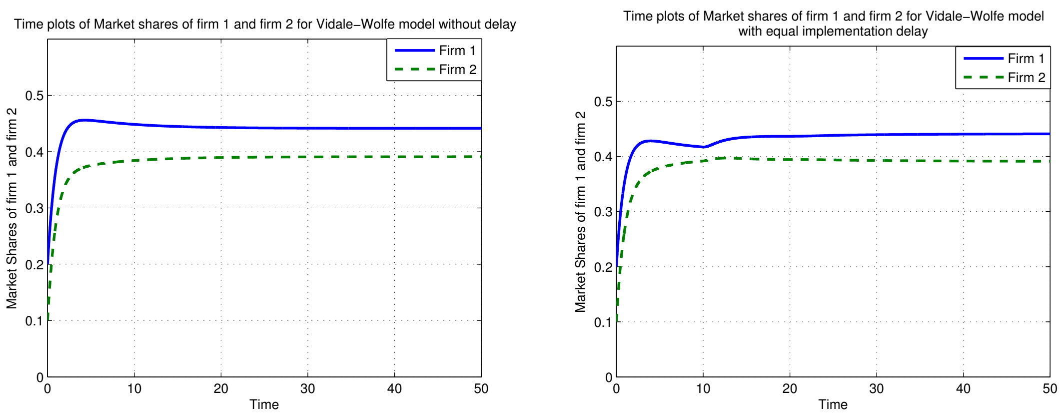
Hence, according to Proposition 1 it can be said that equilibrium point is stable.

Next, for the case when $\tau > 0$, the parameter values in equation (4.12) are replaced leading to

$$P_{vw}(\psi, \tau) = \Psi^2 + 1.39\Psi - 0.09\Psi e^{-\psi\tau} + 0.001e^{-2\psi\tau} - 0.06e^{-\psi\tau} + 0.238 \tag{4.21}$$

Then, solving the equation for $\lambda = i\omega$, it is noted that the characteristic equation has no positive root. Therefore, considering Proposition 2 it is concluded that the model has no Hopf bifurcation.

In figure 4.1 numerical simulations for the Vidale-Wolfe model with implementation delay are presented. Figure 4.1(a) shows the dynamics of the model without delay ($\tau = 0$). Next, figure 4.1(b) ilustrates the dynamics of the model for delay value $\tau = 10$. Note that the equilibrium point maintains its stability.



(a) (b)

Figure 4.1: (a) Vidale-Wolfe model without delay and (b) Vidale-Wolfe with implementation delay for $\tau_1 = \tau_2 = \tau = 10$.

4.2.3 Vidale-Wolfe model with adoption delay

From subsection 2.4.2, the Vidale-Wolfe model with adoption delay can be formulated as follows:

$$\begin{aligned}\dot{x}_1 &= u_{1\tau}\left(1 - x_{1\tau} - x_{2\tau}\right) - \lambda_1 x_1 \\ \dot{x}_2 &= u_{2\tau}\left(1 - x_{1\tau} - x_{2\tau}\right) - \lambda_2 x_2\end{aligned} \tag{4.22}$$

where:
$u_{1\tau} = k_1 x_{1\tau} + c_1 = k_1 x_1\left(t - \tau_1\right) + c_1$
$u_{2\tau} = k_2 x_{2\tau} + c_2 = k_2 x_2\left(t - \tau_2\right) + c_2$
Substituting the above expressions into (4.22), the Vidale-Wolfe model with adoption delay can be expressed by:

$$\begin{aligned}\dot{x}_1 &= -x_{1\tau}^2 k_1 - x_{1\tau}x_{2\tau}k_1 - x_{1\tau}c_1 + x_{1\tau}k_1 - x_{2\tau}c_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau}x_{2\tau}k_2 - x_{2\tau}^2 k_2 - x_{1\tau}c_2 - x_{2\tau}c_2 + x_{2\tau}k_2 - \lambda x_2 + c_2\end{aligned} \tag{4.23}$$

Considering that $\tau_1 = \tau_2 = \tau$, the Jacobian matrix with respect to equilibrium point for model (4.23) is given by:

$$J_{vw} = \begin{bmatrix} A_{vw} & B_{vw} \\ C_{vw} & D_{vw} \end{bmatrix} \tag{4.24}$$

where:
$A_{vw} = -\lambda + e^{-\psi\tau}\left(-2x_{1\tau}k_1 - x_{2\tau}k_1 - c_1 + k_1\right)$
$B_{vw} = e^{-\psi\tau}\left(-x_{1\tau}k_1 - c_1\right)$
$C_{vw} = e^{-\psi\tau}\left(-x_{2\tau}k_2 - c_2\right)$
$D_{vw} = -\lambda + e^{-\psi\tau}\left(-x_{1\tau}k_2 - 2x_{2\tau}k_2 - c_2 + k_2\right)$
At equilibrium it must hold that:

$$(x_1^*(t), x_2^*(t)) = (x_1^*(t-\tau), x_2^*(t-\tau)) \tag{4.25}$$

Therefore, the stability of equilibrium points will be determined by the characteristic equation expressed by:

$$P_{vw}(\psi, \tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{4.26}$$

where:
$P_{1vw} = 2\lambda^2$
$P_{2vw} = \lambda^2$
$P_{3vw} = 2k_1k_2x_1^2 + 4k_1k_2x_1x_2 + 2k_1k_2x_2^2 + c_1k_2x_1 + c_1k_2x_2 + c_2k_1x_1 + c_2k_1x_2 - 3k_1x_2 - 3k_1k_2x_2 - c_1k_2 - c_2k_1 + k_1k_2$
$P_{4vw} = 2k_1x_1 + k_1x_2 + k_2x_1 + 2k_2x_2 + c_1 + c_2 - k_1 - k_2$
$P_{5vw} = 2k_1\lambda x_1 + k_1\lambda x_2 + k_2\lambda x_1 + 2k_2\lambda x_2 + c_1\lambda + c_2\lambda - k_1\lambda - k_2\lambda$

Considering the special case when: $k_1 = k_2 = k$ and $c_1 = c_2 = c$ we obtain:

$$x_1^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{c^2 + 2c\left(c + \lambda - k\right) + 8ck + \left(c + \lambda - k\right)^2}}{k} \tag{4.27}$$

$$x_2^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{c^2 + 2c\left(c + \lambda - k\right) + 8ck + \left(c + \lambda - k\right)^2}}{k} \tag{4.28}$$

Therefore, the characteristic equation is given by:

$$P_{vw}(\psi, \tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{4.29}$$

where:
$P_{1vw} = 2\lambda^2$
$P_{2vw} = \lambda^2$
$P_{3vw} = -\left(c - \lambda - \frac{1}{2}k\right)\left(\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}\right) + 2c^2 + 3c\lambda + \lambda^2 - \frac{1}{2}k\lambda + \frac{1}{2}k^2 + 2ck$
$P_{4vw} = -c - \frac{3}{2} - \frac{1}{2}k + \frac{3}{2}\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}$
$P_{5vw} = -c\lambda - \frac{3}{2}\lambda^2 - \frac{1}{2}k\lambda + \frac{3}{2}\lambda\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}$

4.2.3.1 Stability analysis of the characteristic equation for $\tau = 0$

In this case, the characteristic equation is given by:

$$P_{vw}(\psi, \tau) = \Psi^2 + P_{6vw}\Psi + P_{7vw} \tag{4.30}$$

where:
$P_{6vw} = P_{1vw} + P_{4vw}$
$P_{7vw} = P_{2vw} + P_{3vw} + P_{5vw}$
Thus the stability is given when:

$$\begin{aligned} P_{6vw} &> 0 \\ P_{7vw} &> 0 \end{aligned} \tag{4.31}$$

Therefore, from the previous analysis we can conclude

Proposition 3. *The equilibrium point* (x_1^*, x_2^*) *of model* (4.23) *is a stable equilibrium point for* $\tau = 0$ *when the conditions expressed in* (4.31) *hold.*

4.2.3.2 Stability analysis of the characteristic equation for $\tau > 0$

In this case, the characteristic equation is given by equation (4.26). Now, considering $\Psi = iw$ and substituting in (4.26), we obtain:

$$P_{vw}(i\omega, \tau) = (i\omega)^2 + P_{1vw}(i\omega) + P_{2vw} + P_{3vw}e^{-2(i\omega)\tau} + P_{4vw}(i\omega)e^{-(i\omega)\tau} + P_{5vw}e^{-(i\omega)\tau} \tag{4.32}$$

Then, separating the real and imaginary parts, we have:

$$P_{3vw}\cos(2\omega\tau) + \omega P_{4vw}\sin(\omega\tau) = \omega^2 - P_{2vw} - P_{5vw}\cos(\omega\tau) \tag{4.33}$$

$$-P_{3vw}\sin(2\omega\tau) + \omega P_{4vw}\cos(\omega\tau) = -P_{1vw}\omega + P_{5vw}\sin(\omega\tau) \tag{4.34}$$

Solving equations (4.33) and (4.34), we get:

$$\begin{aligned}0 = &- P_{4vw}\omega^3 + P_{4vw}^2\sin(\omega\tau)\,\omega^2 + P_{4vw}P_{2vw}w + P_{4vw}P_{3vw}\cos(2\omega\tau)\,\omega \\ &- P_{1vw}P_{5vw}\omega + P_{5vw}^2\sin(\omega\tau) + P_{3vw}P_{5vw}\sin(2\omega\tau)\end{aligned} \tag{4.35}$$

After rearrangement the characteristic equation becomes:

$$P_{vw}(\psi,\tau) = \varPsi^2 + P_{1vw}\varPsi + P_{2vw} + e^{-\psi\tau}\left(P_{4vw}\varPsi + P_{3vw}e^{-\psi\tau} + P_{5vw}\right) \tag{4.36}$$

Then, the second necessary condition for Hopf Bifurcation existence [85] is formulated as:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \tag{4.37}$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (4.36) we obtain:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{vw} + B_{vw}i}{C_{vw} + D_{vw}i} \tag{4.38}$$

where:
$A_{vw} = -w\left(4P_{3vw}\cos^2(\omega\tau) + P_{4vw}\omega\sin(\omega\tau) + P_{5vw}\cos(\omega\tau) - 2P_{3vw}\right)$
$B_{vw} = -w\left(4P_{3vw}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)P_{4vw}\omega + P_{5vw}\sin(\omega\tau)\right)$
$C_{vw} = -4P_{3vw}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4vw}\omega\cos(\omega\tau) - \tau P_{5vw}\sin(\omega\tau) + P_{4vw}\sin(\omega\tau) - 2w$
$D_{vw} = 4P_{3vw}\tau\cos^2(\omega\tau) + P_{4vw}\tau\omega\sin(\omega\tau) + P_{5vw}\tau\cos(\omega\tau) - P_{4vw}\cos(\omega\tau) - 2P_{3vw}\tau - P_{1vw}$
Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{vw}C_{vw} + B_{vw}D_{vw}}{C_{vw}^2 + D_{vw}^2} \neq 0 \tag{4.39}$$

Thus, from the previous analysis, we have:

Proposition 4. *The model* (4.23) *has Hopf bifurcation for delay value* $\tau > 0$ *when the equation* (4.35) *has a positive solution and condition* (4.39) *holds.*

4.2.4 Numerical results for Vidale-Wolfe model with adoption delay

Some numerical results for the Vidale-Wolfe model with adoption delay are now presented. The following parameter values are considered: $x_1(0) = 0.2$, $x_2(0) = 0.1$, $\lambda = 0.25$, $k_1 = 0.25$, $c_1 = 0.15$, $k_2 = 0.2$ and $c_2 = 0.1$. For these parameters, the equilibrium point is given by: $x_1^* = 0.383$, $x_2^* = 0.226$.

Now, analyzing for $\tau = 0$, we substitute the parameter values in equation (4.30) and we get:

$$P_{vw}(\psi, \tau) = \Psi^2 + 0.71\Psi + 0.09 \tag{4.40}$$

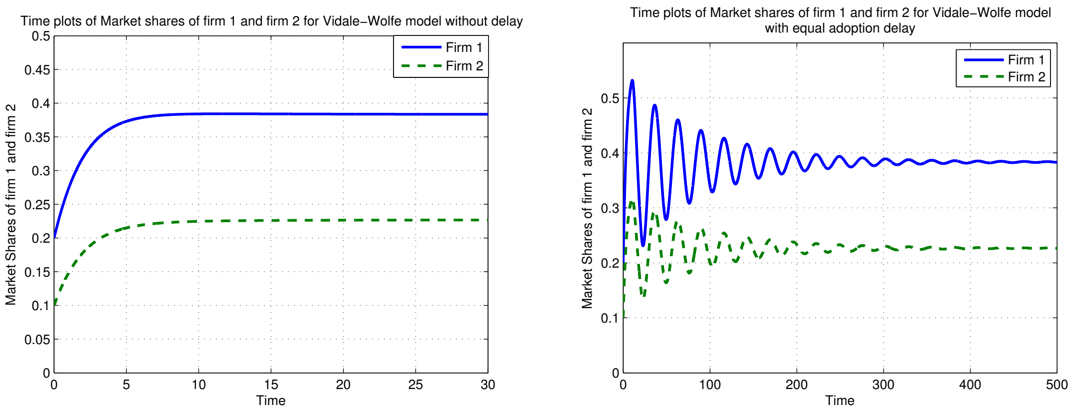
Hence, from Proposition 3 it can be affirmed that equilibrium point is stable.

Then, for the case when $\tau > 0$, substituting the parameter values in equation (4.32) we obtain:

$$P_{vw}(\psi, \tau) = \Psi^2 + 0.5\Psi + 0.21\Psi e^{-\psi\tau} - 0.02e^{-2\psi\tau} + 0.05e^{-\psi\tau} + 0.06 \tag{4.41}$$

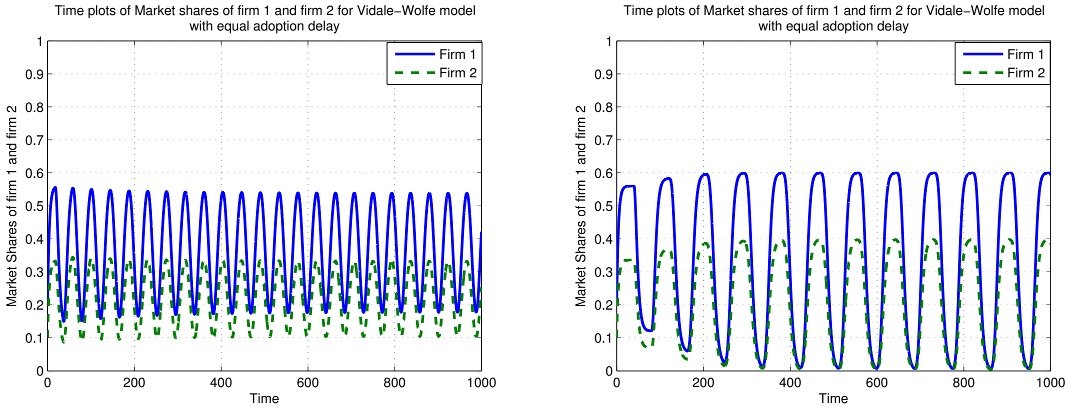
Solving the equation for $\lambda = i\omega$ we have that one solution is given by $\omega = 0.17$ and $\tau = 15.28$. Now, substituting these values in equation (4.39) we have that $\left(\frac{d\lambda}{d\tau}\right) \neq 0$. Therefore, according to Proposition 4 it can be stated that the model has Hopf bifurcation.

Figures 4.2 and 4.3 illustrate the numerical results for the Vidale-Wolfe model with adoption delay. Thus, in figure 4.2(a) the dynamics of the model without delay ($\tau = 0$) is shown. Then, figure 4.2(b) shows the dynamics of the model for $\tau = 10$. Note that in this case, the dynamics of the model has oscillations but the equilibrium point remains stable. Figure 4.3(a) illustrates the dynamics of the model for $\tau = 15.28$. This delay value is the critical delay value (τ_c) that produces existence of Hopf bifurcation. Finally, figure 4.3(b) shows the dynamics of the model for $\tau = 40$. Notice that the model for this delay value has oscillations and the equilibrium point loses stability.



(a) (b)

Figure 4.2: Evolution of market shares of firms x_1 and x_2 for: (a) Vidale Wolfe model without delay $\tau = 0$ and (b) Vidale Wolfe model with adoption delay $\tau = 10$



(a) (b)

Figure 4.3: Evolution of market shares of firms x_1 and x_2 for: (a) Vidale Wolfe model with adoption delay $\tau_c = 15.28$ and (b) Vidale Wolfe model with adoption delay $\tau = 40$

4.3 Extended Lanchester model with delays under affine advertising control policy

4.3.1 Extended Lanchester model with implementation delay

Once again, considering the analysis presented in the previous chapter and subsection 2.4.1, the extended Lanchester model with implementation delay can be expressed by:

$$\begin{aligned} \dot{x}_1 &= u_{1\tau}(1 - x_1) - u_{2\tau}x_1 - \lambda_1 x_1 \\ \dot{x}_2 &= u_{2\tau}(1 - x_2) - u_{1\tau}x_2 - \lambda_2 x_2 \end{aligned} \tag{4.42}$$

where:
$u_{1\tau} = k_1 x_{1\tau} + c_1 = k_1 x_1 (t - \tau_1) + c_1$
$u_{2\tau} = k_2 x_{2\tau} + c_2 = k_2 x_2 (t - \tau_2) + c_2$
Substituting the above expressions into model (4.42) we obtain:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 x_{1\tau} - k_2 x_1 x_{2\tau} + k_1 x_{1\tau} - c_1 x_1 - c_2 x_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_1 x_2 x_{1\tau} - k_2 x_2 x_{2\tau} + k_2 x_{2\tau} - c_1 x_2 - c_2 x_2 - \lambda x_2 + c_2 \end{aligned} \tag{4.43}$$

Considering that $\tau_1 = \tau_2 = \tau$, the Jacobian matrix for model (4.43) is given by:

$$J_{el} = \begin{bmatrix} A_{el} & B_{el} \\ C_{el} & D_{el} \end{bmatrix} \tag{4.44}$$

where:
$A_{el} = -k_1 x_{1\tau} - k_2 x_{2\tau} - c_1 - c_2 - \lambda + e^{-\psi\tau}(-k_1 x_1 + k_1)$

$B_{el} = -e^{-\psi\tau}\left(k_2 x_1\right)$
$C_{el} = -e^{-\psi\tau}\left(k_1 x_2\right)$
$D_{el} = -k_1 x_{1\tau} - k_2 x_{2\tau} - c_1 - c_2 - \lambda + e^{-\psi\tau}\left(-k_2 x_2 + k_2\right)$
Given the conditions of equilibrium points, we get:

$$(x_1^*(t), x_2^*(t)) = (x_1^*(t-\tau), x_2^*(t-\tau)) \tag{4.45}$$

Therefore, the stability of equilibrium points will be determined by the characteristic equation expressed by:

$$\Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{4.46}$$

where:
$P_{1vw} = 2k_1x_1 + 2k_2x_2 + 2c_1 + 2c_2 + 2\lambda$
$P_{2vw} = k_1^2x_1^2 + 2k_1k_2x_1x_2 + k_2^2x_2^2 + 2c_1k_1x_1 + 2c_1k_2x_2 + 2c_2k_1x_1 + 2c_2k_2x_2 + 2k_1\lambda x_1 + 2k_2\lambda x_2 + c_1^2 + 2c_1c_2 + 2c_1\lambda + c_2^2 + 2c_2\lambda + \lambda^2$
$P_{3vw} = -k_1k_2x_1 - k_1k_2x_2 + k_1k_2$
$P_{4vw} = k_1x_1 + k_2x_2 - k_1 - k_2$
$P_{5vw} = k_1x_1^2 + 2k_1k_2x_1x_2 + k_2x_2^2 + c_1k_1x_1 + c_1k_2x_2 + c_2k_1x_1 + c_2k_2x_2 - k_1^2x_1 - k_1k_2x_1 - k_1k_2x_2 + k_1\lambda x_1 - k_2^2x_2 + k_2\lambda x_2 - c_1k_1 - c_1k_2 - c_2k_1 - c_2k_2 - k_1\lambda - k_2\lambda$

Considering the special case when: $k_1 = k_2 = k$ and $c_1 = c_2 = c$ we have:

$$x_1^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{c^2 + 2c\left(c + \lambda - k\right) + 8ck + \left(c + \lambda - k\right)^2}}{k} \tag{4.47}$$

$$x_2^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{c^2 + 2c\left(c + \lambda - k\right) + 8ck + \left(c + \lambda - k\right)^2}}{k} \tag{4.48}$$

In this case, the characteristic equation is given by:

$$P_{el}(\psi,\tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + P_{3el}e^{-2\psi\tau} + P_{4el}\Psi e^{-\psi\tau} + P_{5el}e^{-\psi\tau} \tag{4.49}$$

where:
$P_{1el} = 2c + k + \lambda + \sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2} +$
$P_{2el} = 2c\lambda + \frac{1}{2}\lambda^2 + \frac{1}{2}k^2 + 2c^2 + 2ck + \left(c + \frac{1}{2}\lambda + \frac{1}{2}k\right)\left(\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}\right)$
$P_{3el} = ck + \frac{1}{2}k\lambda + \frac{1}{2}k^2 - \frac{1}{2}k\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}$
$P_{4el} = -c - \frac{1}{2}\lambda - \frac{3}{2}k + \frac{1}{2}\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}$
$P_{5el} = -\frac{1}{2}k^2 - ck - \frac{3}{2}k\lambda - \frac{1}{2}k\sqrt{4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2}$

4.3.1.1 Stability analysis of the characteristic equation for $\tau = 0$

In this case, the characteristic equation is given by:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{6el}\Psi + P_{7el} \tag{4.50}$$

where:
$P_{6el} = P_{1el} + P_{4el}$
$P_{7el} = P_{2el} + P_{3el} + P_{5el}$
Thus, stability conditions are:

$$\begin{aligned} P_{6el} &> 0 \\ P_{7el} &> 0 \end{aligned} \tag{4.51}$$

From the previous analysis, it can be concluded that:

Proposition 5. *The equilibrium point (x_1^*, x_2^*) of model (4.43) is a stable equilibrium point for delay value $\tau = 0$ when the conditions expressed in (4.51) hold.*

4.3.1.2 Stability analysis of the characteristic equation for $\tau > 0$

In this case, the characteristic equation is given by equation (4.49). Now, considering $\Psi = iw$ and substituting in (4.49), we obtain:

$$P_{el}(i\omega, \tau) = (i\omega)^2 + P_{el}\,(i\omega) + P_{2el} + P_{3el}e^{-2(i\omega)\tau} + P_{4el}(i\omega)e^{-(i\omega)\tau} + P_{5el}e^{-(i\omega)\tau} \tag{4.52}$$

Solving the equation and next separating the real and imaginary parts, we get:

$$P_{3el}\cos(2\omega\tau) + \omega P_{4el}\sin(\omega\tau) = \omega^2 - P_{2el} - P_{5el}\cos(\omega\tau) \tag{4.53}$$

$$-P_{3el}\sin(2\omega\tau) + \omega P_{4el}\cos(\omega\tau) = -P_{1el}\omega + P_{5el}\sin(\omega\tau) \tag{4.54}$$

Solving equations (4.53) and (4.54), we have:

$$\begin{aligned} 0 = &- P_{4el}\omega^3 + P_{4el}^2\sin(\omega\tau)\,\omega^2 + P_{4el}P_{2el}\omega + P_{4el}P_{3el}\cos(2\omega\tau)\,\omega \\ &- P_{1el}P_{5el}\omega + P_{5el}^2\sin(\omega\tau) + P_{3el}P_{5el}\sin(2\omega\tau) \end{aligned} \tag{4.55}$$

Now, rearranging the characteristic equation becomes:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + e^{-\psi\tau}\left(P_{4el}\Psi + P_{3el}e^{-\psi\tau} + P_{5el}\right) \tag{4.56}$$

Then, the second necessary condition for Hopf Bifurcation existence [85] is formulated as:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \tag{4.57}$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (4.56) we get:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{el} + B_{el}i}{C_{el} + D_{el}i} \tag{4.58}$$

where:
$A_{el} = -w\left(4P_{3el}\cos^2(\omega\tau) + P_{4el}\omega\sin(\omega\tau) + P_{5el}\cos(\omega\tau) - 2P_{3el}\right)$
$B_{el} = -w\left(4P_{3el}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)P_{4el}\omega + P_{5el}\sin(\omega\tau)\right)$
$C_{el} = -4P_{3el}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4el}\omega\cos(\omega\tau) - \tau P_{5el}\sin(\omega\tau) + P_{4el}\sin(\omega\tau) - 2w$
$D_{el} = 4P_{3el}\tau\cos^2(\omega\tau) + P_{4el}\tau\omega\sin(\omega\tau) + P_{5el}\tau\cos(\omega\tau) - P_{4el}\cos(\omega\tau) - 2P_{3el}\tau - P_{1el}$
Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{el}C_{el} + B_{el}D_{el}}{C_{el}^2 + D_{el}^2} \neq 0 \tag{4.59}$$

From the previous analysis it can be concluded that:

Proposition 6. *Model* (4.43) *has Hopf bifurcation for delay value $\tau > 0$ when equation* (4.55) *has a positive solution and condition* (4.59) *holds.*

4.3.2 Numerical results for extended Lanchester model with implementation delay

In this section, some numerical results for the extended Lanchester model with implementation delay are presented. The parameter values considered are similar to the Vidale-Wolfe model, that is: $x_1(0) = 0.2$, $x_2(0) = 0.1$, $\lambda = 0.2$, $k_1 = 0.4$, $c_1 = 0.35$, $k_2 = 0.17$, $c_2 = 0.4$. The equilibrium point for these parameters results: $x_1^* = 0.44$, $x_2^* = 0.39$.

Now, analyzing for $\tau = 0$, the parameter values in equation (4.50) are substituted and it is obtained:

$$P_{el}(\psi, \tau) = \Psi^2 + 2.0569\Psi + 1.042 \tag{4.60}$$

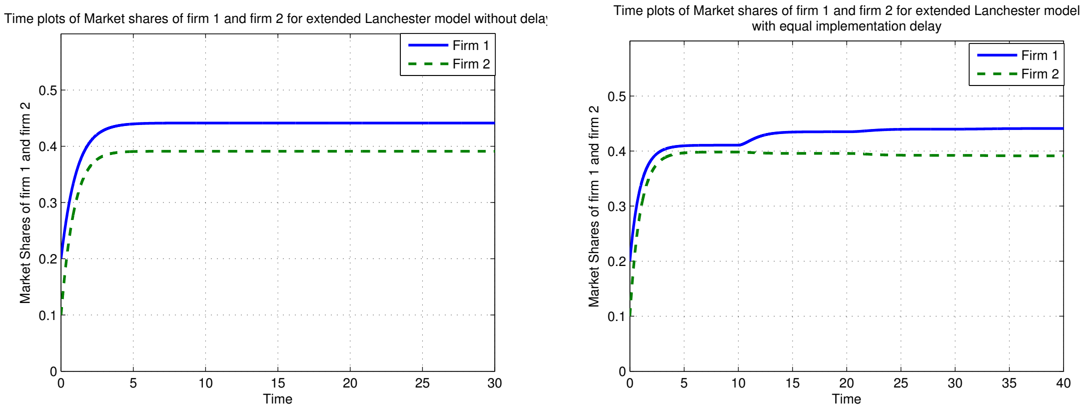
Taking into account Proposition 5 it can be said that equilibrium point is stable.

Then, when $\tau > 0$, the parameter values in equation (4.52) are replaced. Hence we have:

$$P_{el}(\psi, \tau) = \Psi^2 + 2.38\Psi - 0.32\Psi e^{-\psi\tau} + 0.01e^{-2\psi\tau} - 0.39e^{-\psi\tau} + 1.42 \tag{4.61}$$

Next, solving the equation for $\lambda = i\omega$ we find that the characteristic equation has no positive root. Therefore, considering Proposition 6 it is concluded that the model has no Hopf bifurcation.

In figure 4.4 numerical simulations for the extended Lanchester model with implementation delay are shown. Figure 4.4(a) illustrates the dynamics of the model without delay ($\tau = 0$). Figure 4.4(b) shows the dynamics of the model for delay value $\tau = 10$. Notice that the equilibrium point maintains its stability.



(a) (b)

Figure 4.4: Evolution of market shares of firms x_1 and x_2 for:(a)extended Lanchester model without delay ($\tau = 0$) and (b) extended Lanchester model with implementation delay $\tau = 10$

4.3.3 Extended Lanchester model with adoption delay

From subsection 2.4.2, the extended Lanchester model with adoption delay can be formulated as follows:

$$\begin{aligned} \dot{x}_1 &= u_{1\tau}\left(1 - x_{1\tau}\right) - u_{2\tau}x_{1\tau} - \lambda_1 x_1 \\ \dot{x}_2 &= u_{2\tau}\left(1 - x_{2\tau}\right) - u_{1\tau}x_{2\tau} - \lambda_2 x_2 \end{aligned} \tag{4.62}$$

where:
$u_{1\tau} = k_1 x_{1\tau} + c_1 = k_1 x_1\left(t - \tau_1\right) + c_1$
$u_{2\tau} = k_2 x_{2\tau} + c_2 = k_2 x_2\left(t - \tau_2\right) + c_2$
Substituting the above expressions, model (4.62) can be expressed by:

$$\begin{aligned} \dot{x}_1 &= -x_{1\tau}^2 k_1 - x_{1\tau}x_{2\tau}k_2 - x_{1\tau}c_1 - x_{1\tau}c_2 + x_{1\tau}k_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau}x_{2\tau}k_1 - x_{2\tau}^2 k_2 - x_{2\tau}c_1 - x_{2\tau}c_2 + x_{2\tau}k_2 - \lambda x_2 + c_2 \end{aligned} \tag{4.63}$$

Once again, considering that $\tau_1 = \tau_2 = \tau$, the Jacobian matrix for model (4.63) is given by:

$$J_{el} = \begin{bmatrix} A_{el} & B_{el} \\ C_{el} & D_{el} \end{bmatrix} \tag{4.64}$$

where:
$A_{el} = -\lambda + e^{-\psi\tau}\left(-2x_{1\tau}k_1 - x_{2\tau}k_2 - c_1 - c_2 + k_1\right)$
$B_{el} = -e^{-\psi\tau}x_{1\tau}k_2$
$C_{el} = -e^{-\psi\tau}x_{2\tau}k_1$
$D_{el} = -\lambda + e^{-\psi\tau}\left(2x_{1\tau}k_1 - 2x_{2\tau}k_2 - c_1 - c_2 + k_2\right)$

At equilibrium it must hold that:

$$(x_1^*(t), x_2^*(t)) = (x_1^*(t-\tau), x_2^*(t-\tau)) \tag{4.65}$$

Therefore, the stability of equilibrium points will be determined by the characteristic equation expressed by:

$$P_{el}(\psi,\tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + P_{3vw}e^{-2\psi\tau} + P_{4el}\Psi e^{-\psi\tau} + P_{5el}e^{-\psi\tau} \tag{4.66}$$

where:
$P_{1el} = 2\lambda$
$P_{2el} = \lambda^2$
$P_{3el} = 2k_1^2x_1^2 + 4k_1k_2x_1x_2 + 2k_2x_2^2 + 3c_1k_1x_1 + 3c_1k_2x_2 + 3c_2k_1x_1 + 3c_2k_2x_2 - k_1^2x_1 - 2k_1k_2x_1 - 2k_1k_2x_2 - k_2^2x_2 + c_1^2 + 2c_1c_2 - c_1k_1 - c_1k_2 + c_2^2 - c_2k_1 - c_2k_2 + k_1k_2$
$P_{4el} = 3k_1x_1 + 3k_2x_2 + 2c_1 + 2c_2 - k_1 - k_2$
$P_{5el} = 3k_1\lambda x_1 + 3k_2\lambda x_2 + 2c_1\lambda + 2c_2\lambda - k_1\lambda - k_2\lambda$
Considering the special case when: $k_1 = k_2 = k$ and $c_1 = c_2 = c$ we obtain:

$$x_1^* = -\frac{1}{4}\frac{-2c-\lambda+k+\sqrt{c^2+2c\,(c+\lambda-k)+8ck+(c+\lambda-k)^2}}{k} \tag{4.67}$$

$$x_2^* = -\frac{1}{4}\frac{-2c-\lambda+k+\sqrt{c^2+2c\,(c+\lambda-k)+8ck+(c+\lambda-k)^2}}{k} \tag{4.68}$$

In this case, the characteristic equation is given by:

$$P_{el}(\psi,\tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + P_{3el}e^{-2\psi\tau} + P_{4el}\Psi e^{-\psi\tau} + P_{5el}e^{-\psi\tau} \tag{4.69}$$

$P_{1el} = 2\lambda$
$P_{2el} = \lambda^2$
$P_{3el} = -\left(c-\lambda-\frac{1}{2}k\right)\left(\sqrt{4c^2+4ck+4c\lambda+k^2-2k\lambda+\lambda^2}\right)+2c^2+c\lambda+\lambda^2-\frac{1}{2}k\lambda+\frac{1}{2}k^2+2ck$
$P_{4el} = c-\frac{3}{2}\lambda-\frac{1}{2}k+\frac{3}{2}\sqrt{4c^2+4ck+4c\lambda+k^2-2k\lambda+\lambda^2}$
$P_{5el} = c\lambda-\frac{3}{2}\lambda^2-\frac{1}{2}k\lambda+\frac{3}{2}\lambda\sqrt{4c^2+4ck+4c\lambda+k^2-2k\lambda+\lambda^2}$

4.3.3.1 Stability analysis of the characteristic equation for $\tau = 0$

In this case, the characteristic equation is given by:

$$P_{el}(\psi,\tau) = \Psi^2 + P_{6el}\Psi + P_{7el} \tag{4.70}$$

where:
$P_{6el} = P_{1el} + P_{4el}$
$P_{7el} = P_{2el} + P_{3el} + P_{5el}$

Thus the stability is established when:

$$\begin{aligned} P_{6el} &> 0 \\ P_{7el} &> 0 \end{aligned} \tag{4.71}$$

From the previous analysis, it can be inferred

Proposition 7. *The equilibrium point* (x_1^*, x_2^*) *of model* (4.63) *is a stable equilibrium point for delay value* $\tau = 0$ *when the conditions expressed in* (4.71) *hold.*

4.3.3.2 Stability analysis of the characteristic equation for $\tau > 0$

In this case, the characteristic equation is given by equation (4.69). Now, considering $\Psi = iw$ and substituting in (4.69), we get:

$$P_{el}(i\omega, \tau) = (i\omega)^2 + P_{1el}\,(i\omega) + P_{2el} + P_{3el}e^{-2(i\omega)\tau} + P_{4el}(i\omega)e^{-(i\omega)\tau} + P_{5el}e^{-(i\omega)\tau} \tag{4.72}$$

Then, separating the real and imaginary parts, we have:

$$P_{3el}\cos(2\omega\tau) + \omega P_{4el}\sin(\omega\tau) = \omega^2 - P_{2el} - P_{5el}\cos(\omega\tau) \tag{4.73}$$

$$-P_{3el}\sin(2\omega\tau) + \omega P_{4el}\cos(\omega\tau) = -P_{1el}\omega + P_{5el}\sin(\omega\tau) \tag{4.74}$$

Solving equations (4.73) and (4.74), we obtain:

$$\begin{aligned} 0 = &- P_{4el}\omega^3 + P_{4el}^2\sin(\omega\tau)\,\omega^2 + P_{4el}P_{2el}\omega + P_{4el}P_{3el}\cos(2\omega\tau)\,\omega \\ &- P_{1el}P_{5el}\omega + P_{5el}^2\sin(\omega\tau) + P_{3el}P_{5el}\sin(2\omega\tau) \end{aligned} \tag{4.75}$$

Now, rearranging the characteristic equation we get:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + e^{-\psi\tau}\left(P_{4el}\Psi + P_{3el}e^{-\psi\tau} + P_{5el}\right) \tag{4.76}$$

Then, the second necessary condition for Hopf Bifurcation existence [85] is formulated as:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \tag{4.77}$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (4.76) we obtain:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{el} + B_{el}i}{C_{el} + D_{el}i} \tag{4.78}$$

where:

$A_{el} = -w\left(4P_{3el}\cos^2(\omega\tau) + P_{4el}\omega\sin(\omega\tau) + P_{5el}\cos(\omega\tau) - 2P_{3el}\right)$

$B_{el} = -w\left(4P_{3el}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)P_{4el}\omega + P_{5el}\sin(\omega\tau)\right)$

$C_{el} = -4P_{3el}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4el}\omega\cos(\omega\tau) - \tau P_{5el}\sin(\omega\tau) + P_{4el}\sin(\omega\tau) - 2w$

$D_{el} = 4P_{3el}\tau\cos^2(\omega\tau) + P_{4el}\tau\omega\sin(\omega\tau) + P_{5el}\tau\cos(\omega\tau) - P_{4el}\cos(\omega\tau) - 2P_{3el}\tau - P_{1el}$

Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{el}C_{el} + B_{el}D_{el}}{C_{el}^2 + D_{el}^2} \neq 0 \tag{4.79}$$

From the previous analysis, it can be concluded that

Proposition 8. *Model* (4.63) *has Hopf bifurcation for delay value* $\tau > 0$ *when equation* (4.75) *has a positive solution and condition* (4.79) *holds.*

Remark: Since the existence and the value of ω depends on the delay value τ and the control parameters (k_i, c_i) there is also the possibility of considering the control parameters (k_i, c_i) as bifuraction parameters. In appendix D we present numerical results for a particular case where c_1 is considered to be the bifurcation parameter.

4.3.4 Numerical results for extended Lanchester model with adoption delay

Similiar to the previous section, numerical results for the extended Lanchester model with adoption delay are now presented. Once again, the parameter values considered are similar to Vidale-Wolfe model: $x_1(0) = 0.2$, $x_2(0) = 0.1$, $\lambda = 0.25$, $k_1 = 0.25$, $c_1 = 0.15$, $k_2 = 0.2$, $c_2 = 0.1$. For these parameters the equilibrium point is given by: $x_1^* = 0.383$, $x_2^* = 0.226$.

First, analyzing for $\tau = 0$, the parameter values of simulation in equation (4.70) are substituted and it is obtained that:

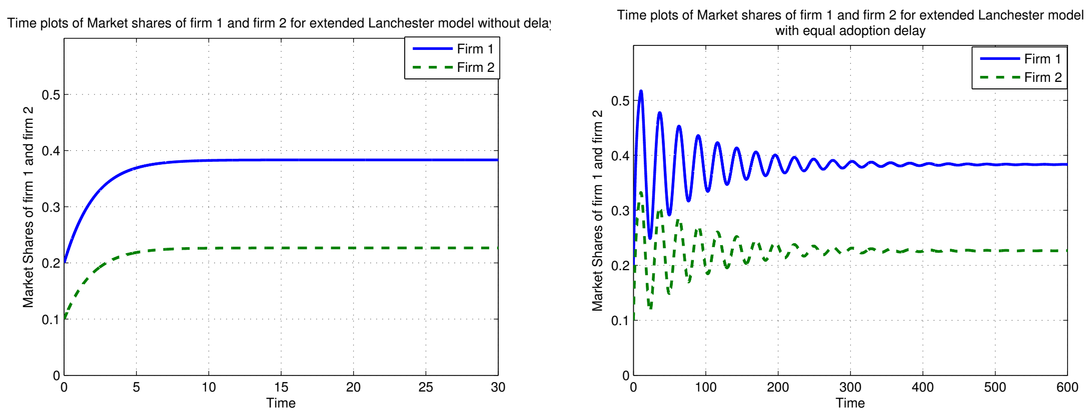
$$P_{vw}(\psi, \tau) = \Psi^2 + 0.97\Psi + 0.23 \tag{4.80}$$

Then, taking into account Proposition 7 it is concluded that equilibrium point is stable. Next, analyzing when $\tau > 0$, substituting the parameter values in equation (4.72) we have:

$$P_{vw}(\psi, \tau) = \Psi^2 + 0.5\Psi - 0.47\Psi e^{-\psi\tau} + 0.005e^{-2\psi\tau} - 0.11e^{-\psi\tau} + 0.06 \tag{4.81}$$

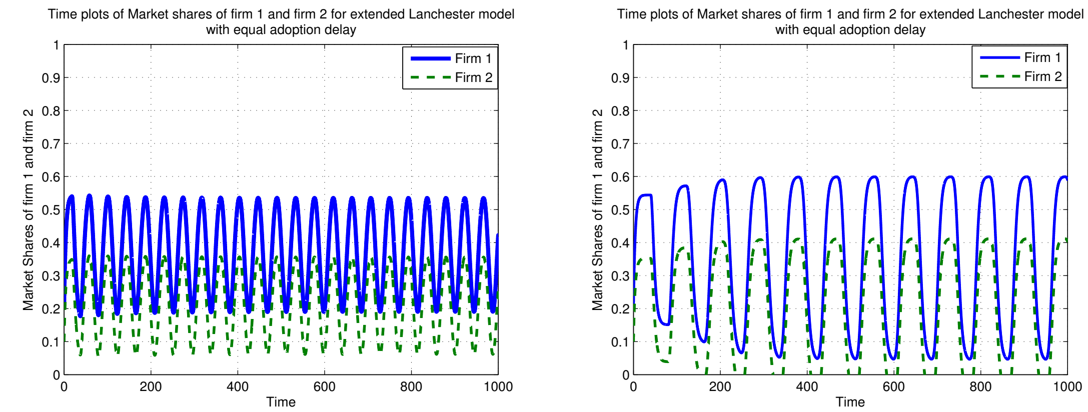
Then, solving equation (4.81) for $\lambda = i\omega$ we have that one solution is given by $\omega = 0.17$ and $\tau = 14.98$. Now, substituting these values in equation (4.79) we have that $\left(\frac{d\lambda}{d\tau}\right) \neq 0$. Therefore, according to Proposition 8 it can be affirmed that the model has Hopf bifurcation.

Figures 4.5 and 4.6 illustrate the numerical results for the extended Lanchester model with adoption delay. Thus, in figure 4.5(a) the dynamics of the model without delay ($\tau = 0$) is shown. Next, figure 4.5(b) illustrates the dynamics of the model for $\tau = 10$. Notice that the dynamics of the model has oscillations however the equilibrium point prevails stable. Figure 4.6(a) shows the dynamics of the model for $\tau = 15.5$. This delay value is the critical delay value (τ_c) which produces existence of Hopf bifurcation. Then, figure 4.6(b) illustrates the dynamics of the model for $\tau = 40$. Note that the model has oscillations and the equilibrium point becomes unstable.



(a) (b)

Figure 4.5: Evolution of market shares of firms x_1 and x_2 for: (a) Extended Lanchester model without delay $\tau = 0$ and (b) Extended Lanchester model with adoption delay $\tau = 10$



(a) (b)

Figure 4.6: Evolution of market shares of firms x_1 and x_2 for: (a) Extended Lanchester model with adoption delay $\tau_c = 14.98$ and (b) Extended Lanchester model with adoption delay $\tau = 40$

4.4 Vidale-Wolfe model and extended Lanchester model with unequal delays under affine advertising control policy

In the previous sections, the Vidale-Wolfe and extended Lanchester models were analyzed considering the existence of two types of delays (implementation and adoption). In order to find analytical solutions equal delay values for both firms ($\tau_1 = \tau_2$) were considered, however, this assumption is often not true in real situations. Thus, in this section, some numerical simulations are shown with unequal delay values for each firm ($\tau_1 \neq \tau_2$).

4.4.1 Vidale-Wolfe model with unequal delay values under affine advertising control policy

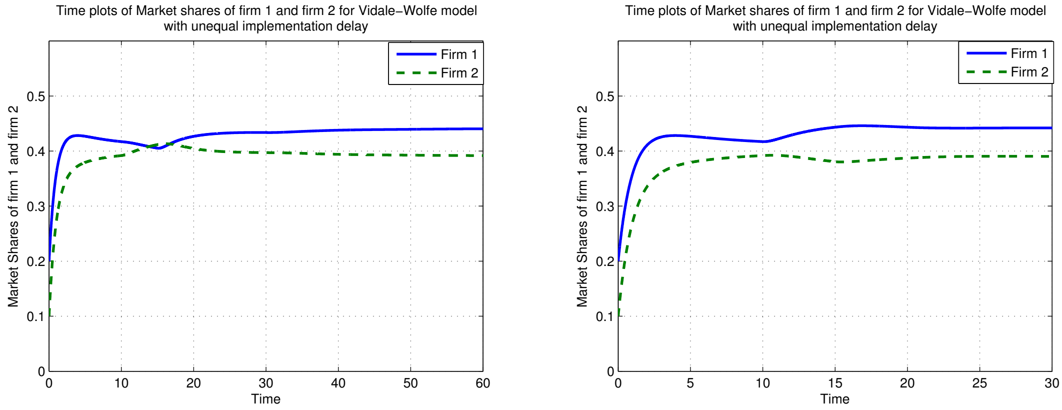
Rewriting the equation (4.2) for the Vidale-Wolfe model with unequal implementation delays and defining $x_{1\tau}$ and $x_{2\tau}$ as follows:
$x_{1\tau} = x_1(t-\tau_1)$
$x_{2\tau} = x_2(t-\tau_2)$
we have:

$$\begin{aligned} \dot{x}_1 &= -k_1x_1x_{1\tau} - k_1x_2x_{1\tau} - k_1x_{1\tau} - c_1x_1 - c_1x_2 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_2x_1x_{2\tau} - k_2x_2x_{2\tau} - k_2x_{2\tau} - c_2x_1 - c_2x_2 - \lambda x_2 + c_2 \end{aligned} \tag{4.82}$$

Likewise, rewriting the equation (4.23) for the Vidale-Wolfe model with unequal adoption delays we obtain:

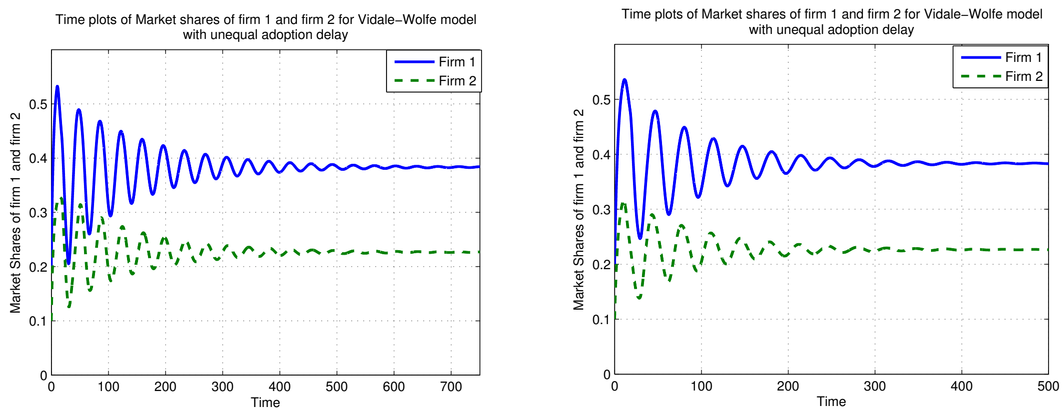
$$\begin{aligned} \dot{x}_1 &= -x_{1\tau}^2k_1 - x_{1\tau}x_{2\tau}k_1 - x_{1\tau}c_1 + x_{1\tau}k_1 - x_{2\tau}c_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau}x_{2\tau}k_2 - x_{2\tau}^2k_2 - x_{1\tau}c_2 - x_{2\tau}c_2 + x_{2\tau}k_2 - \lambda x_2 + c_2 \end{aligned} \tag{4.83}$$

Assume the same parameters as in section 4.2 and unequal delay values ($\tau_1 \neq \tau_2$). Figures 4.7 and 4.8 show the numerical results for the Vidale-Wolfe model with implementation and adoption delay respectively but with different delay values for each firm. Figure 4.7(a) illustrates the dynamics of the model with implementation delay considering $\tau_1 > \tau_2$. In figure 4.7(b) the dynamics of the model when $\tau_1 < \tau_2$ is presented. Note that in both cases the equilibrium point remains stable and for $\tau_1 > \tau_2$ the dynamics has a crossing in time plots of market share trajectories. Figure 4.8(a) shows the dynamics of the model with adoption delay when $\tau_1 > \tau_2$. Finally, in figure 4.8(b) the dynamic of the model for $\tau_1 < \tau_2$ is ilustrated. Note that both models exhibit oscillations, but in both cases, the equilibrium points remain stable.



(a) (b)

Figure 4.7: Evolution of market shares of firms x_1 and x_2 for: (a) Vidale-Wolfe model with implementation delay $\tau_1 = 15$, $\tau_2 = 10$ (b) Vidale-Wolfe model with implementation delay $\tau_1 = 10$ and $\tau_2 = 15$



(a) (b)

Figure 4.8: Evolution of market shares of firms x_1 and x_2 for: (a) Vidale-Wolfe model with adoption delay for $\tau_1 = 18$ and $\tau_2 = 10$ (b) Vidale-Wolfe model with adoption delay for $\tau_1 = 10$ and $\tau_2 = 18$

4.4.2 Extended Lanchester model unequal delay values under affine advertising control policy

Similar to the previous subsection, equation (4.43) which represents the extended Lanchester model with implementation delay is initially rewritten:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 x_{1\tau} - k_2 x_1 x_{2\tau} + k_1 x_{1\tau} - c_1 x_1 - c_2 x_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_1 x_2 x_{1\tau} - k_2 x_2 x_{2\tau} + k_2 x_{2\tau} - c_1 x_2 - c_2 x_2 - \lambda x_2 + c_2 \end{aligned} \tag{4.84}$$

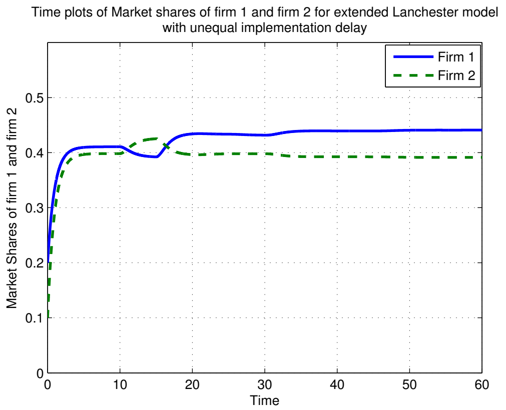
Likewise, equation (4.63) for extended Lanchester model with adoption delay is rewritten:

$$\begin{aligned}\dot{x}_1 &= -x_{1\tau}^2 k_1 - x_{1\tau} x_{2\tau} k_2 - x_{1\tau} c_1 - x_{1\tau} c_2 + x_{1\tau} k_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau} x_{2\tau} k_1 - x_{2\tau}^2 k_2 - x_{2\tau} c_1 - x_{2\tau} c_2 + x_{2\tau} k_2 - \lambda x_2 + c_2\end{aligned} \tag{4.85}$$

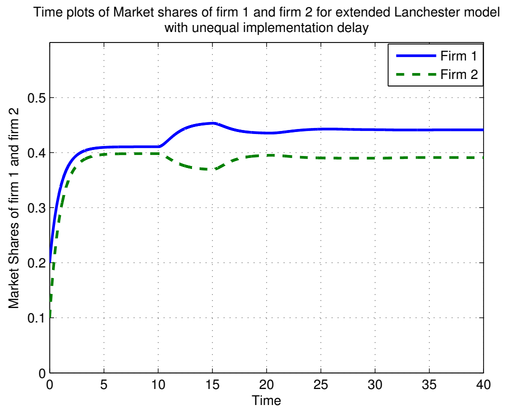
Then, suppose for numerical analysis the same parameters from section 4.3 and unequal delay values ($\tau_1 \neq \tau_2$) are considered.

Therefore, figures 4.9 and 4.10 illustrate the numerical results for the extended Lanchester model with implementation and adoption delay respectively for different delay values for each firm. Figure 4.9(a) shows the dynamics of the model with implementation delay for $\tau_1 > \tau_2$. Then, in figure 4.9(b) we present the dynamics of the model considering $\tau_1 < \tau_2$. Notice that in both cases the equilibrium point remains stable and for $\tau_1 > \tau_2$ the dynamics present two crosses between the trajectories.

Figure 4.10(a) presents the dynamics of the model with adoption delay for $\tau_1 > \tau_2$. Finally, figure 4.10(b) shows the dynamics of the model when $\tau_1 < \tau_2$. Note that both models present oscillations but in both cases, the equilibrium points remain stable.

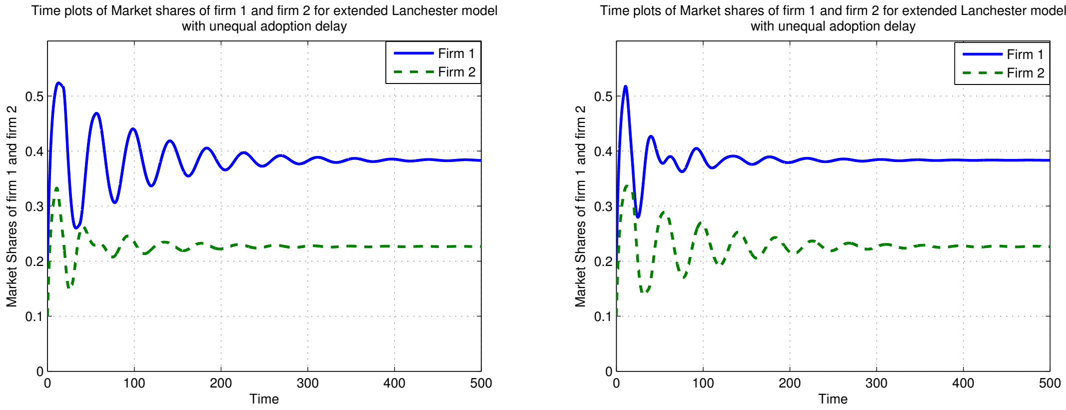


(a)



(b)

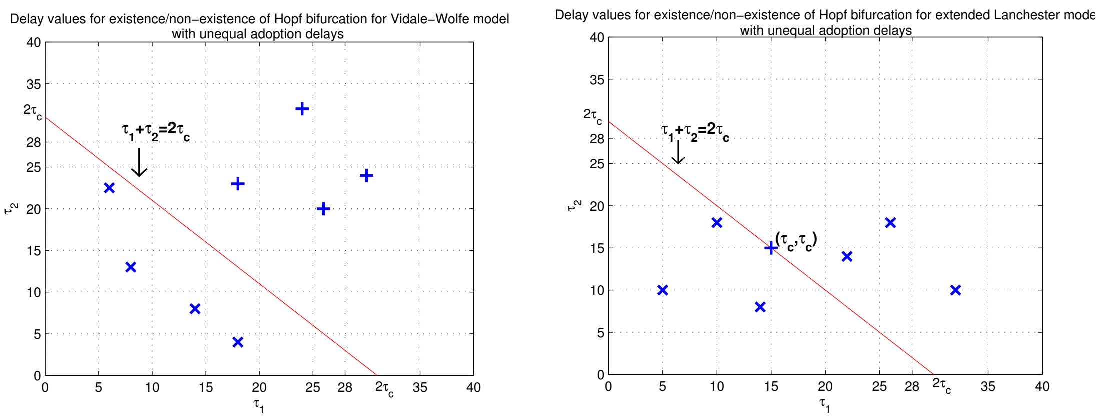
Figure 4.9: Evolution of market shares of firms x_1 and x_2 for: (a) Extended Lanchester model with implementation delay for $\tau_1 = 15$ and $\tau_2 = 10$ (b) Extended Lanchester model with implementation delay for $\tau_1 = 10$ and $\tau_2 = 15$



(a) (b)

Figure 4.10: Evolution of market shares of firms x_1 and x_2 for: (a) Extended Lanchester model with adoption delay for $\tau_1 = 18$ and $\tau_2 = 10$ (b) Extended Lanchester model with adoption delay for $\tau_1 = 10$ and $\tau_2 = 18$

Finally, figure 4.11 shows summarizes the results of numerical simulations performed for different values of τ_1 and τ_2 and leads to the formulation of conjectures about conditions for the existence of Hopf bifurcation in Vidale-Wolfe and extended Lanchester models with unequal adoption delays.



(a) (b)

Figure 4.11: Existence of Hopf bifurcation for unequal adoption delay values for x_1 (τ_1) and x_2 (τ_2) for: (a) Vidale-Wolfe model and (a) Extended Lanchester model. The symbol $\times$ denotes non-existence while the symbol $+$ denotes existence of Hopf bifurcation.

4.5 Chapter conclusions

- In the presence of implementation delays the Vidale-Wolfe and extended Lanchester models present stable dynamics regardless of the delay values.

- For the case of identical adoption delays for both firms, the behavior of the Vidale-Wolfe and extended Lanchester models was analysed mathematically and can be summarized as follows:

  - For $\tau < \tau_c$, in both models, market shares may have an oscillatory transient, but eventually settle at a equilibrium.
  - For both models, there exists a critical value of delay τ_c for Hopf bifurcation.
  - As τ becomes progressively larger than τ_c, bounded and eventually unbounded oscilaltions of market share occur and the equiibrium point becomes unstable.

- The case of unequal delay values for each firm is very complex and was therefore studied through numerical simulations, leading to the formulation of the following conjectures:

  - For implementation delay: The Vidale-Wolfe and extended Lanchester models show stable dynamics regardless of delay values τ_1 and τ_2.
  - For adoption delay: The Vidale-Wolfe model can present Hopf bifurcation when the condition $\tau_1 + \tau_2 > 2\tau_c$ is satisfied. For extended Lanchester model, the existence of Hopf bifurcations seems to be possible only when the condition $\tau_1 = \tau_2 = \tau_c$ is satisfied.

Chapter 5

The Replicator-Mutator model under affine advertising control policy

In this chapter, a new model of duopolies under the approach of the evolutionary dynamics is formulated. Thus, we first present in section 5.2 the main characteristics of an evolutionary model. Subsequently, in section 5.3 we present the proposal of the use of the Replicator-Mutator model in the study of the dynamics of interaction between clients and firms in the particular case of a duopolistic market. Next, in section 5.4 an equilibria and stability analysis of the model proposed in the previous section is developed. Afterward, in section 5.5 parametric sensitivity is analyzed. Then, in section 5.6 we formulate considerations regarding the use of the Replicator-Mutator model under an affine advertising control policy control. Finally, in section 5.7 numerical results of Replicator-Mutator model are shown.

5.1 Motivation for the introduction of the Replicator-Mutator model

We observe that in both the Vidale-Wolfe model (3.2) and the extended Lanchester model (3.6), in the absence of advertising (i.e. setting $u_1 = u_2 = 0$), from equations (3.3) and (3.7), the market shares of both firms decay to the zero equilibrium. In other words, both models only display nontrivial dynamics in the presence of advertising effort. It is known however that interaction between sets of clients of different firms can lead to an intrinsic dynamics of market share – these dynamics are often referred to by the terms word of mouth or diffusion effects [86]. As pointed out in section 2.3 there have been some efforts to model such intrinsic dynamics using population biology (predator-prey) models [40]. Another class of models that describes dynamics of interacting sets of individuals, is that of the replicator and replicator-mutator models from the field of evolutionary game theory

EGT). Such models are now being widely applied outside EGT and the main motivation of this chapter is to propose such models for the advertising and duopoly context.

With this motivation in mind in the next section, we investigate an alternative approach to the problem of modeling finer structure in the manner that advertising might affect client choices and migrations between sets of clients. This alternative approach is based on Evolutionary Game Theory (EGT) and, by making appropriate identifications, we will be able to adapt the Replicator-Mutator model of EGT to our objective of modeling market share dynamics under advertising which affect client preferences.

5.2 Evolutionary Game Theory models

An evolutionary model is characterized by the presence of selection, replication and mutation processes. The selection process is a mechanism of discrimination that favors some specific entities rather than others. In the context of Game Theory, the selection mechanism is determined by the concept of return or payoff. It is assumed that players select strategies with higher payoff values. The replication process ensures that desirable properties of the system entities are preserved from one generation to the next. Finally, the mutation process allows the generation of new diversities, thus preventing stagnation. In the socioeconomic context, the process of mutation is viewed as a procedure of experimentation or innovation which allows the emergence of new identities or new patterns of behavior.

Evolutionary Game Theory can be understood as the study of the evolution of strategies in a population context. In socioeconomic models, it is assumed that players have the ability to adapt their behavior, thus changing their strategy in response to payoff, which in turn, is determined by the behavior of the population as a whole [16].

Evolutionary dynamics provides a mathematical framework appropriate for modeling the principles of natural selection (replication, mutation, competition, and adaptability-dependent strategies) [55], which in turn, allows it to describe how agents can make decisions in complex environments in which interaction with other agents is present [46]. The representation of evolutionary dynamics can be based on the Replicator-Mutator equation, which offers a general description of deterministic evolutionary dynamics including frequency-dependent selection processes and mutation processes [87].

To apply this class of models to the dynamics of advertising in duopolies, it is proposed to identify the concept of strategy with the action of choosing a particular firm. So, the fraction of the population choosing a strategy becomes the fraction of clients who choose a particular firm (i.e. the firm's market share). The mutation process represents a spontaneous change of a client from one firm to another or to neither of the two competing firms (in this case, the client becomes a member of the set of undecided clients). Finally, client preferences regarding firms can be represented in terms of payoff coefficients.

Based on these correspondences, an evolutionary model of the dynamics of interaction between clients and firms is formalized, taking into account the following aspects:

- Client decision-making is determined according to its preference coefficients for each firm involved in the market.

- Clients can change their preferences for firms.

- Through advertising, participating firms can foster increased client preference with regard to themselves.

In order to advance, the following notation and terminology is needed:
n is the number of firms that make up the market.
x_i is the state variable representing the market share of firm i, ($x_i \in [0,1]$).
$\mathbf{x}$ is the vector which represents the market shares of the participating firms.
a_{ij} is the element of the preference matrix representing the clients' preference for firm i in relation to firm j.
$\mathbf{A} = (a_{ij})$ is the preference matrix of the clients with respect to the firms.
q_{ji} is the element of the mutation matrix which represents the probability of change of choice of the clients of firm j to firm i. Since the diagonal element q_{ii} represents the probability of sticking to firm i, it can be thought of as a fidelity parameter.
$\mathbf{Q} = (q_{ij})$ is the mutation matrix of the clients in relation to the firms.
$f_j = \sum\limits_{i=1}^{n} a_{ji}x_i$ is the fitness in relation to the choice of firm j.
$\mathbf{f}$ is the vector which represents the fitness of the participating firms in the market.
$\mu = q_{ii} \in [0,1]$ is the fidelity parameter will be used to parametrize the matrix Q.
$\phi = \sum\limits_{i=1}^{n} f_i x_i$ is the average fitness of the population.
$\mathbf{F} = diag(\mathbf{f}) = diag(\mathbf{Ax})$

Using this notation, a proposal to model the interaction dynamics between clients and firms is made using the Replicator-Mutator equations (see figure 5.1)

$$\dot{x}_i = \sum_{j=1}^{n} x_j f_j q_{ji}(\mu) - x_i\phi \tag{5.1}$$

We can express the model in matrix form:

$$\dot{\mathbf{x}} = \mathbf{Q}^T\mathbf{F}\mathbf{x} - \phi\mathbf{x} \tag{5.2}$$

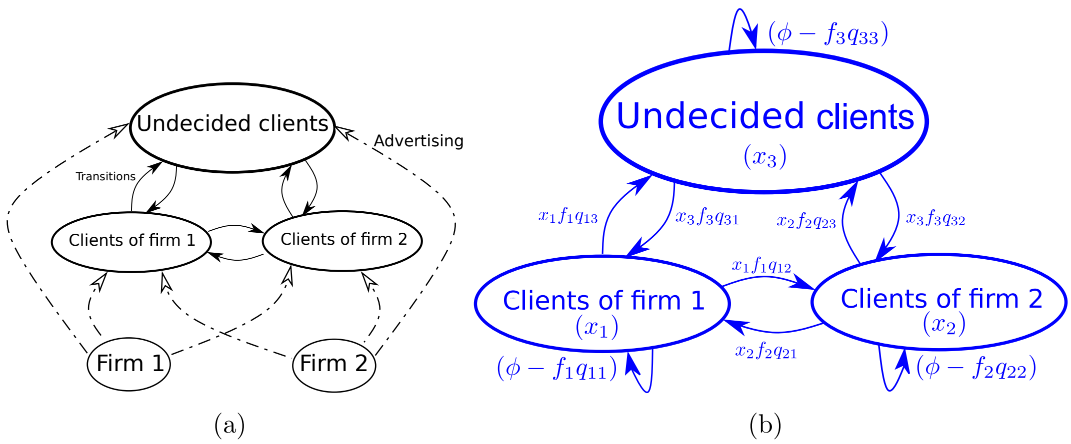


Figure 5.1: Duopoly markets and consumer behavior: (a) the interaction between economic agents, (b) Replicator-Mutator model, showing the terms that determine transitions between sets of clients

5.3 Replicator-Mutator Model

Considering that the total population remains constant, the evolution of the dynamics of the Replicator-Mutator equations represented in equation (5.1) occurs in a simplex of order $n-1$ [87]. For the particular case of a market composed of 2 firms, strategy 1 corresponds to the choice of firm 1, strategy 2 with the choice of firm 2, and strategy 3 to a state of indecision in which neither firm has been chosen. Thus, considering equation (5.1) for the case of $n = 3$ we obtain the following system of equations, written in full:

$$\dot{x}_1 = x_1 f_1 q_{11} + x_2 f_2 q_{21} + x_3 f_3 q_{31} - x_1 \phi \tag{5.3}$$

$$\dot{x}_2 = x_1 f_1 q_{12} + x_2 f_2 q_{22} + x_3 f_3 q_{32} - x_2 \phi \tag{5.4}$$

$$\dot{x}_3 = x_1 f_1 q_{13} + x_2 f_2 q_{23} + x_3 f_3 q_{33} - x_3 \phi \tag{5.5}$$

For the terms f_i and ϕ, we have:

$$f_1 = a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \tag{5.6}$$

$$f_2 = a_{21} x_1 + a_{22} x_2 + a_{33} x_3 \tag{5.7}$$

$$f_3 = a_{31} x_1 + a_{32} x_2 + a_{33} x_3 \tag{5.8}$$

$$\phi = f_1 x_1 + f_2 x_2 + f_3 x_3 \tag{5.9}$$

In relation to the elements of matrix $\mathbf{Q}$ the literature establishes several forms of definition [58], [60], [88], however, for the model proposed in this work with the purpose of simplifying the structure of equations, we consider the work of Komarova *et al.* [89], which proposes a mutation matrix $\mathbf{Q}$ that is independent of the preference matrix $\mathbf{A}$.

In order to define the form of the mutation matrix, we assume that, for all i, a client of firm i has the fidelity parameter μ. This means that the $q_{ii} = \mu$ for all i, and the matrix Q has a constant diagonal. In addition, for each i, we introduce a parameter p_i that weights the "left over" probability $(1-\mu)$ (i.e., the probability of changing firms) between the firms $j, k \neq i$, so that the remaining elements of row i (probabilities of the transitions $i \to j$ and $i \to k$ are $p_i(1-\mu)$ and $(1-p_i)(1-\mu)$ maintaining the ith row sum equal to one. Thus the mutation matrix has the form:

$$Q = \begin{pmatrix} \mu & p_1 (1-\mu) & (1-p_1)(1-\mu) \\ p_2 (1-\mu) & \mu & (1-p_2)(1-\mu) \\ p_3 (1-\mu) & (1-p_3)(1-\mu) & \mu \end{pmatrix} \tag{5.10}$$

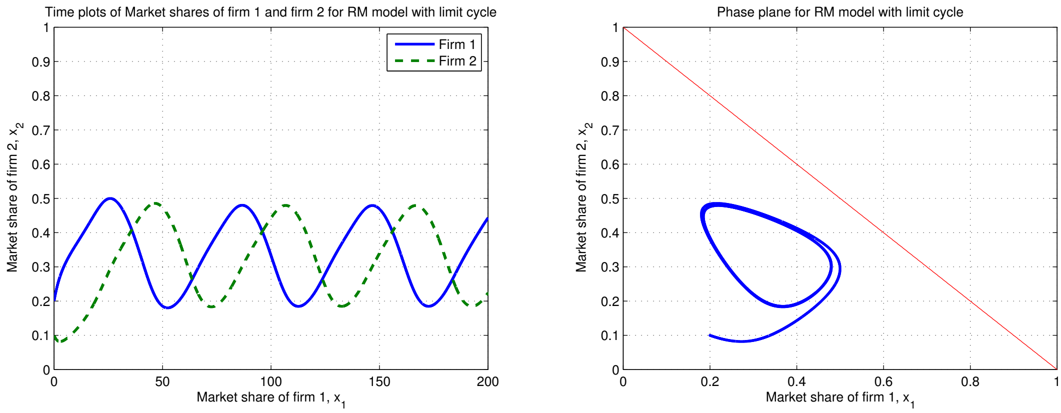
where, p_i is a factor $\in (0,1]$ that determines the values of the off-diagonal entries of row i of matrix $\mathbf{Q}$.

5.3.1 Example showing nontrivial dynamics in the absence of exogenous advertising effort

In this section, to illustrate the flexibility and importance of an appropriate choice of parameters of the Replicator-Mutator model, we present a market scenario in which a limit cycle is present. Consider a market in which the clients have the same preference for the firms and for remaining undecided and also assuming that clients have high loyalty regarding the chosen firm, the matrices of preference and mutation should be defined as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0.78 & 0.2 & 0.02 \\ 0.02 & 0.78 & 0.2 \\ 0.2 & 0.02 & 0.78 \end{pmatrix} \tag{5.11}$$

Figure 5.2 shows the numerical results for market scenario expressed in (5.11). Figures 5.2(a) and 5.2(b) present the evolution of the market share of the firms for initial conditions $x_1(0) = 0.2$ and $x_2(0) = 0.1$ and the phase plane respectively. Note that for this configuration of the matrices $\mathbf{A}$ and $\mathbf{Q}$, the dynamics of the market scenario presents a limit cycle.



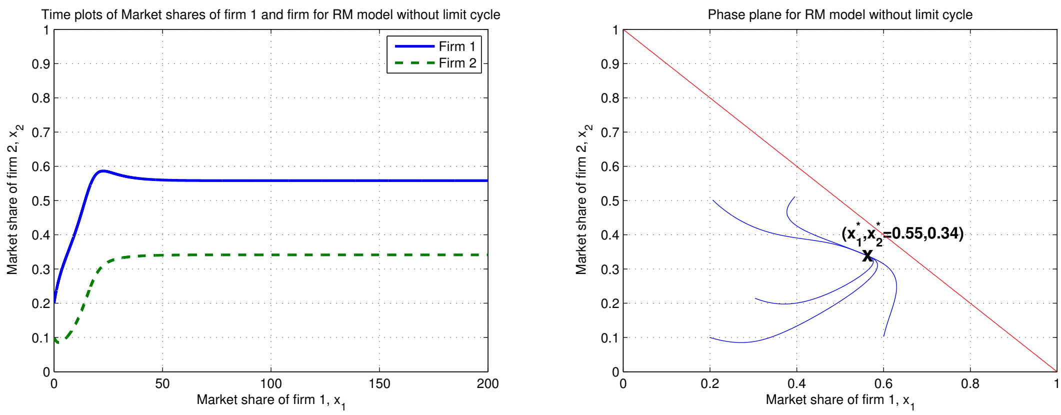
(a) (b)

Figure 5.2: (a) Evolution of the market shares of firms x_1 and x_2 (b) Phase plane of the Replicator-Mutator model. Note the existence of a limit cycle in this particular market scenario defined by (5.11).

Now, suppose that an increase (e.g constant effort with value of 0.4) occurs only in the clients' preference of the firm 1 with regard to firm 2 (a_{12}) and that the remaining elements of the matrices $\mathbf{A}$ and $\mathbf{Q}$ remain unchanged, that is:

$$\mathbf{A} = \begin{pmatrix} 1 & \mathbf{0.6} & 0.2 \\ 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0.78 & 0.2 & 0.02 \\ 0.02 & 0.78 & 0.2 \\ 0.2 & 0.02 & 0.78 \end{pmatrix} \tag{5.12}$$

Figure 5.3 presents the numerical results for market scenario described in (5.12). Similarly to previous case, figures 5.3(a) and 5.3(b) show the evolution of the market share of the firms for initial conditions $x_1(0) = 0.2$ and $x_2(0) = 0.1$ and the phase plane, respectively. Note that the increase in the element a_{12} results in the disappearance of the limit cycle.



(a) (b)

Figure 5.3: (a) Evolution of the market shares of firms x_1 and x_2 (b) Phase plane of the Replicator-Mutator model. Note that, in this case, the limit cycle vanishes.

Remark: Using the Bendixson criterion [90] it is easy to see that the VW model and the extended Lanchester model under constant effort (c_i) in advertising do not have limit cycles.

5.3.2 Comments

1. Comparing the growth terms $C(x_i, u)$ of the RM model (5.1) and the VW model (3.2), it is clear that they would be the same if, in the VW model, $u_1 = u_2 =: u_{vw}$, is defined as an appropriate function of a_{ij}, q_{ij}, μ and, furthermore, the $D(x) = \lambda_i x_i$ term in the VW model has λ_i set to ϕ.

2. In the particular context of electronic markets and on-line advertising, the work developed by Wang [61], presents an evolutionary model for an online advertising ecosystem, also covering some aspects of advertising strategies. Details of this model are given in Appendix B.

3. The Replicator-Mutator model proposed for the duopoly modeling in this thesis is inspired by the work of Wang [61]. However, there are significant differences with regard to the proposal in this chapter:

   - Wang's work [61] considers the dynamics of users and firms using internet advertising, contemplating several specific forms of interaction in this context.
   - We propose to use the Replicator-Mutator equations without modification, in order to model the aggregated dynamics of client decision-making. On the other hand, the Wang model [61] parameterizes the behavior of the individual client, which results in a very large number of parameters (see Appendix B).
   - Wang's work [61] considers only aspects of modeling and simulation of the dynamics proposed therein, while this thesis carries out a more comprehensive study that contemplates an equilibria and stability analysis as well as the formulation of advertising policies for the model.

5.4 Equilibria and stability analysis of Replicator Mutator model

The analytical study of the Replicator-Mutator model for a large number of strategies is complex because of the nonlinear nature of the equations, the number of parameters involved and the possibility of dependence among its elements.

For this reason, several studies on the fixed points of the Replicator-Mutator equations established hypotheses which made analysis possible. For example, in the study by Komarova *et al.* [89], a symmetric matrix $\mathbf{A}$, a symmetric and independent matrix $\mathbf{Q}$, and equal population fractions for the fixed points are considered. In Olfati-Saber's work [58] the particular case of mutation parameter μ tending to zero is considered. In the study of Hussein [59] the existence of dominant strategies is used. In the work of Izquierdo and Izquierdo [91] the condition of dominant strategies is established and the use of a mutation factor μ tending to zero as well. Finally, in the study by Pais and Leonard [92], the condition of a circulating preference matrix $\mathbf{A}$ was stipulated. In this thesis, we propose to use a slight generalization of Komarova's method.

Rewriting the equations of the evolutionary model proposed in section 5.3 for the particular case of a duopoly, we have:

$$\dot{x}_i = \sum_{j=1}^{3} x_j f_j q_{ji}(\mu) - x_i \phi \tag{5.13}$$

where:

$$f_j = \sum_{j=1}^{3} a_{ij} x_j \tag{5.14}$$

$$\phi = \sum_{i=1}^{3} f_i x_i \tag{5.15}$$

Example 1:

For simplicity, we start with a symmetric preference matrix $\mathbf{A}$ and a mutation matrix $\mathbf{Q}$ independent of the matrix $\mathbf{A}$. Thus, considering a particular case where the matrices $\mathbf{A}$ and $\mathbf{Q}$ represent a type of market where the consumers have a greater preference for the acquisition of a product regardless of the firm chosen, in other words, the consumers can change brands but will not stop acquiring the product. It is assumed that the undecided users have similar preferences for both firms. Finally, in regard to the matrix $\mathbf{Q}$ it is assumed that the clients have high fidelity for the firms they have chosen and undecided clients have equal probabilities of choosing either firm.

Given these assumptions the matrices $\mathbf{A}$ and $\mathbf{Q}$ are defined as follows:

$$A = \begin{pmatrix} 1 & a & \frac{a}{3} \\ a & 1 & \frac{a}{3} \\ \frac{a}{3} & \frac{a}{3} & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} \mu & 0.8\,(1-\mu) & 0.2\,(1-\mu) \\ 0.8\,(1-\mu) & \mu & 0.2\,(1-\mu) \\ 0.5\,(1-\mu) & 0.5\,(1-\mu) & \mu \end{pmatrix}$$

In order to analyse the fixed points of (5.13), we first observe from (5.3) to (5.9) that, for all i, $\dot{x}_i$ is given by a polynomial of degree three in x_i. Thus, in order to find a fixed point of (5.13), equivalently of (5.3)- (5.5), we need to find values of $x_i, i = 1, 2, 3$ that are simultaneously zeros of three polynomial of degree three (right hand sides of (5.3)- (5.5)). Since the x_i are nonnegative variables in the interval $[0, 1]$ and sum to one, we can introduce a parametrization similar to that used for the elements of the mutation matrix, as follows:

$$x_1 = \alpha \tag{5.16}$$

$$x_2 = \beta\,(1-\alpha) \tag{5.17}$$

$$x_3 = (1-\beta)\,(1-\alpha) \tag{5.18}$$

where $\alpha, \beta \in [0,1]$ parametrize the fixed points being sought.

Suppose that the market shares of firm 2 and undecided clients are equal, i.e, $\beta = 0.5$.

For Example 1, substituting equations (5.16) , (5.17), and (5.18) into the right hand side of equation (5.13) and setting it to zero, the cubic equation for determining the fixed points is:

$$\begin{aligned} 0 = {} & \alpha^3\,(1.16a - 1.5) + \alpha^2\,(1.325 + 0.675\mu - 1.37a - 0.296\mu a) \\ & + \alpha\,(0.1a - 1.150 + 0.650\mu) + 0.1083a - 0.325\mu - 0.1083a\mu \end{aligned} \tag{5.19}$$

This equation can have up to three real roots, which are denoted $\alpha_i, i = 1,2,3$, so that the first coordinate of the fixed point x_1 can be equal to $\alpha_i, i = 1,2,3$ and the corresponding values of x_2, x_3 are found from (5.17) and (5.18).

Local stability of the equilibrium points is determined through linearization [90]. From (5.13):

$$\begin{aligned} \dot{x}_1 &= g_1\,(x_1, x_2) \\ \dot{x}_2 &= g_2\,(x_1, x_2) \end{aligned} \tag{5.20}$$

The Jacobian matrix for model (5.20) with respect to x_1 and x_2 is given by:

$$J_{RM} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{5.21}$$

where the symbolic expressions for A, B, C, D in (5.21) and for the equilibria are given in Appendix E.

In terms of A, B, C, D the eigenvalues for the Jacobian are:

$$\Gamma = \frac{1}{2}\left[(A+D) \pm \sqrt{4BC + (A-D)^2}\right] \tag{5.22}$$

Now, considering equations (5.19) and (5.22) for some values of element of preference matrix a in order to analyze the existence and stability of the equilibrium points of the Replicator-Mutator model we obtain.

- For $a = 0.3$

In this case, equation (5.19) is given by:

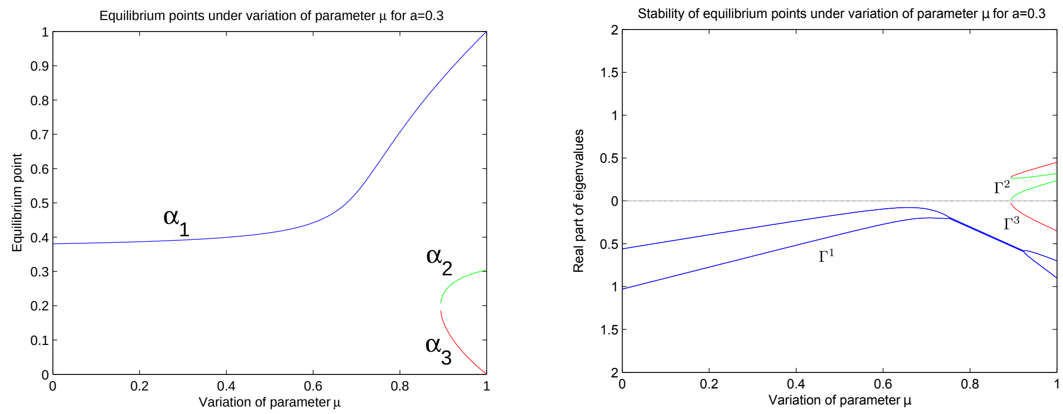
$$0 = -1.15\alpha^3 + \alpha^2\,(0.91 + 0.59\mu) + \alpha\,(-1.12 + 0.77\mu) + 0.36 - 0.36\mu \tag{5.23}$$

- For $a = 0.6$

For this value of element a, equation (5.19) is given by:

$$0 = -0.79\alpha^3 + \alpha^2 (0.5 + 0.5\mu) + \alpha (-1.09 + 0.88\mu) + 0.39 - 0.39\mu \qquad (5.24)$$

Figures 5.4 and 5.5 show the location and stability of the equilibrium point x_1 for equations (5.23) and (5.24). Figures 5.4(a) and 5.5(a) illustrate the loci of solutions α_1, α_2 and α_3 as μ varies. From the figures, it can be observed that solution α_1 increases monotonically with μ. On the other hand, it is observed that additional real solutions α_2 and α_3 appear after the parameter μ crosses a certain threshold value. This threshold value is proportional to the value of element a, that is, a greater value of element a requires a higher value of parameter μ to ensure the existence of the solutions. Additionally, it is possible to say that solution α_2 has monotonic increasing behavior and α_3 has monotonic decreasing behavior as parameter μ increases. In regard to the stability of solutions, figures 5.5(a) and 5.5(b) show that solution α_1 is stable for all range of variations of the parameter μ given that the eigenvalues have negative real parts. For the case of solutions, α_2 and α_3 the figures illustrate that both solutions are unstable because the eigenvalues have positive real part.

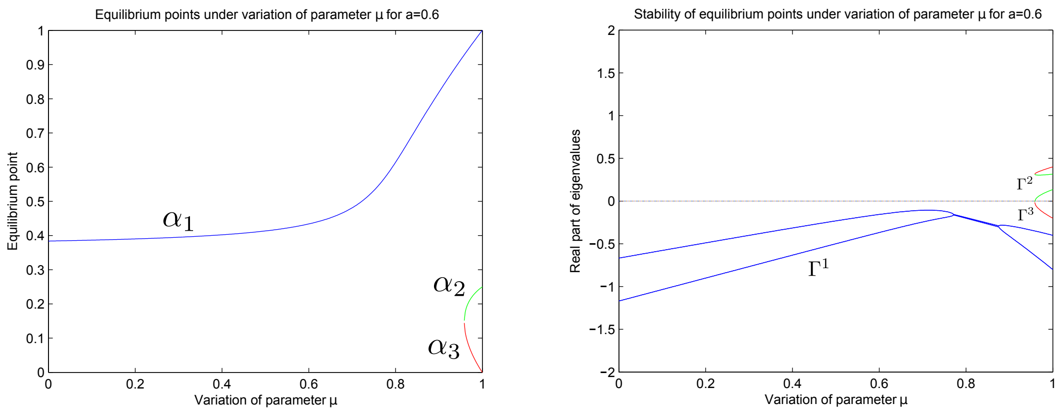


(a) (b)

Figure 5.4: (a) locus of equilibrium points (b) eigenvalue plot showing local stability under variation of parameter μ for $a = 0.3$. Note that in this case an increase in the fidelity parameter results in the existence of three solutions for x_1.

5.5 Parametric sensitivity of equilibrium points of Replicator Mutator Model

In order to analyze the parametric sensitivity of the fixed points of the Replicator-Mutator model expressed by equation (5.13), the fixed points of the model are calculated under variations in the elements of matrices **A** and **Q**.



(a) (b)

Figure 5.5: (a) locus of equilibrium points (b) eigenvalue plot showing local stability under variation of parameter μ for $a = 0.6$. Observe that in this case again an increase in the fidelity parameter results in the existence of three solutions for x_1. In this case ($a = 0.6$) the threshold value for the existence of three solutions is higher than the previous case ($a = 0.3$)

To simplify the analysis, it is assumed that the elements of the initial preference matrix of clients $\mathbf{A}$ and the elements of the initial mutation matrix of clients $\mathbf{Q}$ have a structure similar to matrices $\mathbf{A}$ and $\mathbf{Q}$ from the last section. Hence for $a = 0.6$ and $\mu = 0.8$, we obtain:

$$\mathbf{A} = \begin{pmatrix} 1 & 0.6 & 0.2 \\ 0.6 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{pmatrix} \tag{5.25}$$

$$\mathbf{Q} = \begin{pmatrix} 0.8 & 0.16 & 0.04 \\ 0.16 & 0.8 & 0.04 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \tag{5.26}$$

5.5.1 Parametric sensitivity of the equilibrium points of Replicator-Mutator Model under variations in the elements of client preference matrix A

In this section, the parametric sensitivity of the equilibrium points of the Replicator-Mutator model is analyzed under variations in the elements of the preference matrix of clients $\mathbf{A}$ assuming a given mutation matrix $\mathbf{Q}$ and an initial preference matrix of clients $\mathbf{A}$ defined by equations (5.25) and (5.26). The variations in the elements of the clients' preference matrix $\mathbf{A}$ are determined by the following equation:

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{pmatrix} \tag{5.27}$$

$$a_{ij} = r_{ij} + \Delta a_{ij} \tag{5.28}$$

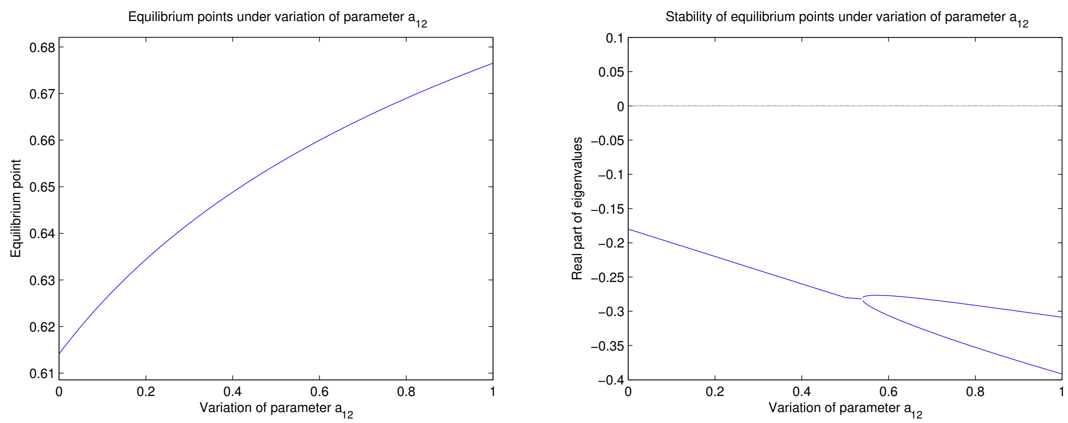
where:
a_{ij} is the final value of element a_{ij}
r_{ij} is the initial value of element a_{ij}
Δa_{ij} is the variation of element a_{ij}

5.5.1.1 Variation of element a_{12}

In this case, the variation in element a_{12} conforming equation (5.28) is considered. Then, the expression for equation (5.19) which determines the fixed points is given by:

$$0 = -0.8\alpha^3 + 0.9\alpha^2 - 0.378\alpha + 0.5a_{12}\alpha^3 - 0.9a_{12}\alpha^2 + 0.4a_{12}\alpha + 0.078 \tag{5.29}$$

Figure 5.6 shows that the equilibrium point x_1 has monotonic increasing behavior as element a_{12} increases. Also, the equilibrium point is stable for the whole interval of variation of the element a_{12} since that Figure 5.6(b) shows the eigenvalues have negative real part over this range of variation.



(a) (b)

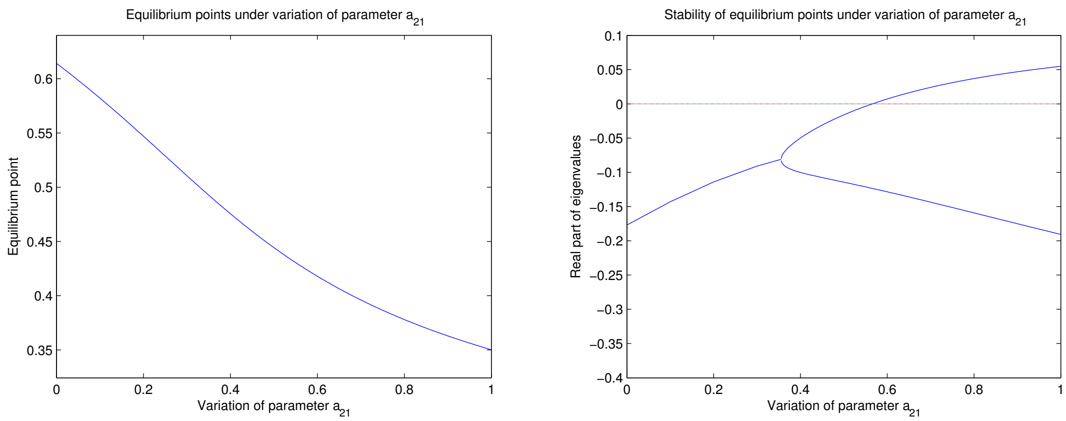
Figure 5.6: Parametric sensitivity under variation of element a_{12}: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Note that an increase of element a_{12} means an increase in the preference to firm 1 over firm 2 and this leads to an increase in the market share of firm 1.

5.5.1.2 Variation of element a_{21}

Proceeding with the sensibility analysis, we now consider variation of element a_{21}. Thus, equation (5.19) is determined as:

$$0 = -0.8\alpha^3 + 0.9\alpha^2 - 0.38\alpha + 0.5a_{21}\alpha^3 - 0.58a_{21}\alpha^2 + 0.08a_{21}\alpha + 0.08 \quad (5.30)$$

The solution for equation (5.30) is shown in figure 5.7. The figure illustrates that equilibrium market share x_1 has monotonic decreasing behavior under increase of element a_{21}. Figure 5.7(b) shows that equilibrium point x_1 is stable up to the value of $\Delta a_{21} = 0.56$.



(a) (b)

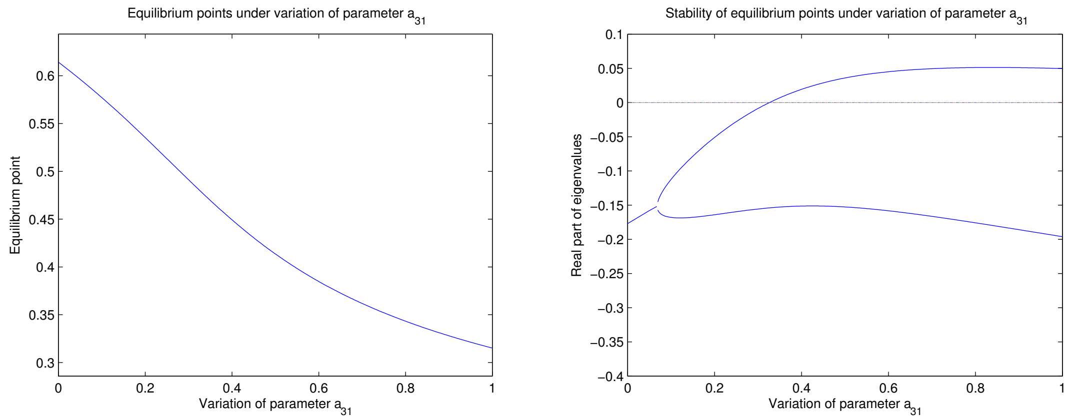
Figure 5.7: Parametric sensitivity under variation of element a_{21}: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Observe that an increase of element a_{21} means a decrease in the preference to firm 1 over firm 2 and this leads to a decrease in the market share of firm 1.

5.5.1.3 Variation of element a_{31}

In this instance, we consider variation in element a_{31}. So, equation (5.19) is expressed by:

$$0 = -0.8\alpha^3 + 0.9\alpha^2 - 0.378\alpha + 0.5a_{31}\alpha^3 - 0.55a_{31}\alpha^2 + 0.05a_{31}\alpha + 0.078 \quad (5.31)$$

Figure 5.8 shows the change in the equilibrium point. It is observed the equilibrium market share x_1 has monotonic decreasing behavior under increase of element a_{31}. Figure 5.8 also shows the equilibrium point x_1 is stable in the interval of variation $[0, 0.3307]$ of element a_{31}.



(a) (b)

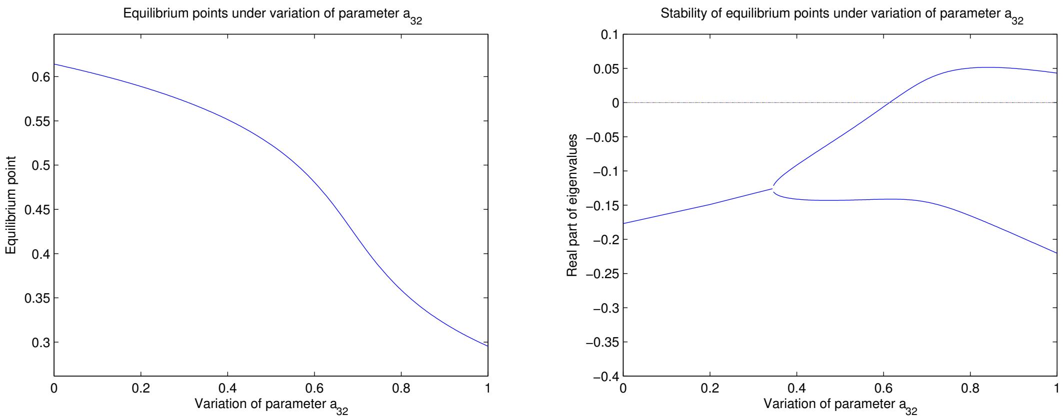
Figure 5.8: Parametric sensitivity under variation of element a_{31}: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Note that an increase of element a_{31} means a decrease in the preference to firm 1 over being undecided and this leads to a decrease in the market share of firm 1.

5.5.1.4 Variation of element a_{32}

Continuing with the sensibility analysis, the variation of element a_{32} is now examined, equation (5.19) becomes:

$$0 = -0.8\alpha^3 + 0.9\alpha^2 - 0.378\alpha - 0.25a_{32}\alpha^3 + 0.525a_{32}\alpha^2 - 0.3a_{32}\alpha + 0.078 \qquad (5.32)$$

Figure 5.9 shows that equilibrium point x_1 has monotonic decreasing behavior under increase of element a_{32}. In addition, it is observed in figure 5.9 that equilibrium point x_1 is stable up to the value of $\Delta a_{32} = 0.6212$.



(a) (b)

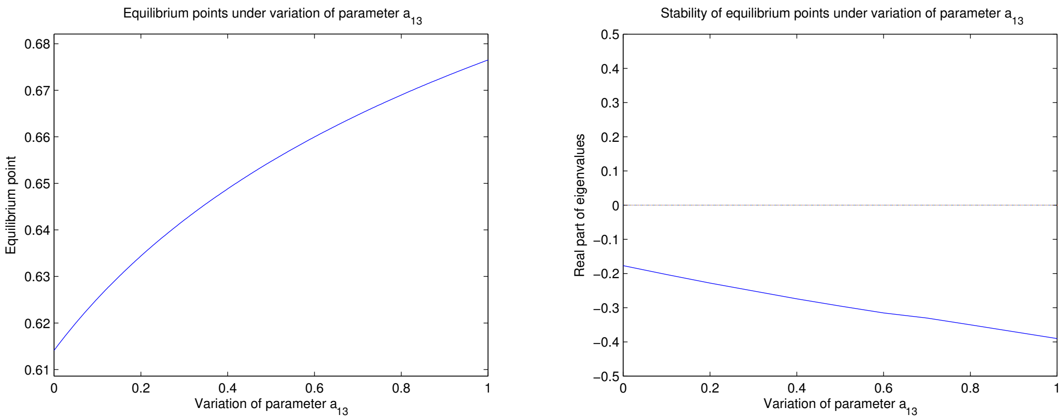
Figure 5.9: Parametric sensitivity under variation of element a_{32}: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Note that an increase of element a_{32} means an increase in the preference to being undecided over firm 2 and this leads indirectly to a decrease in the market share of firm 1.

5.5.1.5 Variation of element a_{13}

Variation in element a_{13} is now contemplated. Thus, equation (5.19) is given as:

$$0 = -0.8\alpha^3 + 0.9\alpha^2 - 0.378\alpha - 0.5a_{13}\alpha^3 - 0.9a_{13}\alpha^2 - 0.4a_{13}\alpha + 0.078 \tag{5.33}$$

The solution for equation (5.33) is shown in figure 5.10. The figure illustrates that equilibrium point x_1 has monotonic increasing behavior under variations of element a_{13}. Concerning the stability of the equilibrium point, it is possible to note in figure 5.10 that equilibrium point x_1 is stable for the whole interval of variation of element a_{13}.



(a) (b)

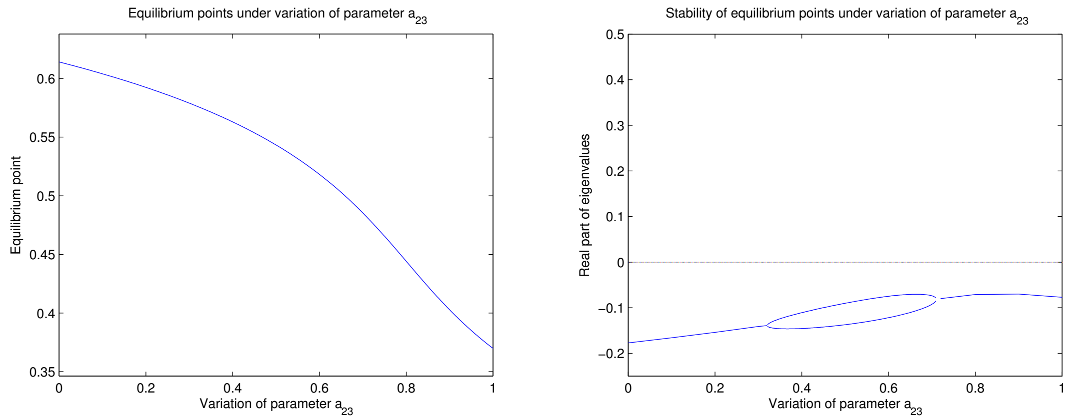
Figure 5.10: Parametric sensitivity under variation of element a_{13}: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Note that an increase of element a_{13} means an increase in the preference to firm 1 over being undecided and this leads to an increase in the market share of firm 1.

5.5.1.6 Variation of element a_{23}

To finalize the sensitivity analysis under variations in the elements of preference matrix $\mathbf{A}$, the change in element a_{23} is studied. So, equation (5.19) is expressed by:

$$0 = -0.8\alpha^3 + 0.9\alpha^2 - 0.378\alpha - 0.25a_{23}\alpha^3 + 0.54a_{23}\alpha^2 - 0.33a_{23}\alpha + +0.04a_{23} + 0.078 \tag{5.34}$$

Figure 5.11 shows the variation in the equilibrium point. Note that equilibrium point x_1 has monotonic decreasing behavior under increase of the element a_{23}. Furthermore, figure 5.11 shows that equilibrium point x_1 is stable for the whole interval of variation of element a_{23}.



(a) (b)

Figure 5.11: Parametric sensitivity under variation of element a_{23}: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Note that an increase of element a_{23} means an increase in the preference to firm 2 over being undecided and this leads indirectly to a decrease in the market share of firm 1 since the flow of undecided clients to firm 1 is adversely affected.

Table 5.1 shows a summary of parametric sensitivity of equilibrium points of Replicator-Mutator model under increases, **one element at a time**, in the elements of the clients' preference matrix **A**.

Element a_{ij} $\uparrow$ / Behavior (x_1)	a_{12}	a_{13}	a_{21}	a_{23}	a_{31}	a_{32}
monotonic increasing $\uparrow$	✓	✓				
monotonic decreasing $\downarrow$			✓	✓	✓	✓

Table 5.1: Parametric sensitivity of equilibrium points of Replicator-Mutator model under increase in the elements of preference matrix **A**. Note that, as expected intuitively, only the increases in preferences for firm 1 result in increases of its market share x_1 (i.e., increases in a_{12}, a_{13} result in increased x_1); while increases in all other elements $(a_{21}, a_{23}, a_{31}, a_{32})$ result in decreases in x_1.

5.5.2 Parametric sensitivity of the equilibrium points of the Replicator-Mutator model under variations in the elements of client mutation matrix Q

Similarly to the study of the previous subsection, the parametric sensitivity of the fixed points of the Replicator-Mutator model expressed in equation (5.13) under changes in the elements of the mutation matrix of clients **Q** is now analyzed. For this purpose, the hypothesis of an established preference matrix **A** and an initial mutation matrix **Q** defined by equations (5.25) and (5.26) is considered. Remembering that matrix **Q** depends on the mutation parameter μ and the relationship between elements q_{ij}, the variation in the elements of the matrix $\mathbf{Q}$ can be formulated through the change in parameter μ on the main diagonal or by the variation parameters (p_i) which relate to the off-diagonal elements of the matrix. In the first case, the variation is produced in all the elements of matrix $\mathbf{Q}$ while in the second instance the change is reflected in each row of the elements of matrix $\mathbf{Q}$.

Recall that the mutation matrix $\mathbf{Q}$ is given by the following expression:

$$\mathbf{Q} = \begin{pmatrix} \mu & p_1(1-\mu) & (1-p_1)(1-\mu) \\ p_2(1-\mu) & \mu & (1-p_2)(1-\mu) \\ p_3(1-\mu) & (1-p_3)(1-\mu) & \mu \end{pmatrix} \tag{5.35}$$

$$\mu \in [0,1] \tag{5.36}$$

$$p_i \in [0,1] \tag{5.37}$$

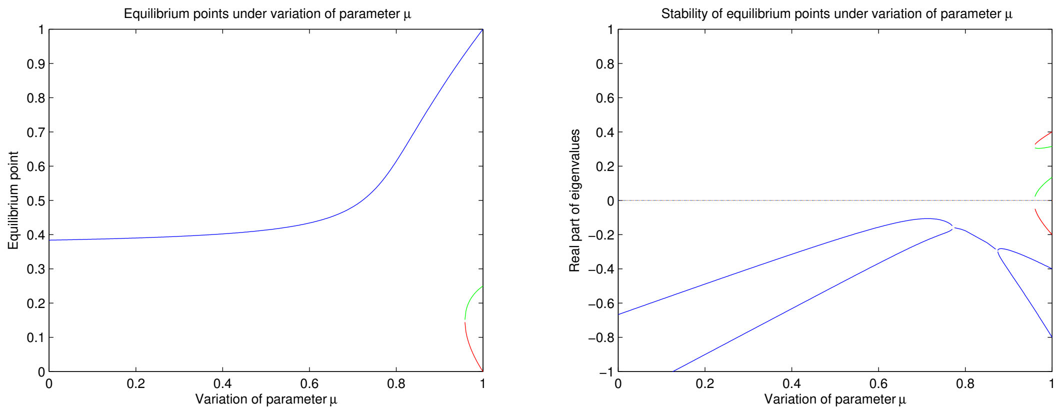
where:
μ is the fidelity parameter of the clients.
p_i is the parameter that establishes the relative importance of the off-diagonal entries q_{ij}.

5.5.2.1 Variation of parameter μ

In this case, the variation of mutation parameter μ is considered. Thus, equation (5.19) is given as:

$$0 = -0.8X^3 + 0.5X^2 - 1.09X + 0.5\mu X^2 + 0.89\mu X - 0.39\mu + 0.39 \tag{5.38}$$

Figure 5.12 shows that equation (5.38) has three solutions. The threshold value of parameter μ for the simultaneous existence the three solutions is ($\mu = 0.958$). One solution (blue) of the equilibrium points has monotonic increasing behavior under increase of mutation parameter μ. In addition it is observed that this solution is stable for all range of variation of parameter μ. The second solution (green) of the equilibrium point has monotonic increasing behavior and a condition of instability throughout the interval of existence. Finally, the third solution (red) of the equilibrium point has monotonic decreasing behavior and similar to the second solution it has a condition of instability for the whole of existence interval.



(a) (b)

Figure 5.12: Parametric sensitivity under variation of the mutation parameter μ: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Note that in this case an increase in the fidelity parameter results in the existence of three solutions for market share x_1. In special case of solution α_1 (blue curve) an increase in the fidelity parameter results in the increase of market share x_1 due to form of matrix Q (5.35) and the structure of equation (5.3).

5.5.2.2 Variation of parameter p_1

The variation in parameter p_1 is now analyzed. Equation (5.19) becomes:

$$0 = -0.8X^3 + 0.9X^2 - 0.378X + 0.078 \tag{5.39}$$

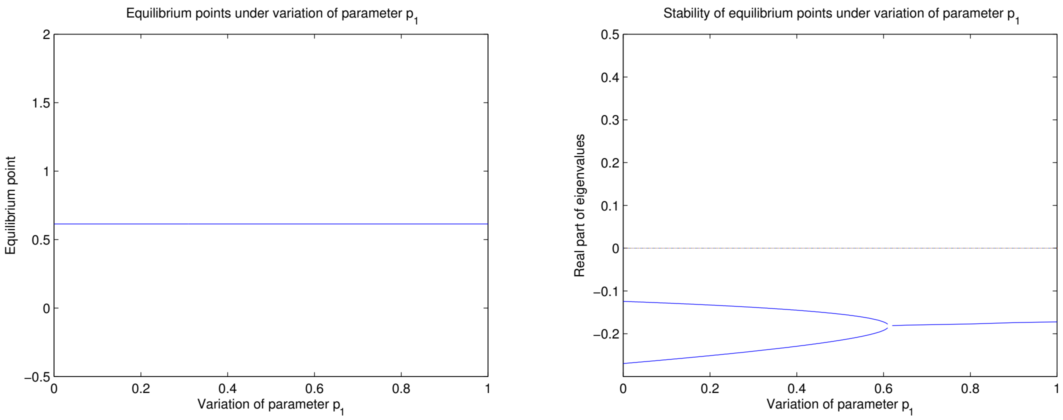
Figure 5.13 shows that equilibrium point x_1 is constant under changes of parameter p_1. In addition it is observed that the equilibrium point is stable for all range of variation of parameter p_1 given that the eigenvalues have negative real part.

5.5.2.3 Variation of parameter p_2

The variation of parameter p_2 is studied. So, equation (5.19) can be expressed as:

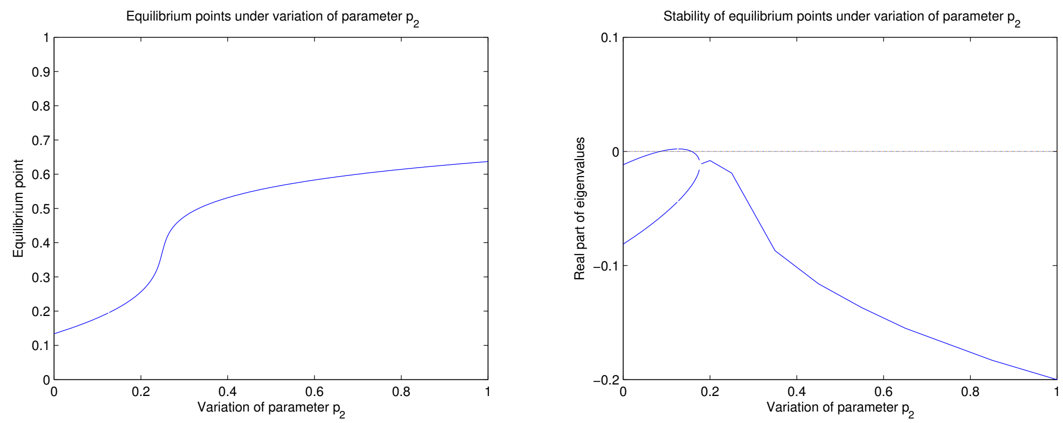
$$0 = -0.8X^3 + 0.9X^2 - 0.33X - 0.06p_2X + 0.06p_2 + 0.03 \tag{5.40}$$

The solution for equation (5.40) is illustrated in figure 5.14. The figure shows that equilibrium point x_1 has monotonic increasing behavior under increase in parameter p_2. It is observed from Figure 5.14(b) that equilibrium point x_1 is stable in the intervals of variation $[0, 0.08]$ and $[0.1563, 1]$ of parameter p_2.



(a) (b)

Figure 5.13: Parametric sensitivity under variation of parameter p_1: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Observe that the variaton in the parameter p_1 does not affect the market share of the firm 1 due to the structure of equation (5.3).



(a) (b)

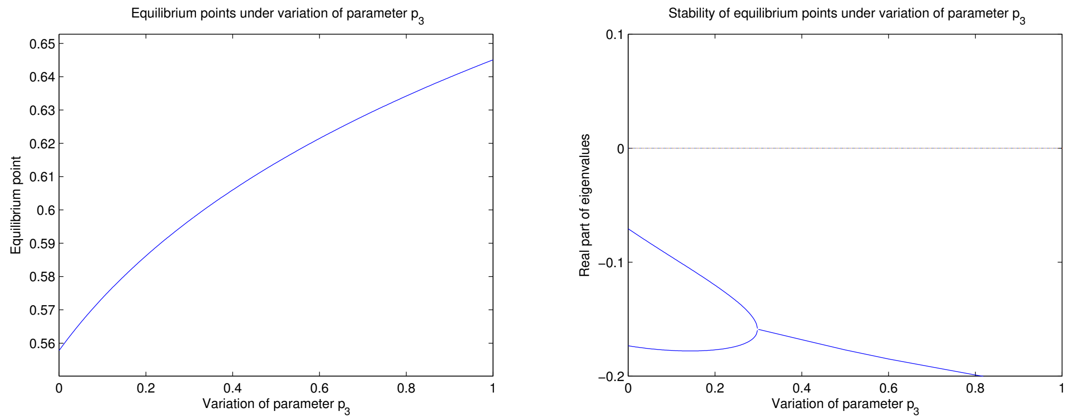
Figure 5.14: Parametric sensitivity under variation of parameter p_2: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Note that an increase in the parameter p_2 increases the market share of the firm 1 due to the structure of equation (5.3).

5.5.2.4 Variation of parameter p_3

To finalize the sensitivity analysis under variations in matrix $\mathbf{Q}$, the variation of parameter p_3 is now regarded. Thus, equation (5.19) is given by:

$$0 = -0.8X^3 + 0.88X^2 - 0.328X + 0.04p_3X^2 - 0.1p_3X + 0.06p_3 + 0.048 \tag{5.41}$$

Figure 5.15 shows that equilibrium point x_1 has monotonic increasing behavior under increase of parameter p_3. Moreover, Figure 5.15 shows that the equilibrium point is stable for the whole range of variation of parameter p_3.



(a) (b)

Figure 5.15: Parametric sensitivity under variation of parameter p_3: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1. Observe that an increase in the parameter p_3 increases the market share of the firm 1 due to the structure of equation (5.3).

Table 5.2 displays a summary of the parametric sensitivity of equilibrium points of Replicator-Mutator model under increases, one parameter at a time in the elements of the mutation matrix $\mathbf{Q}$.

Parameter q_{ij} ↑ / Behavior (x_1)	$\mu(\alpha_i)$	p_1	p_2	p_3
monotonic increasing ↑	✓ (α_1, α_2)		✓	✓
monotonic decreasing ↓	✓ (α_3)			
constant		✓		
multiple equilibrium points	✓			

Table 5.2: Parametric sensitivity of the equilibrium points of Replicator-Mutator model under increase in the parameters defining the mutation matrix $\mathbf{Q}$ (5.35). Note that only variation in fidelity parameter μ produces the existence of multiple equilibrium points for market share of Firm 1 (x_1). With regard to the outcome of x_1 under variation in the parameters (p_1, p_2, p_3) the behavior of the market share x_1 for each parameter variation can be explained by the form of the equations (5.3)- (5.5) that relate the probability of change (q_{ij}) in the choice of the clients.

5.6 Replicator-Mutator under affine advertising control policy

In view of the results of section 5.5 the possibility of changing or altering the equilibrium points (market shares) of the firms participating in the duopolistic market through changes in fidelity parameter μ, variations in the elements of preference matrix $\mathbf{A}$ and modifications in the elements of mutation matrix $\mathbf{Q}$ can be affirmed. However, it is observed that, although fidelity parameter μ and matrix $\mathbf{Q}$ allows modification of the market shares, the literature considers them as intrinsic characteristics of the population [58]. Based on these observations, the following hypotheses are formulated:

1. Advertising performed by a firm does not modify mutation parameter μ and the elements of mutation matrix $\mathbf{Q}$. Specifically, it is assumed that the fidelity rate and mutation matrix $\mathbf{Q}$ are characteristics of the population, not directly affected by advertising.

2. Advertising implemented by a firm can modify the specific elements of preference matrix $\mathbf{A}$, causing them to increase in the case of successful advertising.

5.6.1 Proposal of Control Policy

Based on the above hypotheses, it is possible to represent Δa_{ij} as the variation of element a_{ij} of preference matrix $\mathbf{A}$. Thus, rewriting the equations of the Replicator-Mutator model established in equation (5.13) and allowing variations Δa_{ij} in the elements a_{ij} of the matrix $\mathbf{A}$ we have the following expression:

$$\dot{x}_i = \sum_{j=1}^{3} x_j f_j q_{ji}(\mu) - x_i \phi \tag{5.42}$$

where:

$$f_j = \sum_{j=1}^{3} (a_{ij} + \Delta a_{ij})\, x_j \tag{5.43}$$

Considering the hypotheses formulated above the relationship between publicity and variation of the element of preference matrix $\mathbf{A}$ is represented as follows:

$$\Delta a_{ij} = f(P_{ij}) = u_{ij} \tag{5.44}$$

where:
Δa_{ij} is the variation of element a_{ij}
$f(P_{ij})$ is a function of advertising
u_{ij} is the advertising of the firm i over firm j.

Now, assuming that the impact of the advertising u_i of firm i is reflected only in the entry a_{ij} of the preference matrix we have:

$$A = \begin{pmatrix} 1 & a_{12} + u_1 & a_{13} \\ a_{21} + u_2 & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{pmatrix} \tag{5.45}$$

Thus, equations (5.42), (5.43) and (5.44) model the dynamics of the interactions of the agents involved in a duopoly, taking into account the effect of the advertising by the firms with the purpose of changing their participation in the market.

As in the previous chapters, we propose the use of an affine advertising control policy defined by:

$$u_1 = k_1 x_1 + c_1 \tag{5.46}$$

$$u_2 = k_2 x_2 + c_2 \tag{5.47}$$

5.7 Numerical results

In this section, numerical results for the Replicator-Mutator model under affine advertising control policies are exhibited. The numerical analysis covers the study of three types (scenarios) of markets represented by different configurations of the elements of preference matrix $\mathbf{A}$ and mutation matrix $\mathbf{Q}$. In each market scenario, the implementation of two control policies by the firms with the objective of altering their participation in the market is considered.

5.7.1 Market Scenario 1 (Mobile phone market)

The first scenario models a market where consumers have a greater preference for the acquisition or use of the good or service regardless of the firm chosen. It means the clients can change firms but do not stop having or using the product or service. Concerning the undecided users, we consider that they have similar preferences for both firms as well as the same possibilities of changing their choice. An example of this type of scenario is the mobile phone market or the bank account market [93], [94]. Based on these assumptions, the matrices of preference and mutation are defined as follows:

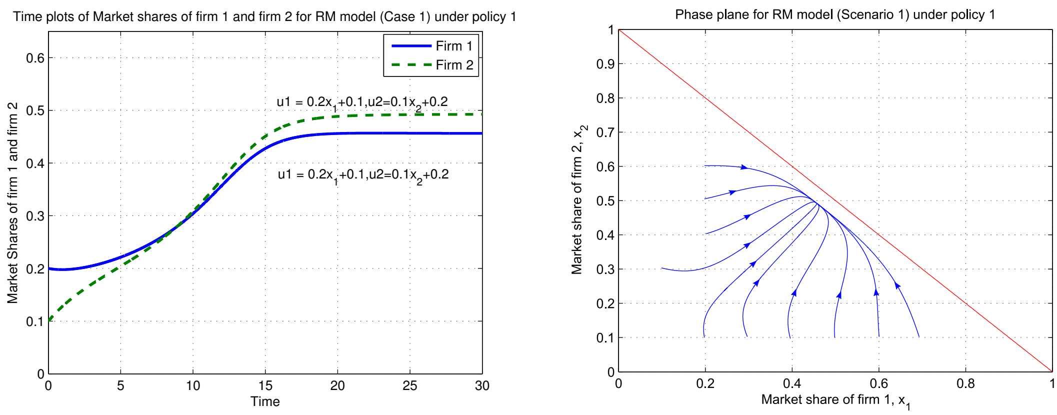
$$\mathbf{A} = \begin{pmatrix} 1 & 0.6 & 0.2 \\ 0.6 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{pmatrix} \tag{5.48}$$

$$\mathbf{Q} = \begin{pmatrix} 0.8 & 0.16 & 0.04 \\ 0.16 & 0.8 & 0.04 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \tag{5.49}$$

Figure 5.16 and figure 5.17 show the numerical results for market scenario 1 under affine control policies formulated by firms 1 and 2. Thus, in figures 5.16(a) and 5.17(a) the evolution of the market shares of the firms for initial conditions $x_1(0) = 0.2$ and $x_2(0) = 0.1$ is observed. Then, figures 5.16(b) and 5.17(b) present the phase planes for both control policies. Note that the equilibrium point for policy 1 is given by $(x_1^* = 0.46, x_2^* = 0.49)$

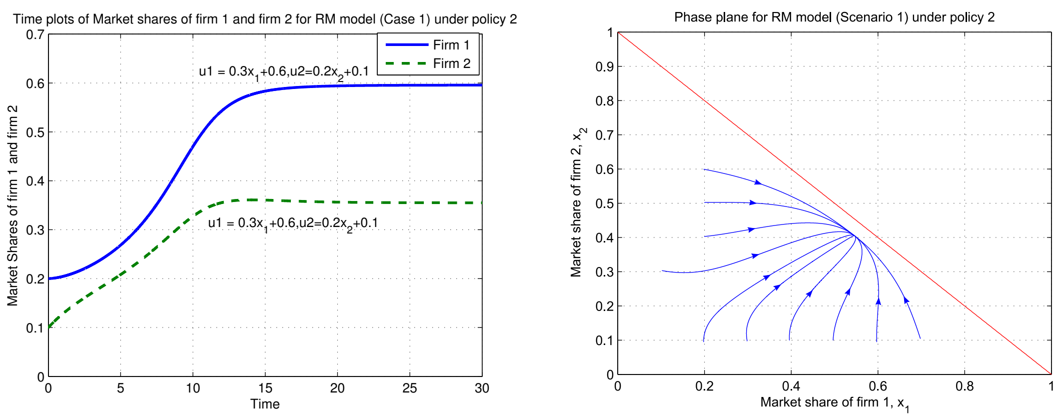
while for policy 2 the equilibrium point is $(x_1^* = 0.54, x_2^* = 0.40)$.

Comparing figures 5.16 and 5.17, it is noted that the increase in advertising of firm 1 produces a large increment in the market share of the company. About firm 2, it is observed that the increase in advertising (lower than firm 1) does not produce an increment in the respective market share. Notice that in this market scenario, one firm has increased its market share and the other has decreased its market share although both companies have increased their advertising. Table 5.3 summarizes the values corresponding for the control parameter, the equilibrium points and the eigenvalues for market scenario 1 considering each control policy.



(a) (b)

Figure 5.16: (a) Evolution of the market shares of firms x_1 and x_2 for market scenario 1 under policy 1 (b) Phase plane for market scenario 1 under policy 1.



(a) (b)

Figure 5.17: (a) Evolution of the market shares of firms x_1 and x_2 for market scenario 1 under policy 2 (b) Phase plane for market scenario 1 under policy 2.

Policy	Control Parameters	Equilibrium Points	Eigenvalues of the Jacobian
P_1	$u_1 = 0.2x_1 + 0.1$ $u_2 = 0.1x_2 + 0.2$	$(x_1^* = 0.46, x_2^* = 0.49)$	-0.21 -0.61
P_2	$u_1 = 0.3x_1 + 0.6$ $u_2 = 0.2x_2 + 0.3$	$(x_1^* = 0.54, x_2^* = 0.40)$	-0.37 -0.78

Table 5.3: Control parameters, equilibrium points and eigenvalues of the Jacobian for market scenario 1 under advertising control policies.

5.7.2 Market Scenario 2 (Credit card market)

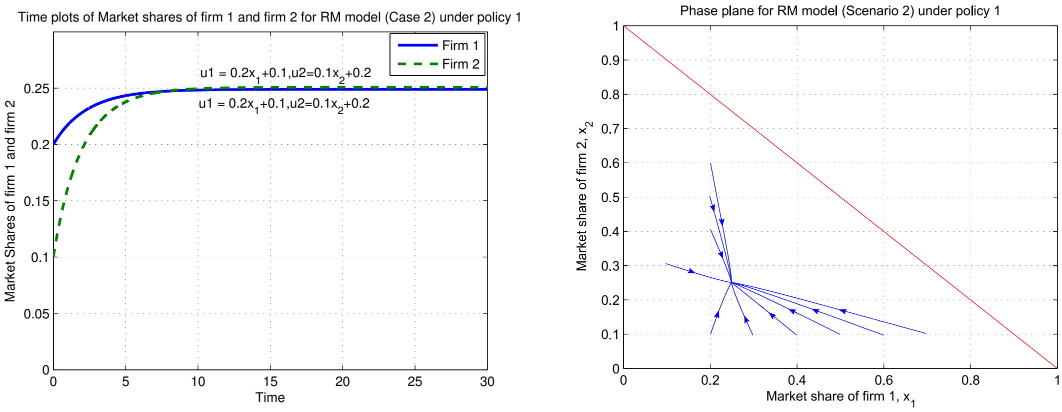
The second scenario attempts to model a market where the preference of consumers to buy or use a product or service regardless of the firm selling it is almost the same as the preference to stop acquiring or using the product or service. Likewise, it is considered that the possibility of changing firms is the same as that of stopping the use of the product (becoming undecided). In other words, the fidelity of the clients to the product or service is relatively low. About undecided users, it is assumed again that they have similar preferences for both firms and they have the same possibilities of changing their choice of using any firm. An example of this type of scenario may be the credit card market [93], [95]. Based on these assumptions, the preference and mutation matrices are defined as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 0.6 & 0.4 \\ 0.6 & 1 & 0.4 \\ 0.3 & 0.3 & 1 \end{pmatrix} \tag{5.50}$$

$$\mathbf{Q} = \begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \tag{5.51}$$

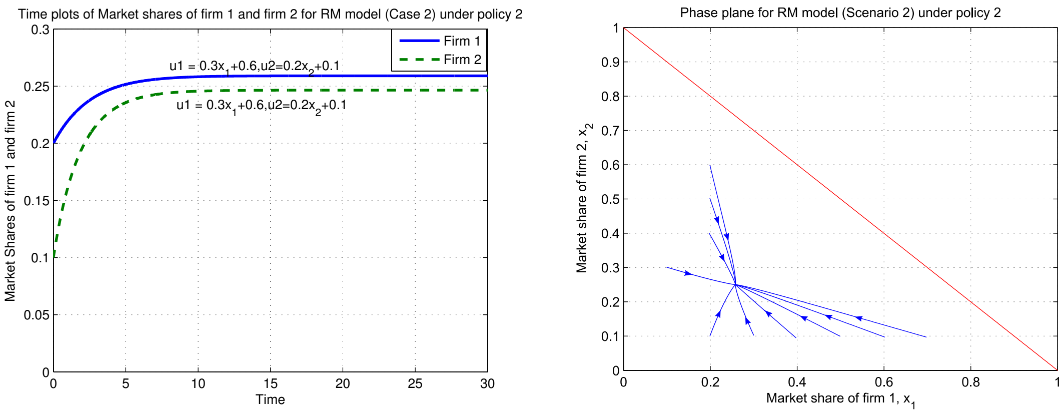
Figure 5.18 and figure 5.19 show the numerical results for market scenario 2 under affine control policy 1 and affine control policy 2 respectively shown in Table 5.4. In figure 5.18(a) and 5.19(a) we show the evolution of the market shares of the firms for initial conditions $x_1(0) = 0.2$ and $x_2(0) = 0.1$. Then, figures 5.18(b) and 5.19(b) present the phase planes for both advertising policies. Note that the equilibrium point for policy 1 is given by $(x_1^* = 0.25, x_2^* = 0.25)$ and in the case of policy 2 the equilibrium point is $(x_1^* = 0.26, x_2^* = 0.25)$.

Evaluating figures 5.18 and 5.19, it can be seen that the market shares of both firms do not change although both companies have increased their advertising (the increase in advertising of firm 1 is greater than the increase in the publicity of company 2). Table 5.4 displays the values for the control parameters, the equilibrium points and the eigenvalues of the Jacobian for each advertising policy.



(a) (b)

Figure 5.18: (a) Evolution of the market shares of firms x_1 and x_2 for market scenario 2 under policy 1 (b) Phase plane for market scenario 2 under policy 1.



(a) (b)

Figure 5.19: (a) Evolution of the market shares of firms x_1 and x_2 for market scenario 2 under policy 2 (b) Phase plane for market scenario 2 under policy 2.

Policy	Control Parameters	Equilibrium Points	Eigenvalues of the Jacobian
P_1	$u_1 = 0.2x_1 + 0.1$ $u_2 = 0.1x_2 + 0.2$	$(x_1^* = 0.25, x_2^* = 0.25)$	-0.46 -0.51
P_2	$u_1 = 0.3x_1 + 0.6$ $u_2 = 0.2x_2 + 0.3$	$(x_1^* = 0.26, x_2^* = 0.25)$	-0.48 -0.54

Table 5.4: Control parameters, equilibrium points and eigenvalues of the Jacobian for the market scenario 2 under advertising control policies.

5.7.3 Market Scenario 3 (Operating system market or Football team market)

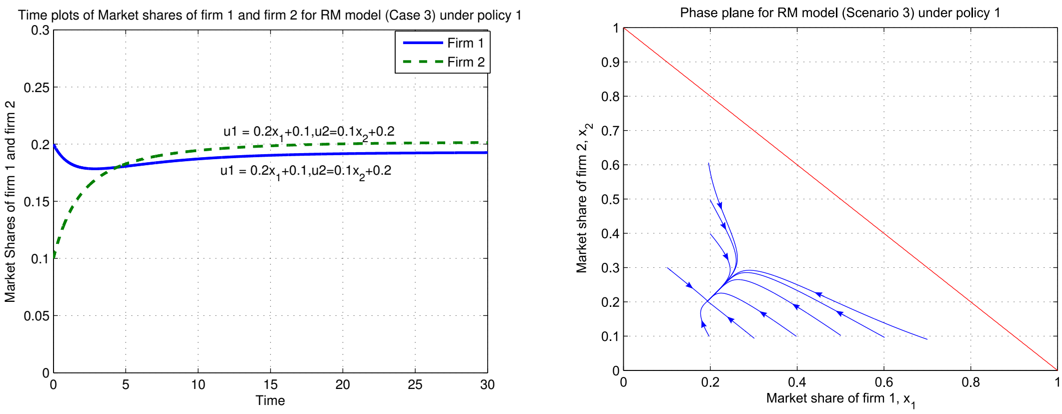
The third scenario seeks to model a market characterized by the high preference of consumers about the product or service that they buy or use in direct relation to the chosen brand. That is, clients have a high preference for a product and a high fidelity for the chosen firm. Concerning the undecided clients, it is supposed that they have the same status of preference and loyalty to both firms. An example of this type of market scenario may be the smartphone market, the football team market, the motorcycle market, or the operating system market [93], [96], [97]. Based on these assumptions it is possible to define the matrices of preference and mutation as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 0.8 & 0.2 \\ 0.8 & 1 & 0.2 \\ 0.4 & 0.4 & 1 \end{pmatrix} \tag{5.52}$$

$$\mathbf{Q} = \begin{pmatrix} 0.8 & 0.04 & 0.16 \\ 0.04 & 0.8 & 0.16 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \tag{5.53}$$

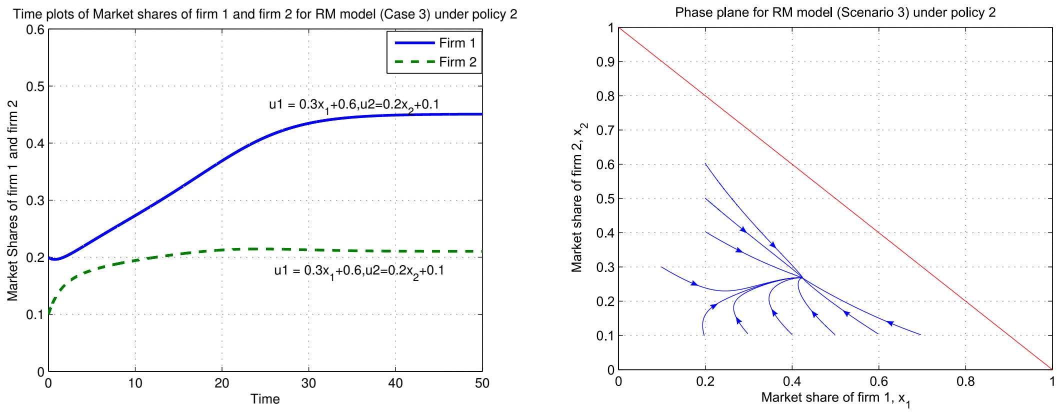
Figure 5.20 and figure 5.21 show the numerical results for market scenario 3 under affine control policies performed by firms 1 and 2. In figure 5.20(a) and 5.21(a) we observe the evolution of the market shares of the firms for initial conditions $x_1(0) = 0.2$ and $x_2(0) = 0.1$. Then, figures 5.20(b) and 5.21(b) present the phase planes for both policies. Note that the equilibrium point for policy 1 is $(x_1^* = 0.19, x_2^* = 0.20)$ and for policy 2 the equilibrium point is $(x_1^* = 0.42, x_2^* = 0.27)$.

Examining figures 5.20 and 5.21, it is observed that the increase in advertising of firm 1 produces a large increment in the market share of the company. About firm 2, it is noted that an increase in advertising (lower than firm 1) produces an increment (lower than firm 1) in the respective market share. Notice that in this market scenario, both firms have increased their market share. Table 5.5 summarizes the values for the control parameters, the equilibrium points and the eigenvalues for this market scenario considering each advertising policy.



(a) (b)

Figure 5.20: (a) Evolution of the market shares of firms x_1 and x_2 for market scenario 3 under policy 1 (b) Phase plane for market scenario 3 under policy 1.



(a) (b)

Figure 5.21: (a) Evolution of the market shares of firms x_1 and x_2 for market scenario 3 under policy 2 (b) Phase plane for market scenario 3 under policy 2.

Policy	Control Parameters	Equilibrium Points	Eigenvalues of the Jacobian
P_1	$u_1 = 0.2x_1 + 0.1$ $u_2 = 0.1x_2 + 0.2$	$(x_1^* = 0.19, x_2^* = 0.20)$	-0.08 -0.27
P_2	$u_1 = 0.3x_1 + 0.6$ $u_2 = 0.2x_2 + 0.3$	$(x_1^* = 0.42, x_2^* = 0.27)$	$-0.18 + 0.016i$ $-0.18 - 0.016i$

Table 5.5: Control parameters, equilibrium points and eigenvalues of the Jacobian for the market scenario 3 under advertising control policies.

5.8 Chapter conclusions

The conclusions of this chapter are:

- The Replicator-Mutator equation can be formulated as a new class of model of the dynamics of a duopolistic market with competition in advertising.

- For the proposed Replicator-Mutator model the analysis of the existence of the equilibrium points and their stability is feasible under certain hypotheses regarding preference matrix **A**, mutation matrix **Q**, mutation parameter μ and the population values for the fixed points.
- The Replicator-Mutator model shows different parametric sensitivity about the equilibrium points and the stability conditions under variations in the elements of preference matrix **A** and mutation matrix **Q**.
- The Replicator-Mutator model allows the representation of different market scenarios through diverse configurations of preference matrix **A** and mutation matrix **Q**.
- The same affine advertising control policies have very different outcomes (market shares) in diverse market scenarios, showing the versatility of the model and the importance of an adequate choice of preference, fidelity and mutation parameters.

Chapter 6

The Replicator-Mutator model with delays under affine advertising control policy

In this chapter, the Replicator-Mutator model formulated in the previous chapter considering the presence of delays is analyzed. Thus, in sections 6.2 and 6.3, the Replicator-Mutator model regarding the two types of delays defined in chapter 2 are examined. Then, section 6.4 carries out a stability analysis of the Replicator-Mutator model for different delay values τ. In section 6.5 a numerical study of the Replicator-Mutator model with delays for the same market scenarios of chapter 5 is performed. Section 6.6 presents a numerical simulation of the Replicator-Mutator model considering different values of delays for each firm. Finally, section 6.7 contains the conclusions of the chapter.

6.1 Delays in the Replicator-Mutator model

In the last two decades, the Replicator-Mutator equations have been studied using different approaches in different areas. However, there are few studies considering the presence of delays. The existing studies essentially address biology [98] and networking [99] themes and in most cases only contemplate the replicator equations [100], [101], [102].

In the context of duopolistic markets, the Replicator-Mutator model formulated in Chapter 5, similarly to the models presented in Chapter 3, considers instant access to market information as well as immediate response to the effects of advertising. However, as explained above, these assumptions do not usually hold in real markets.

Thus in analogy with the analysis performed for the Vidale-Wolfe and extended Lanchester models the inclusion of delays into the Replicator-Mutator model will allow a better description of the dynamics of the market. Nevertheless, as was also mentioned and verified in the Vidale-Wolfe and extended Lanchester models, the presence of substantial changes in the behavior of the system is possible.

The following sections develop an analysis of the Replicator-Mutator model supposing the existence of delays.

6.2 The Replicator-Mutator model with implementation delay under affine advertising control policy

Considering the Replicator-Mutator model formulated in chapter 5 and the assumption exposed in subsection 2.4.1 concerning the delay in the available information the Replicator-Mutator model with implementation delay can be defined the following way:

$$\dot{x}_i = \sum_{j=1}^{3} x_j f_{j\tau} q_{ji} - x_i \phi \tag{6.1}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} + u_{1\tau} & a_{13} \\ a_{21} + u_{2\tau} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{6.2}$$

where:
$f_{j\tau} = f_j \left(t - \tau_j\right)$
$u_{1\tau} = k_1 x_{1\tau} + c_1$
$u_{2\tau} = k_2 x_{2\tau} + c_2$
$x_{1\tau} = x_1 \left(t - \tau_1\right)$
$x_{2\tau} = x_2 \left(t - \tau_2\right)$
Assuming the condition of equal delay values for each firm ($\tau_1 = \tau_2$) the expressions for $x_{1\tau}$ and $x_{2\tau}$ are given by: $x_{1\tau} = x_1 \left(t - \tau_1\right) = x_1 \left(t - \tau\right)$ and $x_{2\tau} = x_2 \left(t - \tau_2\right) = x_2 \left(t - \tau\right)$.

6.3 The Replicator-Mutator model with adoption delay under affine advertising control policy

Similar to the previous section, the Replicator-Mutator model with adoption delay can be expressed as:

$$\dot{x}_i = \sum_{j=1}^{3} x_{j\tau} f_{j\tau} q_{ji} - x_i \phi \tag{6.3}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} + u_{1\tau} & a_{13} \\ a_{21} + u_{2\tau} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{6.4}$$

where:
$x_{j\tau} = x_j\left(t-\tau_j\right)$
$f_{j\tau} = f_j\left(t-\tau_j\right)$
$u_{1\tau} = k_1 x_{1\tau} + c_1$
$u_{2\tau} = k_2 x_{2\tau} + c_2$
$x_{1\tau} = x_1\left(t-\tau_1\right)$
$x_{2\tau} = x_2\left(t-\tau_2\right)$
Again, assuming the condition of equal delay values for each firm ($\tau_1 = \tau_2$) the expressions for $x_{1\tau}$ and $x_{2\tau}$ are given by: $x_{1\tau} = x_1\left(t-\tau_1\right) = x_1\left(t-\tau\right)$ and $x_{2\tau} = x_2\left(t-\tau_2\right) = x_2\left(t-\tau\right)$.

6.4 Stability analysis of the Replicator-Mutator model with delays

This section analyses the stability of equilibria of the Replicator-Mutator model as delay τ varies. Note that the analysis is the same for both types of delays.

For the Replicator-Mutator model with delay:

$$\dot{x}_i = \sum_{j=1}^{3} x_j f_{j\tau} q_{ji} - x_i \phi \tag{6.5}$$

let the equilibrium points be denoted (x_1^*, x_2^*).

The characteristic equation for (6.5) is as follows:

$$P_{rm}(\psi,\tau) = \Psi^2 + P_{1rm}\Psi + P_{2rm} + P_{3rm}e^{-2\psi\tau} + P_{4rm}\Psi e^{-\psi\tau} + P_{5rm}e^{-\psi\tau} \tag{6.6}$$

where:
$P_{irm}, i = 1, \ldots 5$, is a real coefficient.

6.4.1 Stability analysis of the characteristic equation for $\tau = 0$

In this case, assuming $\tau = 0$, equation (6.6) yields

$$P_{rm}(\psi,\tau) = \Psi^2 + P_{6rm}\Psi + P_{7rm} \tag{6.7}$$

where:
$P_{6rm} = P_{1rm} + P_{4rm}$
$P_{7rm} = P_{2rm} + P_{3rm} + P_{5rm}$
Then, the stability criteria [83] can be established when

$$\begin{aligned} P_{6rm} &> 0 \\ P_{7rm} &> 0 \end{aligned} \tag{6.8}$$

Therefore, we may formulate the following proposition:

Proposition 9. *The equilibrium point* (x_1^*, x_2^*) *of model* (6.5) *is a stable equilibrium point for* $\tau = 0$ *when the conditions expressed in* (6.8) *hold.*

6.4.2 Stability analysis of the characteristic equation for $\tau > 0$

Now, analyzing for $\tau > 0$, and replacing $\Psi = i\omega$ in (6.6) we have:

$$P_{rm}(i\omega, \tau) = (i\omega)^2 + P_{1rm}(i\omega) + P_{2rm} + P_{3rm}e^{-2(i\omega)\tau} + P_{4rm}(i\omega)e^{-(i\omega)\tau} + P_{5rm}e^{-(i\omega)\tau} \quad (6.9)$$

Then, separating the real and imaginary parts from (6.9), we obtain:

$$P_{3rm}\cos(2\omega\tau) + \omega P_{4rm}\sin(\omega\tau) = \omega^2 - P_{2rm} - P_{5rm}\cos(\omega\tau) \quad (6.10)$$

$$-P_{3rm}\sin(2\omega\tau) + \omega P_{4rm}\cos(\omega\tau) = -P_{1rm}\omega + P_{5rm}\sin(\omega\tau) \quad (6.11)$$

Solving equations (6.10) and (6.11), we obtain:

$$\begin{aligned} 0 = &- P_{4rm}\omega^3 + P_{4rm}^2\sin(\omega\tau)\,\omega^2 + P_{4rm}P_{2rm}w + P_{4rm}P_{3rm}\cos(2\omega\tau)\,\omega \\ &- P_{1rm}P_{5rm}\omega + P_{5rm}^2\sin(\omega\tau) + P_{3rm}P_{5rm}\sin(2\omega\tau) \end{aligned} \quad (6.12)$$

Now, rearranging the characteristic equation (6.6) we get:

$$P_{rm}(\psi, \tau) = \Psi^2 + P_{1rm}\Psi + P_{2rm} + e^{-\psi\tau}\left(P_{4rm}\Psi + P_{3rm}e^{-\psi\tau} + P_{5rm}\right) \quad (6.13)$$

Then, the second necessary condition for Hopf Bifurcation existence [85] is formulated as:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \quad (6.14)$$

Next, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (6.13) gives:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{E_{rm} + F_{rm}i}{G_{rm} + H_{rm}i} \quad (6.15)$$

where:
$E_{rm} = -w\left(4P_{3rm}\cos^2(\omega\tau) + P_{4rm}\omega\sin(\omega\tau) + P_{5rm}\cos(\omega\tau) - 2P_{3rm}\right)$
$F_{rm} = -w\left(4P_{3rm}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)\,P_{4rm}\omega + P_{5rm}\sin(\omega\tau)\right)$
$G_{rm} = -4P_{3rm}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4rm}\omega\cos(\omega\tau) - \tau P_{5rm}\sin(\omega\tau) + P_{4rm}\sin(\omega\tau) - 2w$
$H_{rm} = 4P_{3rm}\tau\cos^2(\omega\tau) + P_{4rm}\tau\omega\sin(\omega\tau) + P_{5rm}\tau\cos(\omega\tau) - P_{4rm}\cos(\omega\tau) - 2P_{3rm}\tau - P_{1rm}$
Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{E_{rm}G_{rm} + F_{rm}H_{rm}}{G_{rm}^2 + H_{rm}^2} \neq 0 \quad (6.16)$$

Hence, from the previous analysis, the following proposition can be formulated:

Proposition 10. *The model* (6.5) *has Hopf bifurcation for delay value* $\tau > 0$ *when the equation* (6.12) *has a positive solution and condition* (6.16) *holds.*

6.5 Numerical results

This section presents numerical results for the Replicator-Mutator model assuming the existence of delays. For this purpose, the analysis considers the same three market scenarios proposed and studied in chapter 5.

6.5.1 Market Scenario 1 (Mobile phone market)

Recalling subsection 5.7.1 , this scenario represents a type of market where the clients prefer to purchase the product independent of the brand chosen. For convenience, the preference matrix **A** and the mutation matrix **Q** for this market scenario are given below once again:

$$\mathbf{A} = \begin{pmatrix} 1 & 0.6 & 0.2 \\ 0.6 & 1 & 0.2 \\ 0.2 & 0.2 & 1 \end{pmatrix} \tag{6.17}$$

$$\mathbf{Q} = \begin{pmatrix} 0.8 & 0.16 & 0.04 \\ 0.16 & 0.8 & 0.04 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \tag{6.18}$$

6.5.1.1 Market Scenario 1 with implementation delay

Substituting the expressions of matrices **A** and **Q** into equation (6.1) and analyzing the Replicator-Mutator model we have that the characteristic equation for policy 1 is given by:

$$P_{rm1}(\psi, \tau) = \Psi^2 + 0.85\Psi + 0.14 + 0.01e^{-2\psi\tau} - 0.02\Psi e^{-\psi\tau} - 0.01e^{-\psi\tau} \tag{6.19}$$

In the same way, the characteristic equation for policy 2 can be expressed as:

$$P_{rm1}(\psi, \tau) = \Psi^2 + 1.11\Psi + 0.28 + 0.01e^{-2\psi\tau} - 0.03\Psi e^{-\psi\tau} - 0.02e^{-\psi\tau} \tag{6.20}$$

Then, assuming $\tau = 0$ equations (6.19) and (6.20) yield:

$$P_{rm1}(\psi, 0) = \Psi^2 + 0.83\Psi + 0.14 \tag{6.21}$$

$$P_{rm1}(\psi, 0) = \Psi^2 + 1.08\Psi + 0.27 \tag{6.22}$$

Hence, taking into account Proposition 9 it can be said that equilibrium point is stable.

Now, analyzing for $\tau > 0$, we substitute $\varPsi = i\omega$ in (6.19) and (6.20). Thereby, solving the corresponding equations, it is observed that the characteristic equation has no positive root. Therefore, considering Proposition 10 it can be inferred that the market scenario 1 with implementation delay has no Hopf bifurcation.

6.5.1.2 Market Scenario 1 with adoption delay

Replacing the expressions of matrices **A** and **Q** into (6.3) and evaluating the Replicator-Mutator model, we can express the characteristic equation for policy 1 by:

$$P_{rm1}(\psi, \tau) = \varPsi^2 + 2.89\varPsi + 1.73 + 0.91e^{-2\psi\tau} - 2.07\varPsi e^{-\psi\tau} - 2.51e^{-\psi\tau} \tag{6.23}$$

Similarly, for control policy 2, the characteristic equation can be represented as:

$$P_{rm1}(\psi, \tau) = \varPsi^2 + 3.44\varPsi + 2.41 + 1.09e^{-2\psi\tau} - 2.36\varPsi e^{-\psi\tau} - 3.24e^{-\psi\tau} \tag{6.24}$$

Now, assuming $\tau = 0$ equations (6.23) and (6.24) become:

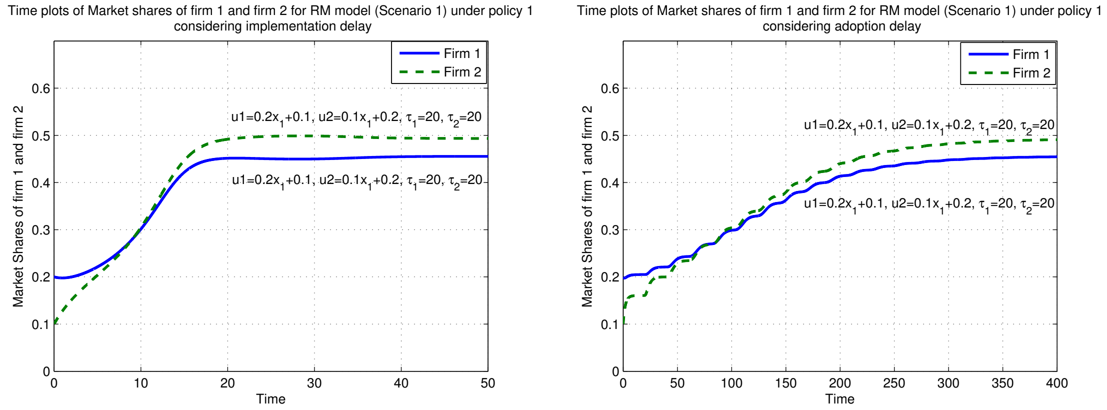
$$P_{rm1}(\psi, 0) = \varPsi^2 + 0.82\varPsi + 0.13 \tag{6.25}$$

$$P_{rm1}(\psi, 0) = \varPsi^2 + 1.08\varPsi + 0.26 \tag{6.26}$$

Therefore, considering Proposition 9 it can be said that equilibrium point is stable.

Then, analyzing for $\tau > 0$, we replace $\varPsi = i\omega$ in (6.23) and (6.24). Thus, solving the respective equations it is noted that the characteristic equation has no positive root. Hence, taking into account Proposition 10 it is concluded that the martket scenario with adoption delay has no Hopf bifurcation.

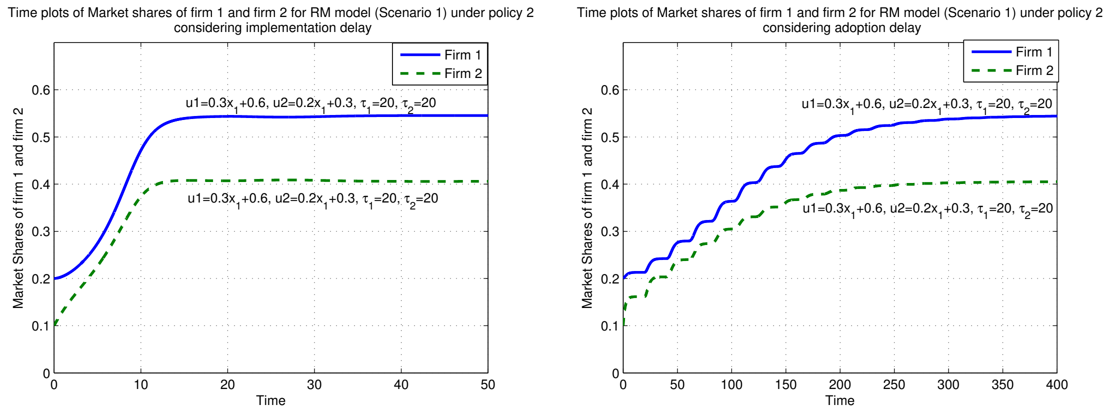
Figure 6.1 shows the numerical results for market scenario 1 under affine control policy 1 considering the two types of delays defined in equations (6.1) and (6.3). In this way, figure 6.1(a) illustrates the Replicator-Mutator model considering the presence of implementation delay and figure 6.1(b) examines the Replicator-Mutator model considering the presence of adoption delay. Analyzing the figures, it is possible to corroborate the propositions formulated about the stability of the equilibrium point for both types of delays. It is also possible to note when comparing the figures that the stabilization time for the case of adoption delay is larger.



(a) (b)

Figure 6.1: Evolution of the market shares of firms 1 and 2 for market scenario 1 under policy 1 with (a) implementation delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$ and (b) adoption delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$.

Next, in figure 6.2 the evolution of the market shares of the firms for market scenario 1 under affine control policy 2 assuming the two types of delays defined in equations (6.1) and (6.3) for initial conditions $x_1(0) = 0.2$ and $x_2(0) = 0.1$ is presented. Thus, figure 6.2(a) illustrates the dynamics of the market considering the presence of implementation delay and figure 6.2(b) displays the dynamics of the market supposing the presence of adoption delay. In a similar way to control policy 1, the figures allow corroboration of the Proposition 9 and Proposition 10 about the stability of the equilibrium points. In this case, again it is noted that the time stabilization for adoption delay is larger.



(a) (b)

Figure 6.2: Evolution of the market shares of firms 1 and 2 for market scenario 1 under policy 2 with (a) implementation delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$ and (b) adoption delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$.

6.5.2 Market Scenario 2 (Credit cards market)

Recalling subsection 5.7.2, this scenario describes a market where the clients have low fidelity. So, the preference matrix $\mathbf{A}$ and the mutation matrix $\mathbf{Q}$ for this market scenario are:

$$\mathbf{A} = \begin{pmatrix} 1 & 0.6 & 0.4 \\ 0.6 & 1 & 0.4 \\ 0.3 & 0.3 & 1 \end{pmatrix} \tag{6.27}$$

$$\mathbf{Q} = \begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \tag{6.28}$$

6.5.2.1 Market Scenario 2 with implementation delay

Following the same procedure as in market scenario 1, we substitute matrices $\mathbf{A}$ and $\mathbf{Q}$ into equation (6.1). Then, analyzing the Replicator-Mutator model, we have that the characteristic equation for policy 1 is given by:

$$P_{rm2}(\psi,\tau) = \Psi^2 + 0.97\Psi + 0.23 + 0.01e^{-2\psi\tau} - 0.01\Psi e^{-\psi\tau} - 0.01e^{-\psi\tau} \tag{6.29}$$

Similarly, the characteristic equation for control policy 2 can be expressed as:

$$P_{rm2}(\psi,\tau) = \Psi^2 + 1.03\Psi + 0.27 + 0.01e^{-2\psi\tau} - 0.01\Psi e^{-\psi\tau} - 0.01e^{-\psi\tau} \tag{6.30}$$

Now, considering $\tau = 0$ equations (6.29) and (6.30) become:

$$P_{rm2}(\psi,0) = \Psi^2 + 0.82\Psi + 0.13 \tag{6.31}$$

$$P_{rm2}(\psi,0) = \Psi^2 + 1.08\Psi + 0.26 \tag{6.32}$$

From Proposition 9 it can be affirmed that the equilibrium point is stable.

Now, analyzing for $\tau > 0$, we substitute $\Psi = i\omega$ in (6.29) and (6.30). Thereby, solving the corresponding equations, we have that the characteristic equation has no positive root. Hence, based on Proposition 10 it is concluded that the market scenario 2 with implementation delay has no Hopf bifurcation.

6.5.2.2 Market Scenario 2 with adoption delay

As in the previous case, we substitute matrices $\mathbf{A}$ and $\mathbf{Q}$ into (6.3) and the characteristic equation for policy 1 is:

$$P_{rm2}(\psi,\tau) = \Psi^2 + 1.24\Psi + 0.39 + 0.02e^{-2\psi\tau} - 0.28\Psi e^{-\psi\tau} - 0.17e^{-\psi\tau} \tag{6.33}$$

Equivalently for control policy 2, the characteristic equation can be written as:

$$P_{rm2}(\psi,\tau) = \Psi^2 + 1.42\Psi + 0.50 + 0.03e^{-2\psi\tau} - 0.39\Psi e^{-\psi\tau} - 0.28e^{-\psi\tau} \tag{6.34}$$

Now, analyzing for $\tau = 0$, equations (6.23) and (6.24) are given by:

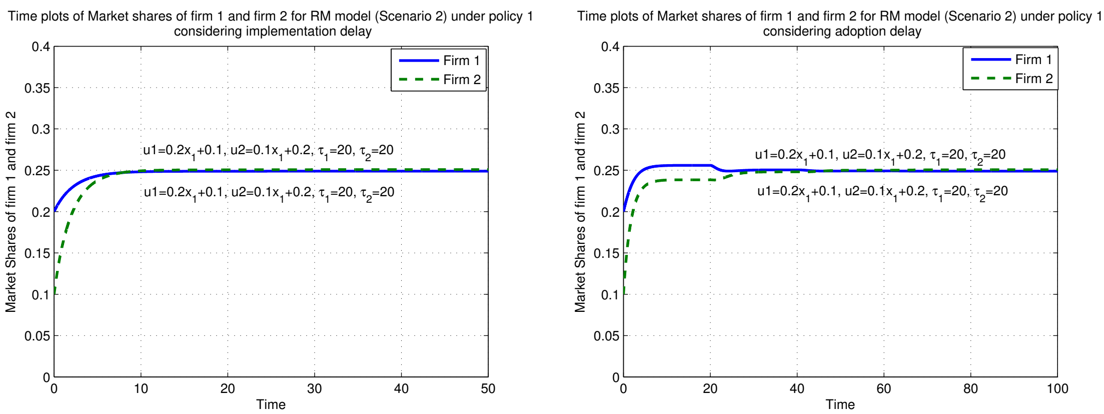
$$P_{rm2}(\psi,0) = \Psi^2 + 0.82\Psi + 0.13 \tag{6.35}$$

$$P_{rm2}(\psi,0) = \Psi^2 + 1.08\Psi + 0.26 \tag{6.36}$$

Therefore, according to Proposition 9 it can be said that the equilibrium point is stable.

Then, considering now $\tau > 0$, and subtituting $\Psi = i\omega$ in (6.33) and (6.34) we can solve the respective equations. Thus, it is noted that the characteristic equation has no positive root. Accordingly, from Proposition 10 follows that market scenario 2 with adoption delay has no Hopf bifurcation.

Figure 6.3 shows the numerical results for market scenario 2 under affine control policy 1 considering the two types of delays defined in equations (6.1) and (6.3). In figure 6.3(a) the dynamics of the model considering the presence of implementation delay is presented. Next, figure 6.3(b) displays the model presuming the presence of adoption delay. Examining the figures, it is possible to confirm Proposition 9 and Proposition 10 formulated concerning the stability of the equilibrium points for both types of delays. Moreover, comparing the figures, we note that the stabilizaton time is larger in the case of the adoption delay.



(a) (b)

Figure 6.3: Evolution of the market shares of firms 1 and 2 for market scenario 2 under policy 1 with (a) implementation delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$ and (b) adoption delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$.

In figure 6.4 the dynamics of the market shares of the firms for market scenario 2 under affine control policy 2 assuming the two types of delays defined in equations (6.1) and (6.3) for initial conditions $x_1(0) = 0.2$ and $x_2(0) = 0.1$ is shown. Figure 6.4(a) illustrates the model with the presence of implementation delay and figure 6.4(b) describes the model suppposing the existence of adoption delay. Similar to the case of control policy 1 the figures indicate that the equilibrium point is stable regardless of the type of delay and that in the case of adoption delay the stabilization time is larger.

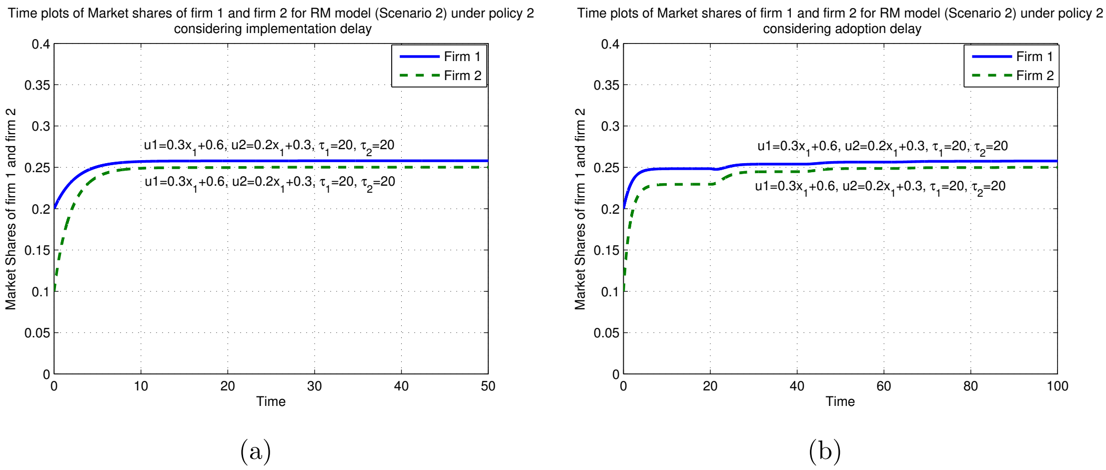


Figure 6.4: Evolution of the market shares of firms 1 and 2 for market scenario 2 under policy 2 with (a) implementation delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$ and (b) adoption delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$.

6.5.3 Market Scenario 3 (Operating system market or Football team market)

To finalize the numerical study of the Replicator-Mutator model with delays, in this instance we consider a market scenario that represents a type of market where the clients have a high predilection and loyalty for the products of the chosen company. The preference matrix $\mathbf{A}$ and the mutation matrix $\mathbf{Q}$ for this type of market are:

$$\mathbf{A} = \begin{pmatrix} 1 & 0.8 & 0.2 \\ 0.8 & 1 & 0.2 \\ 0.4 & 0.4 & 1 \end{pmatrix} \tag{6.37}$$

$$\mathbf{Q} = \begin{pmatrix} 0.8 & 0.04 & 0.16 \\ 0.04 & 0.8 & 0.16 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \tag{6.38}$$

6.5.3.1 Market Scenario 3 with implementation delay

Similarly to the steps utilized in the previous market scenarios, we substitute the expressions of matrices **A** and **Q** into equation (6.1). Then analyzing the Replicator-Mutator model, we have that the characteristic equation for policy 1 is given by:

$$P_{rm3}(\psi,\tau) = \varPsi^2 + 0.36\varPsi + 0.2 + 0.01e^{-2\psi\tau} - 0.01\varPsi e^{-\psi\tau} - 0.01e^{-\psi\tau} \tag{6.39}$$

In like manner, for control policy 2 the characteristic equation is:

$$P_{rm3}(\psi,\tau) = \varPsi^2 + 0.39\varPsi + 0.04 + 0.01e^{-2\psi\tau} - 0.02\varPsi e^{-\psi\tau} - 0.01e^{-\psi\tau} \tag{6.40}$$

Then, considering $\tau = 0$ equations (6.39) and (6.40) are represented as:

$$P_{rm3}(\psi,0) = \varPsi^2 + 0.35\varPsi + 0.2 \tag{6.41}$$

$$P_{rm3}(\psi,0) = \varPsi^2 + 0.37\varPsi + 0.4 \tag{6.42}$$

Thus, according to Proposition 9 it can be concluded that equilibrium point is stable.

Next, analyzing for $\tau > 0$, we substitute $\varPsi = i\omega$ in (6.39) and (6.40). So, solving the corresponding equations, it can be noted that the characteristic equation has no positive root. Therefore, from Proposition 10 it is concluded that the market scenario 3 has no Hopf bifurcation.

6.5.3.2 Market Scenario 3 with adoption delay

Similar to all previous cases, firstly the expressions of matrices **A** and **Q** are replaced into (6.3). Then evaluating the Replicator-Mutator model, we can write the characteristic equation for policy 1:

$$P_{rm3}(\psi,\tau) = \varPsi^2 + 1.21\varPsi + 0.36 + 0.18e^{-2\psi\tau} - 0.85\varPsi e^{-\psi\tau} - 0.52e^{-\psi\tau} \tag{6.43}$$

In the same way for control policy 2, we can express the characteristic equation as:

$$P_{rm3}(\psi,\tau) = \varPsi^2 + 2.08\varPsi + 1.01 + 0.66e^{-2\psi\tau} - 1.71\varPsi e^{-\psi\tau} - 1.64e^{-\psi\tau} \tag{6.44}$$

Now, assuming $\tau = 0$ equations (6.43) and (6.44) become:

$$P_{rm3}(\psi,0) = \varPsi^2 + 0.36\varPsi + 0.02 \tag{6.45}$$

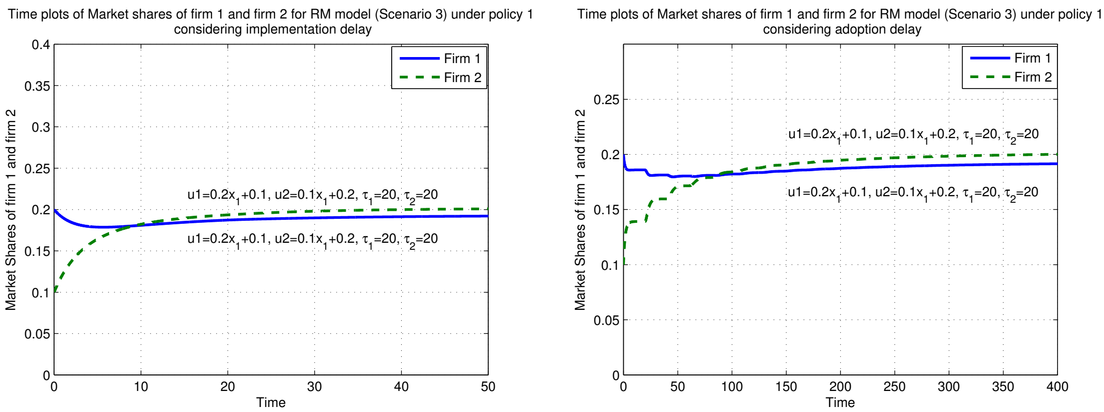
$$P_{rm3}(\psi,0) = \varPsi^2 + 0.37\varPsi + 0.03 \tag{6.46}$$

Hence, from Proposition 9 it can be said that the equilibrium point is stable.

Then, analyzing for $\tau > 0$, we substitute $\varPsi = i\omega$ in (6.43) and (6.44). Solving the respective equations it is observed that the characteristic equation has no positive root. Accordingly, based on Proposition 10 it is determined that the market scenario 3 has no

Hopf bifurcation.

Figure 6.5 presents the numerical results for market scenario 3 under affine control policy 1 supposing the existence of both types of delays formulated in equations (6.1) and (6.3). Figure 6.5(a) shows the dynamics of the market considering the presence of implementation delay and figure 6.5(b) illustrates the dynamics of the market presuming the existence of adoption delay. Observing the figures, it is possible to confirm as in all previous cases the stability of the equilibrium points and the larger stabilization time for adoption delay.



(a) (b)

Figure 6.5: Evolution of the market shares of firms 1 and 2 for market scenario 3 under policy 1 with (a) implementation delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$ and (b) adoption delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$.

Finally, figure 6.6 shows the evolution of the market shares of the firms for market scenario 3 under affine control policy 2 considering the two types of delays defined in equations (6.1) and (6.3). In figure 6.6(a) the evolution of the market shares assuming the presence of implementation delay is shown and in figure 6.6(b) the dynamics of the market shares supposing the presence of adoption delay is illustrated. Similar to all the analyzed cases, the figures allow verifying Proposition 9 and Proposition 10 formulated, which is that the equilibrium points are stable regardless of the type of delay and that stabilization time in the adoption delay is larger.

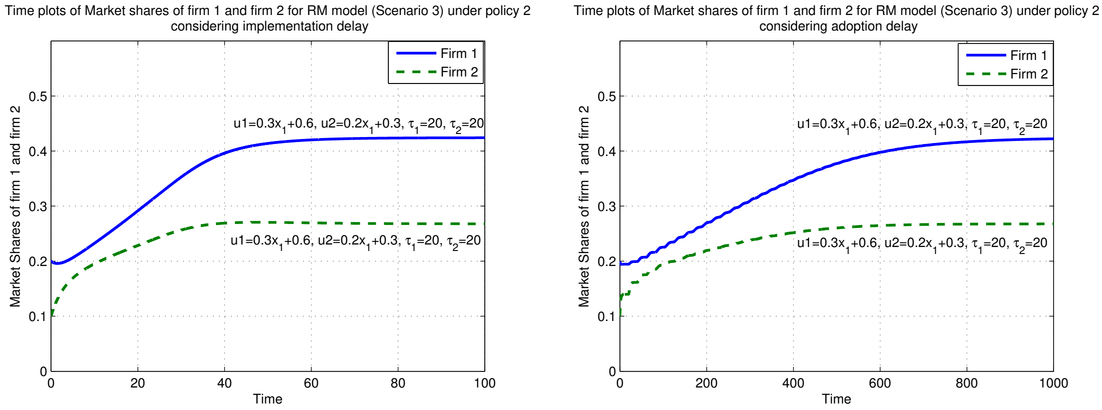


Figure 6.6: Evolution of the market shares of firms 1 and 2 for market scenario 3 under policy 2 with (a) implementation delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$ and (b) adoption delay with delay values $\tau_1 = 20$ and $\tau_2 = 20$.

6.6 The Replicator-Mutator model under affine advertising control policy considering unequal delay values

Likewise to the Vidale-Wolfe and extended Lanchester models from section 4.4 in this section numerical simulations for the Replicator-Mutator model considering unequal delay values for each firm are presented.

6.6.1 The Replicator-Mutator model under affine advertising control policy considering unequal delay values of implementation

Rewriting the Replicator-Mutator model considering the implementation delay we obtain

$$\dot{x}_i = \sum_{j=1}^{3} x_j f_{j\tau} q_{ji} - x_i \phi \tag{6.47}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} + u_{1\tau} & a_{13} \\ a_{21} + u_{2\tau} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{6.48}$$

where: $f_{j\tau} = f_j\,(t - \tau_j)$, $u_{1\tau} = k_1 x_{1\tau} + c_1$ and $u_{2\tau} = k_2 x_{2\tau} + c_2$

Assuming the condition of different delay values for each firm the expressions for $x_{1\tau}$ and $x_{2\tau}$ are given by: $x_{1\tau} = x_1\,(t - \tau_1)$ and $x_{2\tau} = x_2\,(t - \tau_2)$.

6.6.2 The Replicator-Mutator model under affine advertising control policy considering unequal delay values of adoption

Rewriting the Replicator-Mutator model assuming the adoption delay we have

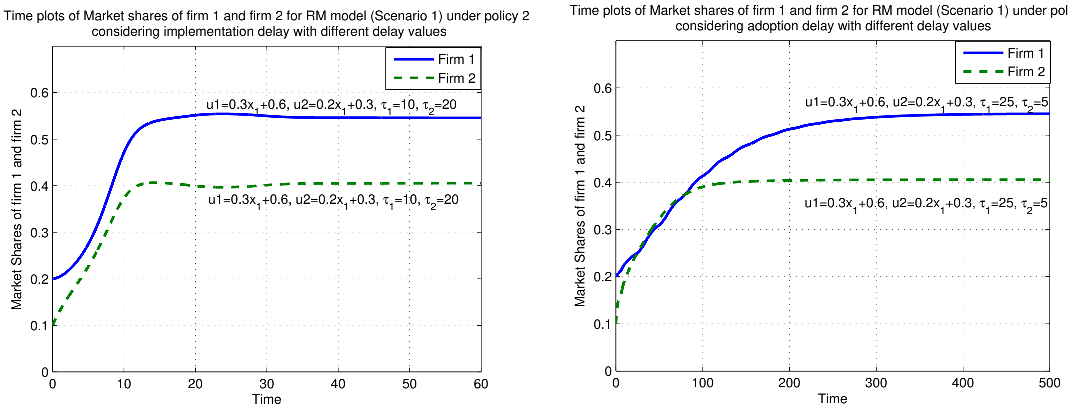
$$\dot{x}_i = \sum_{j=1}^{3} x_{j\tau} f_{j\tau} q_{ji} - x_i \phi \tag{6.49}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} + u_{1\tau} & a_{13} \\ a_{21} + u_{2\tau} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{6.50}$$

where: $x_{j\tau} = x_j\left(t - \tau_j\right)$, $f_{j\tau} = f_j\left(t - \tau_j\right)$, $u_{1\tau} = k_1 x_{1\tau} + c_1$ and $u_{2\tau} = k_2 x_{2\tau} + c_2$
Now, considering different delay values for each firm we get: $x_{1\tau} = x_1\left(t - \tau_1\right)$ and $x_{2\tau} = x_2\left(t - \tau_2\right)$.

Then, based on the assumptions cited, numerical simulations for each market scenario under both types of delays with different delay value for each company are presented. The control parameters for control policy 1 are given by $k_1 = 0.2$, $c_1 = 0.1$,$k_2 = 0.1$, $c_2 = 0.2$ and for the case of control policy 2 by $k_1 = 0.3$, $c_1 = 0.6$, $k_2 = 0.2$, $c_2 = 0.3$. Furthermore, the initial conditions in all cases are setting in $x_1(0) = 0.2$ and $x_2(0) = 0.1$.

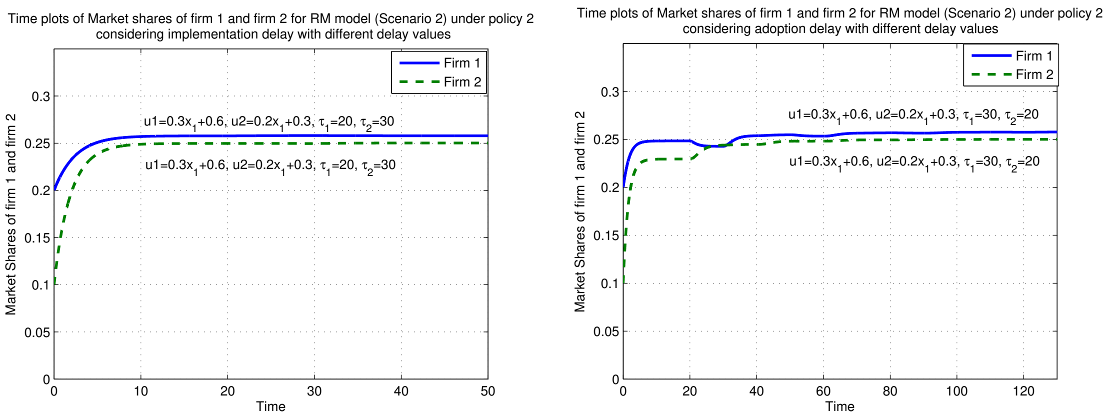
For market scenario 1, figure 6.7 shows the evolution of the market shares of firms 1 and 2 under affine control policy 2 for both types of delays. Specifically, figure 6.7(a) illustrates the case of implementation delay for $\tau_1 = 10$ and $\tau_2 = 20$ while figure 6.7(b) presents the case of adoption delay considering $\tau_1 = 25$ and $\tau_2 = 5$. From the figures, it can be said that the equilibrium point remains stable for both types of delays. Additionally, in the case of adoption delay, the existence of two crossings between the time responses of market shares is observed.



(a) (b)

Figure 6.7: Evolution of the market shares of firms 1 and 2 for market scenario 1 under policy 2 with (a) implementation delay with delay values $\tau_1 = 10$ and $\tau_2 = 20$ and (b) adoption delay with delay values $\tau_1 = 25$ and $\tau_2 = 5$.

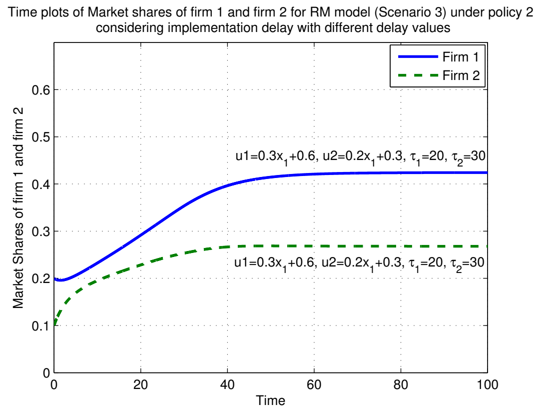
Next, for market scenario 2 figure 6.8 displays the evolution of the market shares of firms 1 and 2 under affine control policy 2 for both types of delays. Expressly, figure 6.8(a) presents the case of implementation delay for $\tau_1 = 20$ and $\tau_2 = 30$ while figure 6.8(b) shows the case of adoption delay considering $\tau_1 = 30$ and $\tau_2 = 20$. From the figures, we can declare that the equilibrium point remains stable for both types of delays. Moreover, it is noted in similar form with market scenario 1 that in the case of adoption delay there are two crosses between the trajectories of the market shares.



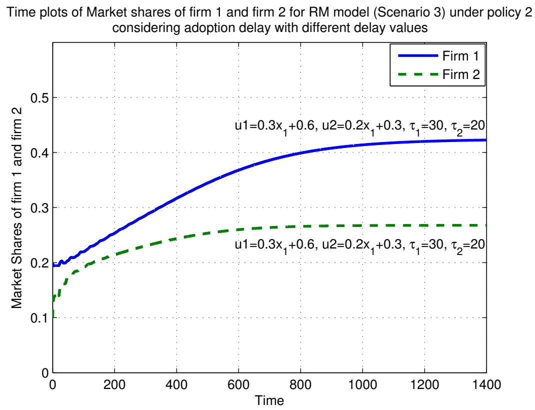
(a) (b)

Figure 6.8: Evolution of the market shares of firms 1 and 2 for market scenario 2 under policy 2 with (a) implementation delay with delay values $\tau_1 = 20$ and $\tau_2 = 30$ and (b) adoption delay with delay values $\tau_1 = 30$ and $\tau_2 = 20$.

Finally, for market scenario 3, figure 6.9 presents the evolution of the market shares of firms 1 and 2 under affine control policy 2 assuming both types of delays. Specifically , figure 6.9(a) shows the case of implementation delay for $\tau_1 = 20$ and $\tau_2 = 30$ while figure 6.9(b) illustrates the case of adoption delay considering $\tau_1 = 30$ and $\tau_2 = 20$. From the figures, it can be affirmed that equilibrium point remains stable for both types of delays. Furthermore, it is observed in both cases of delays there is no existence of crossing between the trajectories of the market shares.



(a)



(b)

Figure 6.9: Evolution of the market shares of firms 1 and 2 for market scenario 3 under policy 2 with (a) implementation delay with delay values $\tau_1 = 20$ and $\tau_2 = 30$ and (b) adoption delay with delay values $\tau_1 = 30$ and $\tau_2 = 20$.

6.7 Chapter conclusions

- The Replicator-Mutator model was formulated considering the existence of delays in the information available in the decision making of the firms and in the response of the clients to the advertising by the firms.
- The Replicator-Mutator model in the three market scenarios considered shows stability with regard to both types of delays, regardless of the market scenario studied, the control policy implemented, and the delay values.
- In the three market scenarios analyzed for Replicator-Mutator model with delays, it is observed that the time of stabilization for adoption delay is larger than the case of implementation delay.
- For both types of delays, and both control policies, the stabilization time of the Replicator-Mutator model considering delay for the 3 market scenarios analyzed is larger than the case of the Replicator-Mutator model without delay.

Chapter 7

Conclusions, Contributions and Future work

This last chapter summarizes the main conclusions and contributions of the thesis and indicates some possibilities for future work in the area.

7.1 Summary of thesis and concluding remarks

This thesis addressed the modeling of the dynamics of a duopolistic market with competition in advertising using the traditional Vidale-Wolfe, Lanchester and Replicator-Mutator models as starting points to propose new models. Chapter 3 presented an extension to the Lanchester model based on the hypothesis of the existence of a third population of undecided clients, in addition to the usual two associated to the two firms in the duopoly. The results showed that the extended Lanchester model allows modeling the duopolistic dynamics considering three populations of clients and explicitly modeling the process of competition between the firms. Chapter 3 also established the interesting result that, despite differences in the trajectories, under identical advertising policies, the final outcome in terms of equilibrium market share is the same for both Vidale-Wolfe and the proposed extended Lanchester models .

In Chapter 4 the existence of delays in the Vidale-Wolfe and extended Lanchester models was contemplated. Two types of delays in the models were considered, the delay in the available information (implementation delay) and the delay in the response of clients to advertising (adoption delay). The analysis showed that the Vidale-Wolfe and extended Lanchester models present stability under implementation delay. In the case of adoption delay, the models show the existence of Hopf bifurcation.

In Chapter 5 a new approach for modeling duopolistic dynamics based on the Replicator-Mutator equations was proposed. The Replicator-Mutator model makes use of preferences and loyalty of the clients to the firms allowing a more flexible modeling of duopolistic markets because it is possible to represent different scenarios or types of markets through the preference matrix $\mathbf{A}$ and the mutation matrix $\mathbf{Q}$.

Finally, in Chapter 6 the existence of delays in the Replicator-Mutator model was proposed, using implementation and adoption delays as in Chapter 4. The results found in the three market scenarios analyzed showed that the Replicator-Mutator model is stable in the presence of both types of delays. Furthermore, the results exhibited that the Replicator-Mutator model with adoption delay shows a stabilization time larger than the model with the same value of implementation delay.

The main contributions of this thesis are:

- Proposal of a model for extending for the Vidale-Wolfe and Lanchester models by explicitly modeling a set of undecided clients who can transition to the set of clients of either of the competing firms or remain undecided. This new set is also the destination of clients that are lost to either firm through the decay terms in conventional Vidale-Wolfe models and its introduction made possible the introduction of a new model (extended Lanchester) that has all the features of both Vidale-Wolfe and Lanchester models.

- Formulation of a new evolutionary model for duopolistic markets through the Replicator-Mutator equations that offers more flexibility in the characterization of existing behaviors in these markets.

- The analysis of the models containing delays that represent lags in availability of information used for feedback as well as delays in the response of clients to advertising.

For the reader's convenience, the main features of existing and proposed models are summarized in Table 7.1 and Table 7.2.

Model \ Features	UC	ECD	D	CD	ASEP
Vidale-Wolfe (3.10)	✓	X	X	X	✓
Lanchester (3.4)	X	✓	X	X	✓
Extended Lanchester (3.17)	✓	✓	✓	X	✓
Replicator-Mutator (5.13)	✓	✓	✓	✓	X

Table 7.1: Comparison between the duopoly models existing in the literature and those proposed in this thesis with regard to existence of undecided clients (UC), explicitly competitive dynamics (ECD), presence of delays (D), complex dynamics (CD) and analytical solutions for equilibrium points (ASEP), where X denotes non-existence and ✓denotes existence.

Model \ Delay type	Implementation delay	Adoption delay
Vidale-Wolfe (3.10)	X	✓
Extended Lanchester (3.17)	X	✓
Replicator-Mutator (5.13)	X	X

Table 7.2: Comparison between the models with delays in regard to the existence of Hopf bifurcations where X denotes non-existence and ✓denotes existence.

7.2 Future work

- Future research relating to the Vidale-Wolfe and extended Lanchester models should include the formulation of a topological conjugacy between the trajectories of the Vidale-Wolfe and extended Lanchester dynamical systems.

- Proposals for other advertising control policies and extensions to the models to represent targeted policies and the inclusion of other parameters in the modeling (such as price, quality) should also be considered.

- For the Replicator-Mutator model a systematic study of the dependence of the mutation matrix $\mathbf{Q}$ on the client's preference matrix $\mathbf{A}$ as well as of the consideration of the effect of advertising on the mutation matrix $\mathbf{Q}$ and on the mutation parameter μ. Other studies may involve the formulation and demonstration of general properties of the model regarding the existence of equilibrium points and their stability conditions.

- Development of analytical results for the case of unequal delay values for each firm and the discussion of the existence of the other types of delays (for example delay in the undecided population x_3).

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Appendix A

Vidale-Wolfe and extended Lanchester models

A.1 Equilibrium points for special cases of affine control

Policy	Control Parameters	Equilibrium Points	Conditions of Existence
P_1	$u_1 = c_1$,$u_2 = c_2$	$\left(\frac{c_1}{c_1+c_2+\lambda}, \frac{c_2}{c_1+c_2+\lambda}\right)$	Always exists
P_2	$u_1 = k_1 x_1$,$u_2 = c_2$	$\left(0, \frac{c_2}{c_2+\lambda}\right)$ $\left(\frac{k_1-c_2-\lambda}{k_1}, \frac{c_2}{k_1}\right)$	Always exists $k_1 > c_2 + \lambda$
P_3	$u_1 = k_1 x_1$,$u_2 = k_2 x_2$	$(0, 0)$ $\left(0, \frac{k_2-\lambda}{k_2}\right)$ $\left(\frac{k_1-\lambda}{k_1}, 0\right)$	Always exists $k_2 > \lambda$ $k_1 > \lambda$
P_4	$u_1 = k_1 x_1 + c_1$,$u_2 = c_2$	$\left(\frac{2c-p}{2c}, \frac{p}{2(c+\lambda)}\right)$ $\left(\frac{2c-q}{2c}, \frac{q}{2(c+\lambda)}\right)$	Negative$\Rightarrow$Never exists $c + \lambda < g < 3c + \lambda$
P_5	$u_1 = k_1 x_1 + c_1$,$u_2 = k_2 x_2$	$\left(\frac{-\lambda-f}{2k}, 0\right)$ $\left(\frac{-\lambda+f}{2k}, 0\right)$ $\left(\frac{k}{k_2-k}, \frac{2kk_2-k\lambda+k_2\lambda-k_2^2}{kk_2-k_2^2}\right)$	Negative$\Rightarrow$Never exists $\lambda < f < 2k + \lambda$ $k < \frac{k_2\lambda}{\lambda-k_2} \quad \wedge \quad k2 < \frac{\lambda}{2}$
P_6	$u_1 = k_1 x_1 + c_1$,$u_2 = k_2 x_2 + c_2$	$\left(\frac{-e-c-2f}{4k}, \frac{-e-c-2f}{4k}\right)$ $\left(\frac{-e-c+2f}{4k}, \frac{-e-c+2f}{4k}\right)$	Negative$\Rightarrow$Never exists $\frac{2c+\lambda-k}{2} < f < \frac{3k+\lambda+2c}{2}$

Table A.1: Equilibrium points and conditions of the control parameters for existence of the equilibrium points under special cases of affine control

Policy	Determinant for Vidale-Wolfe model	Trace for Vidale-Wolfe model
P_1	$\lambda (c_1 + c_2 + \lambda)$	$-c_1 - c_2 - 2\lambda$
P_2	$c_2k_1x_1 + c_2k_1x_2 + 2\lambda k_1x_1 + \lambda k_1x_2 + c_2\lambda - c_2k_1 + \lambda^2 - \lambda k_1$	$-2k_1x_1 - k_1x_2 - c_2 - 2\lambda + k_1$
P_3	$2k_1k_2x_1^2 + 4k_1k_2x_1x_2 + 2k_1k_2x_2^2 + 2\lambda k_1x_1 + \lambda k_1x_2 + \lambda k_2x_1 + 2\lambda k_2x_2 - 3k_1k_2x_1 - 3k_1k_2x_2 + \lambda^2 - \lambda k_1 - \lambda k_2 + k_1k_2$	$-2k_1x_1 - k_1x_2 - k_2x_1 - 2k_2x_2 - 2\lambda + k_1 + k_2$
P_4	$c^2x_1 + c^2x_2 + 2c\lambda x_1 + c\lambda x_2 - c^2 + c\lambda + \lambda^2$	$-2cx_1 - cx_2 - c - 2\lambda$
P_5	$2kk_2x_1^2 + 4kk_2x_1x_2 + 2kk_2x_2^2 + 2\lambda kx_1 + \lambda kx_2 + +\lambda k_2x_1 + +2\lambda k_2x_2 - 2kk_2x_1 - 2kk_2x_2 + \lambda^2 - \lambda k_2$	$-2kx_1 - kx_2 - k_2x_1 - 2k_2x_2 - 2\lambda + k_2$
P_6	$(kx_1 + kx_2 + \lambda - k)(2kx_1 + 2kx_2 + 2c + \lambda - k)$	$-3kx_1 - 3kx_2 - 2c - 2\lambda + 2k$

Table A.2: Expressions for determinants and traces in Vidale-Wolfe model (3.10) under special cases of affine control

Policy	Determinant for extended Lanchester model	Trace for extended Lanchester model
P_1	$(c_1 + c_2 + \lambda)^2$	$-2c_1 - 2c_2 - 2\lambda$
P_2	$(k_1x_1 + c_2 + \lambda)(2k_1x_1 + c_2 + \lambda - k_1)$	$-3k_1x_1 - 2c_2 - 2\lambda + k_1$
P_3	$2k_1^2x_1^2 + 4k_1k_2x_1x_2 + 2k_1^2x_2^2 + 3\lambda k_1x_1 + 3\lambda k_2x_2 - k_1^2x_1 - 2k_1k_2x_1 - 2k_1k_2x_2 - k_2^2x_2 + \lambda^2 - \lambda k_1 - \lambda k_2 + k_1k_2$	$-3k_1x_1 - 3k_2x_2 - 2\lambda + k_1 + k_2$
P_4	$(2cx_1 + c + \lambda)(cx_1 + 2c + \lambda)$	$-3cx_1 - 3c - 2\lambda$
P_5	$2k^2x_1^2 + 4kk_2x_1x_2 + 2k_2^2x_2^2 + 3\lambda kx_1 + 3\lambda k_2x_2 + 2k^2x_1 - 2kk_2x_1 + kk_2x_2 - k_2^2x_2 + \lambda^2 + \lambda k - \lambda k_2$	$-3kx_1 - 3k_2x_2 - 2\lambda - k + k_2$
P_6	$(2kx_1 + 2kx_2 + 2c + \lambda - k)(kx_1 + kx_2 + 2c + \lambda - k)$	$-3kx_1 - 3kx_2 - 4c - 2\lambda + 2k$

Table A.3: Expressions for determinants and traces in extended Lanchester model (3.17) under special cases of affine control

A.2 Particular Conditions for the control parameters in special cases of affine control

Policy	Particular control parameters	Expressions of Variables
P_4	$k_1 = c_1 = c_2 = c$	$p = 3c + \lambda + g$, $q = 3c + \lambda - g$ $g = \sqrt{5c^2 + 2c\lambda + \lambda^2}$
P_5	$k_1 = c_1 = k$	$f = \sqrt{\lambda^2 + 4k^2}$
P_6	$k_1 = k_2 = k$ $c_1 = c_2 = c$	$f = \sqrt{\frac{e^2+2ec+c^2+8kc}{4}}$ $e = c + \lambda - k$

Table A.4: Particular conditions for control parameters and expressions of variables under special cases of affine control used in section 3.3 to analyse the equilibria and stability of Vidale-Wolfe (3.10) and extended Lanchester models (3.17).

Appendix B

Wang's model of online advertising ecosystem

Wang's study [61] starts by defining the clients, the advertisers, and the website publishers as agents involved in an online advertising ecosystem exposed to complex interaction processes. Wang contemplates the modeling individual of the clients' choice about advertisers and modeling of the choice of advertisers concerning the website publishers. The model proposed by Wang [61] takes into account the dynamics of the clients and the advertisers, the preferences of the clients and the advertisers, the influence of advertising on the clients, the positions of the advertisers in a ranking of websites and the interaction between other advertisers. The model presented by Wang [61], characterizing the dynamics of clients and advertisers utilizing the Replicator-Mutator dynamics is expressed as follows:

$$\dot{\mathbf{x}}^u = [\mathbf{Q}^u]^T \mathbf{F}^u \mathbf{x}^u - \phi^u \mathbf{x}^u + \sum_{v=1}^{N} \lambda_{uv} \rho_{uv} \mathbf{y}^v - \sum_{v=1}^{N} \alpha^v \mathbf{y}^v \tag{B.1}$$

$$\dot{\mathbf{y}}^v = [\mathbf{P}^v]^T \mathbf{H}^v \mathbf{y}^v - \phi^v \mathbf{y}^v + \sum_{l=1}^{M} \gamma_{vl} \xi_{vl} \mathbf{y}^v - \beta^v \mathbf{y}^v \tag{B.2}$$

where N is the number of the clients, M is the number of the advertisers in the ecosystem, $\mathbf{x}^u$ are the states which represent the individual assignment of the clients to the firms, and $\mathbf{y}^v$ are the states which depict individual assignment of the advertisers to the ranking of websites, $\mathbf{Q}^u$ is the mutation matrix individual of the clients about the choice of the firms, $\mathbf{P}^v$ is the mutation matrix individual of the advertisers to the ranking of websites, $\mathbf{F}^u$ is the diagonal matrix of individual fitness associated with the clients concerning to firms, $\mathbf{H}^v$ is the diagonal matrix of individual fitness associated with the firms about the ranking of websites, ϕ^u is the individual average fitness associated with the clients, ϕ^v is the individual mean fitness associated with the firms, α^u is the total capacity of the clients, β^v is the total capacity of the advertisers, λ_{uv} is the link between clients u and v, ρ_{uv} is the influence of the client u about the client v, γ_{vl} is the influence of the advertiser v to the advertiser l, and ξ_{vl} is the link between advertisers v and l.

Equation (B.1) models the evolution of the individual clients' choice to advertisers by considering individual client's preferences and the advertising effect enforced by the advertisers. Thus, the first two expressions of equation (B.1) represent the individual clients' choice considering their preferences exclusively, while the last two parts of equation (B.1) describe the influence of the advertising done by the advertisers based on the position of a website publisher ranking. On the other hand, equation (B.2) represents the modeling of the dynamics of advertisers to the website publisher. In this case, the Qiuchen Wang model considers that the choice of the advertisers is determined by the individual preferences of the advertisers and by the effect of interaction with other advertisers. So, the first two expressions of equation (B.2) represent the choice of the advertisers only by their preferences whereas the last two expressions of equation (B.2) describe the influence of the other advertisers through the interaction in the ecosystem, taking into consideration the presence of the website publishers, which interconnect clients and advertisers.

Concerning the design of advertising strategies by advertisers, Wang's study [61] mainly develops a verbal description of possible strategies that advertisers could carry out to improve their relationship with the clients. The strategies addressed are essentially advertising strategies based on the acquisition of information and location of the clients.

Appendix C

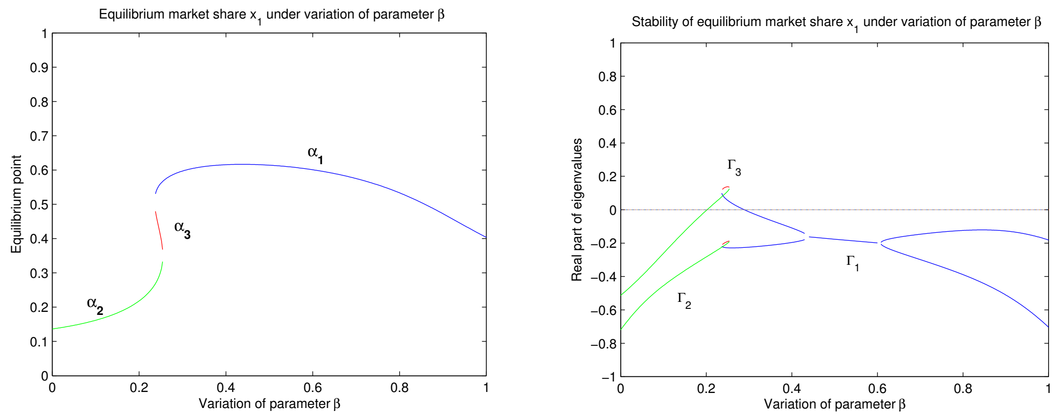
Parametric sensitivity of the equilibrium points under variation of population distribution parameter β

C.1 Variation of population distribution parameter β

In chapter 5 to simplify the analysis of the existence and stability of the equilibrium points, the particular condition of the existence of two strategies with an equal fraction of the population was considered. Now, in this section we examine the variation in parameter β from equations (5.17) and (5.18). Thereby, equation (5.19) is given by:

$$\begin{aligned} 0 = \alpha^3\left(2.4\beta - 1.6\beta^2 - 1.6\right) + \alpha^2\left(3.408\beta^2 - 4.544\beta + 2.32\right) \\ + \alpha\left(2.292\beta - 1.02 - 2.016\beta^2\right) + 0.208\beta^2 - 0.148\beta + 0.1 \end{aligned} \tag{C.1}$$

Figure C.1 shows that equilibrium point x_1 has no monotonic behavior under the whole interval of variations of parameter p_4. Thus, it is possible to observe the existence of three curves for x_1. Then, the first curve (green) exists for the interval $[0, 0.25]$. The second curve (red) exists for the interval $[0.24, 0.25]$ and the third curve (blue) exists for the interval $[0.24, 1]$. The first and third curve have monotonic increasing behavior, while the second curve has monotonic decreasing behavior. Now, about the stability of the equilibrium point figure C.1 displays that the equilibrium point is stable between the intervals $[0, 0.204]$ and $[0.286, 1]$, that is, the rest point x_1 is stable in stretches of the first (green) and third (blue) curve.



(a) (b)

Figure C.1: Parametric sensitivity under variation of parameter β: (a) locus of equilibrium point x_1 (b) eigenvalue plot showing local stability of equilibrium point x_1.

Appendix D

Hopf bifurcation in Vidale-Wolfe and extended Lanchester model

D.1 Hopf bifurcation caused by varying the control parameters

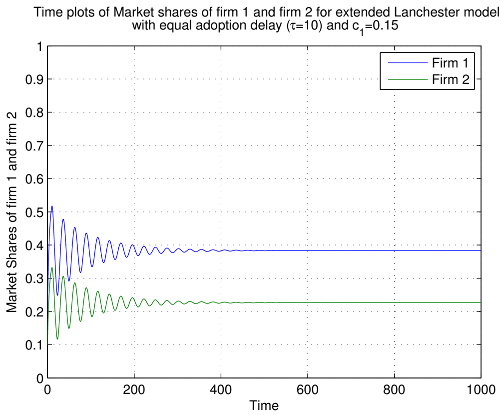
Rewriting the extended Lanchester model with adoption delay from Chapter 4, we have:

$$\begin{aligned}\dot{x}_1 &= u_{1\tau}\left(1 - x_{1\tau}\right) - u_{2\tau}x_{1\tau} - \lambda_1 x_1 \\ \dot{x}_2 &= u_{2\tau}\left(1 - x_{2\tau}\right) - u_{1\tau}x_{2\tau} - \lambda_2 x_2\end{aligned} \tag{D.1}$$

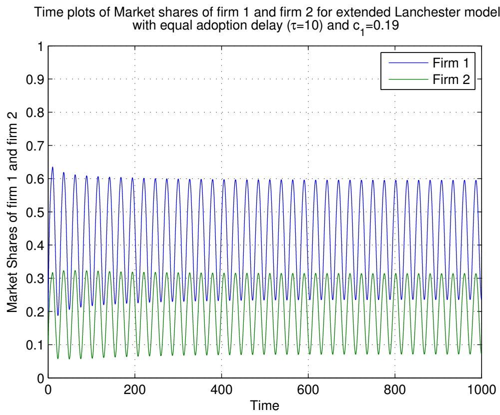
where:
$u_{1\tau} = k_1 x_{1\tau} + c_1 = k_1 x_1\left(t - \tau_1\right) + c_1$
$u_{2\tau} = k_2 x_{2\tau} + c_2 = k_2 x_2\left(t - \tau_2\right) + c_2$
Substituting the above expressions, model (D.1) can be expressed by:

$$\begin{aligned}\dot{x}_1 &= -x_{1\tau}^2 k_1 - x_{1\tau}x_{2\tau}k_2 - x_{1\tau}c_1 - x_{1\tau}c_2 + x_{1\tau}k_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau}x_{2\tau}k_1 - x_{2\tau}^2 k_2 - x_{2\tau}c_1 - x_{2\tau}c_2 + x_{2\tau}k_2 - \lambda x_2 + c_2\end{aligned} \tag{D.2}$$

In order to analyze the existence of Hopf bifurcation as a function of control parameters we consider the following values for numerical simulations: $x_1(0) = 0.2, x_2(0) = 0.1, \lambda = 0.25, k_1 = 0.25, k_2 = 0.2, c_2 = 0.1, \tau_1 = 10, \tau_2 = 10$. Figure D.1 shows the numerical results with c_1 being the bifurcation parameter. Figure D.1(a) presents the dynamics of the model for $c_1 = 0.15$ with oscillatory transient but stable dynamics meanwhile figure D.1(b) shows the dynamics of the model for $c_1 = 0.19$ with the existence of Hopf bifurcation.



(a)



(b)

Figure D.1: Evolution of the market shares of firms 1 and 2 for the extended Lanchester model with adoption delays for $k_1 = 0.25, k_2 = 0.2, c_2 = 0.1$ with (a) equal adoption delay value $\tau = 10$ and $c_1 = 0.15$ and (b) equal adoption delay value $\tau = 10$ and $c_1 = 0.19$. Note that for $c_1 = 0.19$ the model presents Hopf bifurcation, that is, variation of the control parameters can also lead to bifurcations.

Appendix E

Links to Maple code for the models proposed in this thesis

E.1 Maple code for the Vidale-Wolfe model

https://www.dropbox.com/s/ulx24q2gfs8labq/Vidale_Wolfe_Model.mw?dl=0

E.2 Maple code for the extended Lanchester model

https://www.dropbox.com/s/te1vvx73715efm9/Extended_Lanchester_Model.mw?dl=0

E.3 Maple code for the Vidale-Wolfe and extended Lanchester model with delays

https://www.dropbox.com/s/in3a00utheg5pcu/Vidale_Wolfe_extended_Lanchester_Models_with_Delays.mw?dl=0

E.4 Maple code for the Replicator-Mutator model

https://www.dropbox.com/s/5pt0lrhxeu6j17o/Replicator_Mutator_Model.mw?dl=0

E.5 Maple code for the Replicator-Mutator model with delays

E.5.1 Maple code for the Replicator-Mutator model with implementation delay

https://www.dropbox.com/s/t54nkrgjz1x5co8/Replicator_Mutador_Model_with_Implementation_Delay.mw?dl=0

E.5.2 Maple code for the Replicator-Mutator model with adoption delay

https://www.dropbox.com/s/lbssvzt4lcqrnkr/Replicator_Mutador_Model_with_Adoption_Delay.mw?dl=0