Relatório Técnico

Núcleo de Computação Eletrônica

A Generalization of the Helly Property Applied to the Cliques of a Graph

M. C. Dourado
F. Protti
J. L. Szwarcfiter

NCE - 01/02

Universidade Federal do Rio de Janeiro

A Generalization of the Helly Property Applied to the Cliques of a Graph

Mitre C. Dourado*
Fábio Protti†
Jayme L. Szwarcfiter‡

Abstract

Let $p \geq 1$ and $q \geq 0$ be integers. A family $\mathcal{S}$ of sets is (p,q)*-intersecting* when every subfamily $\mathcal{S}' \subseteq \mathcal{S}$ formed by p or less members has total intersection of cardinality at least q. A family $\mathscr{F}$ of sets is (p,q)*-Helly* when every (p,q)-intersecting subfamily $\mathscr{F}' \subseteq \mathscr{F}$ has total intersection of cardinality at least q. A graph G is a (p,q)*-clique-Helly graph* when its family of cliques (maximal complete sets) is (p,q)-Helly. According to this terminology, the usual Helly property and the clique-Helly graphs correspond to the case $p = 2, q = 1$.

In this work we present characterizations for (p,q)-Helly families of sets and (p,q)-clique-Helly graphs. For fixed p, q, those characterizations lead to polynomial-time

*Universidade Federal do Rio de Janeiro, COPPE, Caixa Postal 68511, 21945-970, Rio de Janeiro, RJ, Brasil. Partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, Brasil. E-mail: mitre@cos.ufrj.br

†Universidade Federal do Rio de Janeiro, NCE, Caixa Postal 2324, 20001-970, Rio de Janeiro, RJ, Brasil. Partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, and Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro - FAPERJ, Brasil. E-mail: fabiop@nce.ufrj.br

‡Universidade Federal do Rio de Janeiro, Instituto de Matemática, NCE and COPPE, Caixa Postal 2324, 20001-970, Rio de Janeiro, RJ, Brasil. Partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, and Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro - FAPERJ, Brasil. E-mail: jayme@nce.ufrj.br

recognition algorithms. When p or q is not fixed, it is shown that the recognition of (p, q)-clique-Helly graphs is NP-hard.

We also extend further the notions presented, by defining the (p, q, r)-Helly property (which holds when every (p, q)-intersecting subfamily $\mathscr{F}' \subseteq \mathscr{F}$ has total intersection of cardinality at least r) and giving a way of recognizing (p, q, r)-Helly families in terms of the (p, q)-Helly property.

Keywords: Clique-Helly Graphs, Helly Property, Intersecting Sets

1 Introduction

A well known result by Helly published in 1923 [4, 11] states that if there are given n convex subsets of a d-dimensional euclidean space with $n > d$ and if each family formed by $d + 1$ of the subsets has a point in common, then there exists a common point of the n subsets.

This result inspired the definition of the "Helly property" for families of sets in general, a concept that has been extensively studied in many contexts (see e.g. [7]). We say that a family $\mathscr{F}$ of sets *has the Helly property* (or *is Helly*) when every subfamily $\mathscr{F}' \subseteq \mathscr{F}$ of pairwise intersecting sets has non-empty total intersection.

When the family of cliques of a graph G satisfies the Helly property, we say that G is a *clique-Helly* graph (cfr. [9]). Clique-Helly graphs were characterized via the notion of *extended triangles* [8, 15]. An extended triangle of a graph G is an induced subgraph of G formed by a triangle T together with the vertices which form a triangle with at least one edge of T.

Theorem 1 [8, 15] *G is a clique-Helly graph if and only if every of its extended triangles contains a universal vertex.*

The above characterization leads to a straightforward recognition algorithm for clique-Helly graphs with time complexity $O(\ (\ |V(G)| + t(G)\ )\ |E(G)|\ )$, where $t(G)$ is the number of triangles of G.

We may think of a more general "p-Helly property", which holds when every $\mathscr{F}' \subseteq \mathscr{F}$ of p-wise intersecting sets has non-empty total intersection. Thus, the original result of Helly may be restated by simply saying that any family of convex subsets of a d-dimensional euclidean space is $(d+1)$-Helly.

The p-Helly property has been studied in the context of hypergraphs [2, 3]. In fact, this concept is equivalent to the *Helly number*. A family $\mathscr{F}$ of sets has Helly number p if, for all $\mathscr{F}' \subseteq \mathscr{F}$, $\bigcap_{S \in \mathscr{F}'} S = \emptyset$ implies that there exist p sets $S_1, S_2, \ldots, S_p \in \mathscr{F}'$ such that $S_1 \cap S_2 \cap \ldots \cap S_p = \emptyset$. For instance, any family of paths of a tree has Helly number 2 (see [1], p. 399). It is clear that a family of sets is p-Helly if and only if it has Helly number p. In [12], the Helly number is defined as the minimum p for which $\mathscr{F}$ is p-Helly, and it is shown that the Helly number of the m-convex sets of any connected graph G equals the clique number of G. In [10], a stronger notion is introduced: $\mathscr{F}$ is said to have *strong Helly number p* if, for all $\mathscr{F}' \subseteq \mathscr{F}$, there exist p sets $S_1, S_2, \ldots, S_p \in \mathscr{F}'$ such that $S_1 \cap S_2 \cap \ldots \cap S_p = \bigcap_{S \in \mathscr{F}'} S$. In the same work, it has been shown that the family of cliques of an EPT graph (the edge intersection graph of a family of paths in a tree) has strong Helly number 4.

In this work we propose a new direction in which the p-Helly property can be generalized, by requiring that the subfamilies $\mathscr{F}' \subseteq \mathscr{F}$ satisfy the following property:

"if every group of p members of $\mathscr{F}'$ have q elements in common, then $\mathscr{F}'$ has total intersection of cardinality at least q."

This leads naturally to the formal definition of the *(p, q)-Helly property*, as we shall see in Section 2, where we give a characterization for (p, q)-Helly families of sets. For fixed integers p and q, this characterization leads to a recognition algorithm whose time complexity is polynomial on the size of the family. Still in Section 2, we consider a slightly generalized form of this property, called the *(p, q, r)-Helly property*. A family $\mathscr{F}$ is said to be *(p, q, r)-Helly* when, for every $\mathscr{F}' \subseteq \mathscr{F}$, if every group of p members of $\mathscr{F}'$ have q

elements in common, then $\mathscr{F}'$ has total intersection of cardinality at least r. We describe a characterization of (p, q, r)-Helly families in terms of the (p, q)-Helly property.

In Section 3, we study the (p, q)-Helly property applied to the case of the family of cliques of a graph. We say that a graph G is (p, q)-*clique-Helly* when its family of cliques is (p, q)-Helly. We show some examples and properties of (p, q)-clique-Helly graphs and give a characterization for them by means of the $(p + 1)$-*expansions* of the intersection graph of the complete sets with size q. The definition of p-expansion is a generalization of the definition of extended triangle.

Since the number of cliques of a graph G may be exponential on the size of G [13], the recognition algorithm for (p, q)-Helly families of sets cited in Section 2 cannot be applied in general to the cliques of G in order to obtain a polynomial method for deciding whether G is (p, q)-clique-Helly, in the case where p and q are fixed. However, the characterization of (p, q)-clique-Helly graphs given in Section 3 does lead to a polynomial recognition algorithm for fixed p and q, as we remark in Section 4. We also show in Section 4 that, when p or q is not fixed, recognizing (p, q)-clique-Helly graphs is NP-hard.

Finally, in Section 5 we propose some questions concerning the (p, q, r)-Helly property.

In what follows, we give some definitions and notation. Let G be a graph. A vertex $w \in V(G)$ is a *universal vertex* when w is adjacent to every other vertex of G. If $S \subseteq V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A subgraph H of G is a *spanning subgraph* of G when $V(H) = V(G)$. A *complete* is a subset of pairwise adjacent vertices. A *clique* is a maximal complete.

If S is a set, then $|S|$ denotes the cardinality of S.

The *universe* $\text{Univ}(\mathscr{F})$ of a family $\mathscr{F}$ of sets is defined as the union of its members: $\text{Univ}(\mathscr{F}) = \cup_{S \in \mathscr{F}} S$. The *total intersection* $\text{Int}(\mathscr{F})$ of a family $\mathscr{F}$ of sets is defined as $\text{Int}(\mathscr{F}) = \cap_{S \in \mathscr{F}} S$. A *core* of a family $\mathscr{F}$ of sets is any subset contained in $\text{Int}(\mathscr{F})$.

We say that S is a *q-set* when $|S| = q$, a q^-*-set* when $|S| \leq q$, and a q^+-*set* when $|S| \geq q$. This notation will also be applied to other terms used throughout this work: families, cores, completes and cliques.

2 The Generalized Helly Property

In this section, we first define the (p,q)-Helly property for families of sets in general. This definiton is a generalization of the usual Helly property, which corresponds to the case $p = 2, q = 1$. We also provide a characterization for a family to be (p,q)-Helly. As we shall see, for fixed p and q, this characterization leads to a recognition algorithm whose time complexity is polynomial on the size of the family.

Next, we extend further these notion by defining the (p,q,r)-Helly property, and we study a way of recognizing (p,q,r)-Helly families in terms of the (p,q)-Helly property.

2.1 (p,q)-Helly families of sets

Definition 2 *Let $p \geq 1$ and $q \geq 0$ be integers, and let $\mathscr{F}$ be a family of sets. We say that $\mathscr{F}$ is (p,q)-intersecting when every p^--subfamily $\mathscr{F}' \subseteq \mathscr{F}$ has a q^+-core.*

The following proposition lists some immediate consequences of the above definition:

Proposition 3

(i) *For all* $p \geq 1$ *and* $\mathscr{F}$, $\mathscr{F}$ *is* $(p,0)$*-intersecting.*
(ii) *For all* $p > 1$, *if* $\mathscr{F}$ *is* (p,q)*-intersecting then* $\mathscr{F}$ *is* $(p-1,q)$*-intersecting.*
(iii) *For all* $q > 0$, *if* $\mathscr{F}$ *is* (p,q)*-intersecting then* $\mathscr{F}$ *is* $(p,q-1)$*-intersecting.*

□

We remark that, for itens (ii) and (iii) above, the converse is not true in general.

Definition 4 *Let* $p \geq 1$ *and* $q \geq 0$ *be integers, and let* $\mathscr{F}$ *be a family of sets. We say that* $\mathscr{F}$ *satisfies the* (p,q)-Helly property *when every* (p,q)*-intersecting subfamily* $\mathscr{F}' \subseteq \mathscr{F}$ *has a* q^+*-core. In this case, we also say that* $\mathscr{F}$ *is* (p,q)-Helly.

The next proposition is also easy to proof:

Proposition 5

(i) *For all* $p \geq 1$ *and* $\mathscr{F}$, $\mathscr{F}$ *is* $(p,0)$*-Helly.*
(ii) *For all* $p > 1$, *if* $\mathscr{F}$ *is* $(p-1,q)$*-Helly then* $\mathscr{F}$ *is* (p,q)*-Helly.*
(iii) *For all* $q > 0$, *if* $\mathscr{F}$ *is* $(p,q-1)$*-Helly then* $\mathscr{F}$ *is* (p,q)*-Helly.* □

The following lemma will be useful for the characterization of (p,q)-Helly families of sets.

Lemma 6 *Let* $p \geq 1$ *and* $q \geq 0$ *be integers,* $\mathscr{Q}$ *a* $(p+1)$*-family of* q*-subsets of* U, *and* $\mathscr{F}$ *a* p^-*-family of sets over* U *such that every member of* $\mathscr{F}$ *contains at least* p *members of* $\mathscr{Q}$. *Then* $\mathscr{F}$ *has a* q^+*-core.*

Proof. Consider the bipartite graph $G = (\mathcal{Q} \cup \mathscr{F}, E)$ where there exists an edge (Q, S) in E, for $Q \in \mathcal{Q}$ and $S \in \mathscr{F}$, if and only if S contains Q. Since every $S \in \mathscr{F}$ contains at least p members of $\mathcal{Q}$, we have $p|\mathscr{F}| \leq |E|$.

Assume by contradiction that $\mathscr{F}$ does not have a q^+-core. This means that there is no q-subset Q of U such that every $S \in \mathscr{F}$ contains Q. In particular, no $Q \in \mathcal{Q}$ can be contained in all the members of $\mathscr{F}$. This means that every $Q \in \mathcal{Q}$ is contained in at most $|\mathscr{F}| - 1$ members of $\mathscr{F}$. Then $|E| \leq (p+1)(|\mathscr{F}| - 1)$.

By combining the two inequalities obtained above, we have $|\mathscr{F}| \geq p + 1$, a contradiction. Therefore, the lemma holds. □

The case $q = 1$ in the above lemma has been proved in the context of hypergraphs [2].

Since any family of q^+-sets is $(1, q)$-intersecting, it is easy to see that a family $\mathscr{F}$ is $(1, q)$-Helly if and only if the subfamily formed by the q^+-sets of $\mathscr{F}$ has a q^+-core.

Now let us deal with the case $p > 1$. The following theorem presents a characteriztion for (p, q)-Helly families of sets in such a case:

Theorem 7 *Let $p > 1$ and $q \geq 0$ be integers, and let $\mathscr{F}$ be a family of sets. Then $\mathscr{F}$ is (p, q)-Helly if and only if for every $(p+1)$-family $\mathcal{Q}$ of q-subsets of Univ($\mathscr{F}$), the subfamily $\mathscr{F}'$ formed by the members of $\mathscr{F}$ that contain at least p members of $\mathcal{Q}$ has a q^+-core.*

Proof.

($\Rightarrow$) Suppose that $\mathscr{F}$ is (p, q)-Helly and there exists a $(p+1)$-family $\mathcal{Q}$ of q-subsets of Univ($\mathscr{F}$) such that the subfamily $\mathscr{F}'$ formed by the members of $\mathscr{F}$ that contain at least p members of $\mathcal{Q}$ does not have a q^+-core.

Consider a p^--subfamily $\mathscr{F}'' \subseteq \mathscr{F}'$. By Lemma 6, $\mathscr{F}''$ has a q^+-core. Therefore, $\mathscr{F}'$ is (p,q)-intersecting. Since $\mathscr{F}$ is (p,q)-Helly, we conclude that $\mathscr{F}'$ has a q^+-core. This is a contradiction. Hence, the necessity holds.

($\Leftarrow$) Assume by contradiction that $\mathscr{F}$ is not (p,q)-Helly. Let $\mathscr{F}' = \{S_1, \ldots, S_k\}$ be a minimal (p,q)-intersecting subfamily of $\mathscr{F}$ which does not have a q^+-core. Clearly, $k > p$.

By the minimality of $\mathscr{F}'$, the subfamily $\mathscr{F}' \backslash S_i$ has a q-core Q_i, for $i = 1, \ldots, k$. It is clear that $Q_i \not\subseteq S_i$.

Let $\mathcal{Q} = \{Q_1, \ldots, Q_{p+1}\}$. Let $\mathscr{F}'' \subseteq \mathscr{F}$ formed by the members of $\mathscr{F}$ that contain at least p members of $\mathcal{Q}$. Since $k > p > 1$, every member of $\mathscr{F}'$ contains at least p members of $\mathcal{Q}$. Consequently, $\mathscr{F}' \subseteq \mathscr{F}''$. By hypothesis, $\mathscr{F}''$ has a q^+-core. Therefore, $\mathscr{F}'$ has a q^+-core. This is a contradiction. Hence, the sufficiency holds. □

By setting $q = 1$, we obtain as a corollary of the above theorem the characterization of k-Helly hypergraphs described in [3].

If $|\text{Univ}(\mathscr{F})| = n$, then the number of $(p+1)$-families of q-subsets of $\text{Univ}(\mathscr{F})$ is $O(n^{q(p+1)})$. Hence, for fixed integers $p > 1$ and $q > 0$, Theorem 7 implies that deciding whether $\mathscr{F}$ is (p,q)-Helly can be done in polynomial time on the size of $\mathscr{F}$.

2.2 (p,q,r)-Helly families of sets

Definition 8 *Let $p \geq 1$, $q \geq 0$, $r \geq 0$ be integers, and let $\mathscr{F}$ be a family of sets. We say that $\mathscr{F}$ satisfies the* (p,q,r)-Helly property *when every (p,q)-intersecting subfamily $\mathscr{F}' \subseteq \mathscr{F}$ has an r^+-core. In this case, we also say that $\mathscr{F}$ is* (p,q,r)-Helly.

The above definition has some direct consequences, listed below without proof:

Proposition 9

(i) *For all* $p \geq 1$ *and* $q \geq 0$, $\mathscr{F}$ *is* (p,q)*-Helly if and only if* $\mathscr{F}$ *is* (p,q,q)*-Helly.*
(ii) *For all* $p \geq 1$, $q \geq 0$ *and* $\mathscr{F}$, $\mathscr{F}$ *is* $(p,q,0)$*-Helly.*
(iii) *For all* $p > 1$, *if* $\mathscr{F}$ *is* $(p-1,q,r)$*-Helly then* $\mathscr{F}$ *is* (p,q,r)*-Helly.*
(iv) *For all* $q > 0$, *if* $\mathscr{F}$ *is* $(p,q-1,r)$*-Helly then* $\mathscr{F}$ *is* (p,q,r)*-Helly.*
(v) *For all* $r > 0$, *if* $\mathscr{F}$ *is* (p,q,r)*-Helly then* $\mathscr{F}$ *is* $(p,q,r-1)$*-Helly.*
(vi) *For all* $q,r \geq 0$, $\mathscr{F}$ *is* $(1,q,r)$*-Helly if and only if the subfamily formed by the* q^+*-sets of* $\mathscr{F}$ *has an* r^+*-core.* □

We describe now a characterization of (p,q,r)-Helly families of sets in terms of the (p,q)-Helly property.

Let $p \geq 1$ and $q \geq r \geq 0$ be integers, and let $\mathscr{F}$ be a family of sets. Denote by $X = \{X_1, \ldots, X_{|X|}\}$ the collection of the (p,r)-intersecting subfamilies of $\mathscr{F}$ which are *not* (p,q)-intersecting. Let $I = \{1, 2, \ldots, |X|\}$. For each $F_j \in \mathscr{F}$, denote $I(F_j) = \{i \in I \mid F_j \in X_i\}$. For $i,k \in I$, represent by R_i an r-set formed by chosen elements that satisfy $R_i \cap R_k = \emptyset$ for $i \neq k$ and $R_i \cap \text{Univ}(\mathscr{F}) = \emptyset$. The *augmentation of* $\mathscr{F}$ *relative to* (q,r) is a family $\mathscr{A}$ of sets, obtained from $\mathscr{F}$, as follows. For each $\mathscr{F}_j \in \mathscr{F}$, the corresponding member of $\mathscr{A}$ is $A_j = \mathscr{F}_j \cup (\bigcup_{i \in I(F_j)} R_i)$.

Theorem 10 *Let* $p \geq 1$ *and* $q \geq r \geq 0$ *be integers. A family* $\mathscr{F}$ *of sets is* (p,q,r)*-Helly if and only if the augmentation of* $\mathscr{F}$ *relative to* (q,r) *is* (p,r)*-Helly.*

Proof. Let $\mathscr{F}$ be a (p,q,r)-Helly family of sets. Denote by $\mathscr{A}$ its augmentation relative to (q,r). We show that $\mathscr{A}$ is (p,r)-Helly. Let $\mathscr{A}'$ be a (p,r)-intersecting subfamily of $\mathscr{A}$. Denote by $\mathscr{F}'$ the subfamily of $\mathscr{F}$ formed by the members of $\mathscr{F}$ corresponding to those of $\mathscr{A}'$. We know that $\mathscr{F}'$ must be

(p, r)-intersecting as well. If $\mathscr{F}'$ is (p, q)-intersecting, then $\text{Int}(\mathscr{F}') = \text{Int}(\mathscr{A}')$. Because $\mathscr{F}$ is (p, q, r)-Helly we conclude that $\mathscr{A}'$ has an r^+-core. On the other hand, it follows from the definition of $\mathscr{A}$ that if $\mathscr{F}'$ is not (p, q)-intersecting then $\text{Int}(\mathscr{A}')$ contains an r-set R_i. Consequently, $\mathscr{A}$ is indeed (p, r)-Helly.

Conversely, by hypothesis the augmentation $\mathscr{A}$ of $\mathscr{F}$ relative to (q, r) is (p, r)-Helly. Let $\mathscr{F}'$ be a (p, q)-intersecting subfamily of $\mathscr{F}$. Denote by $\mathscr{A}'$ the subfamily of $\mathscr{A}$ whose sets correspond to those of $\mathscr{F}'$. It follows that $\mathscr{A}'$ is also (p, q)-intersecting, hence (p, r)-intersecting. Because $\mathscr{F}'$ is (p, q)-intersecting, it also follows that $\text{Int}(\mathscr{F}') = \text{Int}(\mathscr{A}')$. Since $\mathscr{A}$ is (p, r)-Helly, we conclude that $\mathscr{F}'$ has an r^+-core. Consequently, $\mathscr{F}$ is (p, q, r)-Helly. □

3 (p, q)-clique-Helly Graphs

3.1 Definition and Examples

We start this section by applying the concepts of the previous section to the family of cliques of a graph:

Definition 11 *Let $p \geq 1$ and $q \geq 0$ be integers, and let G be a graph. We say that G is a (p, q)*-clique-Helly graph *when its family of cliques is (p, q)-Helly.*

In the remainder of this work, we will assume that $p \geq 2$ and $q \geq 1$, unless differently mentioned.

It is clear that $(p - 1, q)$-clique-Helly graphs form a subclass of (p, q)-clique-Helly graphs. The example below shows other relations between classes of (p, q)-clique-Helly graphs:

Example 12 Define the graph $G_{p,q}$ in the following way: $V(G_{p,q})$ is formed by a $(q-1)$-complete Q, a p-complete $Z = \{z_1, \ldots, z_p\}$, and a p-independent set $W = \{w_1, \ldots, w_p\}$. Moreover, there exist the edges (z_i, w_j), for $i \neq j$, and the edges (q, x), for $q \in Q$ and $x \in Z \cup W$. Figure 1 depicts a scheme of the graph $G_{p,q}$, where a dashed line between z_i and w_i means $(z_i, w_i) \notin E(G_{p,q})$.

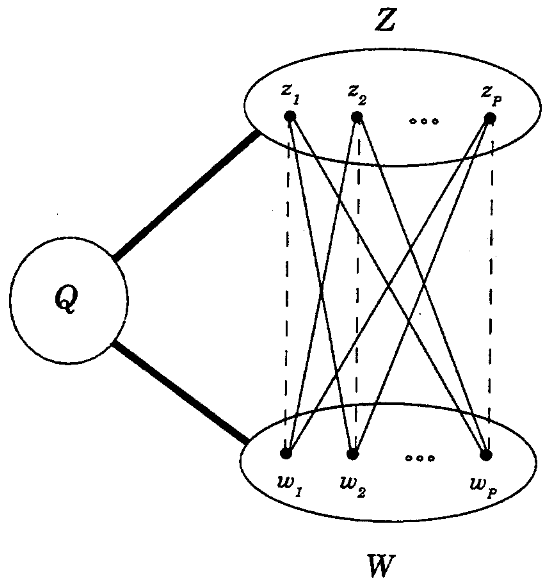

Figure 1: The graph $G_{p,q}$.

The family of cliques of the graph $G_{p,q}$ contains exactly $p+1$ members, each one of size $p+q-1$: $Q \cup \{z_1, \ldots, z_p\}$ and $Q \cup (Z \backslash \{z_i\}) \cup \{w_i\}$, for $1 \leq i \leq p$.

Observe that $G_{p,q}$ is (p,q)-clique-Helly, but it is not $(p-1,q)$-clique-Helly. Therefore, $G_{p,q}$ is (t,q)-clique-Helly for $t \geq p$, and it is not (t,q)-clique-Helly for $t < p$.

Moreover, $G_{p+1,q}$ is not (p,q)-clique-Helly, but it is (p,t)-clique-Helly for any $t \neq q$. Consequently, for distinct q and t, (p,q)-clique-Helly graphs and (p,t)-clique-Helly graphs are incomparable classes. □

It is possible to give a first characterization for (p,q)-clique-Helly graphs, as a direct consequence of Theorem 7:

Observation 13 *A graph G is (p,q)-clique-Helly if and only if for each clique C of G and for every $(p+1)$-family $\mathcal{Q}$ of q-completes contained in C, the subfamily of cliques of G that contain at least p members of $\mathcal{Q}$ has a q^+-core.*

However, the "characterization" above does not lead in general to a polynomial-time recognition algorithm for (p,q)-clique-Helly graphs, since the number of cliques of G may be exponential on the size of G. We will present in the next subsection a more useful characterization for (p,q)-clique-Helly graphs.

Define a graph G to be K_r-free when the size of the maximum clique of G is at most $r-1$. An interesting fact derived from Definition 11 is that every $K_{(p+q)}$-free graph is (p_1,q_1)-clique-Helly for $p_1 \geq p$ and $q_1 \geq q$. In order to prove this fact, we need first the following lemma:

Lemma 14 *Let $\mathcal{Q}$ be a $(p+1)$-family of q-completes of a graph G. If every member of $\mathcal{Q}$ is contained in a same $(p+q-1)^-$-complete of G, then the cliques of G that contain at least p members of $\mathcal{Q}$ have a q^+-core.*

Proof. Let $\mathcal{Q}$ be a $(p+1)$-family of q-completes contained in a $(p+q-1)^-$-complete C, and let $\mathcal{F}$ be the subfamily of cliques of G that contain at least p members of $\mathcal{Q}$. Observe that if a vertex x of C belongs to two members of $\mathcal{Q}$, then x belongs to all the cliques of $\mathcal{F}$. We will show that there exist at least q vertices in C belonging simultaneously to at least two members of $\mathcal{Q}$, which proves the lemma.

Suppose the contrary. Thus at most $q-1$ vertices of C belong simultaneously to more than one member of $\mathcal{Q}$. Assume initially that $|C| = p+q-1$. Then at least $p+q-1-(q-1) = p$ vertices of C have the property of belonging to exactly one member of $\mathcal{Q}$. Let X be the set formed by such vertices, where $|X| = p+r, 0 \leq r \leq q-1$. Observe that every member of $\mathcal{Q}$ must contain at least $r+1$ vertices belonging to X. This implies $|X| \geq (p+1)(r+1) = p+r+pr+1 > pr$, a contradiction.

If C contains strictly less than $p+q-1$ vertices, the same argument above can be used. □

We remark that the above lemma holds not only for the family of cliques of a graph, but also for families of sets in general.

Theorem 15 *Let G be a $K_{(p+q)}$-free graph. Then G is (p_1, q_1)-clique-Helly for all $p_1 \geq p$ and $q_1 \geq q$.*

Proof. Let $p_1 \geq p$ and $q_1 \geq q$. By Observation 13, we have to prove that for every (p_1+1)-family $\mathcal{Q}$ of q_1-completes contained in a same clique of G, the subfamily $\mathscr{F}$ of cliques of G that contain at least p_1 members of $\mathcal{Q}$ must have a q_1^+-core.

Since G is $K_{(p_1+q_1)}$-free, it follows that every member of $\mathcal{Q}$ is contained in a same $(p_1+q_1-1)^-$-complete of G. By Lemma 14, $\mathscr{F}$ has a q_1^+-core, as desired. □

3.2 Characterizing (p,q)-clique-Helly Graphs

In order to give an useful characterization for (p,q)-clique-Helly graphs, we need some further definitions and lemmas, presented in the sequel.

Definition 16 [15] *Let $\mathscr{F}$ be a subfamily of cliques of G. The* clique subgraph induced by $\mathscr{F}$ in G, *denoted by $G[\mathscr{F}]_c$, is the subgraph of G formed exactly by the vertices and edges belonging to the cliques of $\mathscr{F}$.*

Definition 17 *Let G be a graph, and let C be a p-complete of G. The* p-expansion relative to C *is the subgraph of G induced by the vertices w such that w is adjacent to at least $p-1$ vertices of C.*

We remark that the p-expansion for $p = 2$ has been used for characterizing clique-Helly graphs [8, 15]. It is clear that constructing a p-expansion relative to a given p-complete C can be done in polynomial time, for a fixed p.

Lemma 18 *Let G be a graph, C a p-complete of it, H the p-expansion of G relative to C, and $\mathscr{C}$ the subfamily of cliques of G that contain at least $p-1$ vertices of C. Then $G[\mathscr{C}]_c$ is a spanning subgraph of H.*

Proof. We have to show that $V(G[\mathscr{C}]_c) = V(H)$. Let $v \in V(H)$. Then v is adjacent to at least $p-1$ vertices of C. Hence, v together with those $p-1$ vertices form a p-complete, which is contained in a clique that contains at least $p-1$ vertices of C. Therefore, $v \in V(G[\mathscr{C}]_c)$. Now, consider $v \in V(G[\mathscr{C}]_c)$. Then v belongs to some clique containing $p-1$ vertices of C. That is, v is adjacent to at least $p-1$ vertices of C, and hence $v \in V(H)$. Consequently, $V(G[\mathscr{C}]_c) = V(H)$. Furthermore, both H and $G[\mathscr{C}]_c$ are subgraphs of G, but H is induced. Thus $E(G[\mathscr{C}]_c) \subseteq E(H)$. □

Definition 19 *Let G be a graph. The graph $\Phi_q(G)$ is defined in the following way: the vertices of $\Phi_q(G)$ correspond to the q-completes of G, two vertices being adjacent in $\Phi_q(G)$ if the corresponding q-completes in G are contained in a common clique.*

Observe that $\Phi_q(G)$ can be constructed in polynomial time, for a fixed q. We also remark that Φ_q is precisely the operator $\Phi_{q,2q}$, studied in [14]. An interesting property of Φ_q is that it preserves the subfamily of cliques of G containing at least q vertices:

Lemma 20 (Clique Preservation Property) *Let G be a graph. Then there exists a bijection between the subfamily of q^+-cliques of G and the family of cliques of $\Phi_q(G)$.*

Proof. Let C be a q^+-clique of G, and let $c = |C|$. Consider all the q-completes of G contained in $V(C)$. These sets clearly correspond to a $\binom{c}{q}$-complete C' of $\Phi_q(G)$. Assume that C' is not maximal. Then there exists $x \in V(\Phi_q(G))$, $x \notin V(C')$, such that x is adjacent to all the vertices of C'. But x corresponds to a q-complete Q of G such that for every q-complete $Q_1 \subseteq V(C)$, both Q and Q_1 are contained in a same q^+-clique of G. This implies that every vertex v of Q is adjacent to every vertex $w \neq v$ of C. Since $x \notin V(C')$, Q must necessarily contain at least one vertex not belonging to C. In other words, C is not maximal, a contradiction. Hence, C' is a clique of $\Phi_q(G)$.

Conversely, let C' be a clique of $\Phi_q(G)$ and $\mathscr{F}$ be the family of q-completes of G corresponding to the vertices of C'. Since any two vertices of C' are adjacent, any two completes of $\mathscr{F}$ are contained in a same q^+-clique of G. Hence, the union of the q-completes of $\mathscr{F}$ is a q^+-complete C of G.

Suppose by contradiction that C is not maximal. Thus, there exists a vertex $u \notin C$ which is adjacent to all the vertices of C. Consider $v_1, v_2, ..., v_{q-1} \in C$. It is clear that $Q = \{u, v_1, v_2, ..., v_{q-1}\}$ is a q-complete of G, and for every Q_1 in $\mathscr{F}$, both Q and Q_1 are contained in a same q^+-clique of G. Since $u \notin C$, $Q \notin \mathscr{F}$, and this means that Q corresponds to a vertex $x \in V(\Phi_q(G))$ such that $x \notin C'$ and x is adjacent to all the vertices of C'. This implies that C' is not maximal, a contradiction. □

The graph $\Phi_2(G)$ is the *edge clique graph* of G, introduced in [5], where the validity of the Clique Preservation Property was shown to that case.

The following definition is possible due to the Clique Preservation Property:

Definition 21 *Let G be a graph. If C is a q^+-clique of G, denote by $\Phi_q(C)$ the clique that corresponds to C in $\Phi_q(G)$. If C' is a clique of $\Phi_q(G)$, denote by $\Phi_q^{-1}(C')$ the q^+-clique that corresponds to C' in G. If $\mathscr{F}$ is a subfamily of q^+-cliques of G, define $\Phi_q(\mathscr{F}) = \{\Phi_q(C) \mid C \in \mathscr{F}\}$. If $\mathscr{C}$ is a subfamily of cliques of $\Phi_q(G)$, define $\Phi_q^{-1}(\mathscr{C}) = \{\Phi_q^{-1}(C) \mid C \in \mathscr{C}\}$.*

Lemma 22 *Let G be a graph, $\mathscr{F}$ a subfamily of q^+-cliques of it, $\mathscr{C} = \Phi_q(\mathscr{F})$, and $H = \Phi_q(G)$. Then $H[\mathscr{C}]_c$ contains a universal vertex if and only if $G[\mathscr{F}]_c$ contains q universal vertices.*

Proof. If $H[\mathscr{C}]_c$ contains a universal vertex x, then every clique of $\mathscr{F}$ contains the q-complete of G that corresponds to x, that is, $G[\mathscr{F}]_c$ contains q universal vertices. Conversely, if $G[\mathscr{F}]_c$ contains q universal vertices forming a q-complete Q of G, then every clique of $\mathscr{C}$ contains the vertex of H that corresponds to Q, that is, $H[\mathscr{C}]_c$ contains a universal vertex. □

Lemma 23 *Let C be a $(p+1)$-complete of a graph G, and let $\mathscr{C}$ be a p^--subfamily of cliques of G such that every clique of $\mathscr{C}$ contains at least p vertices of C. Then $\mathscr{C}$ has a 1^+-core.*

Proof. This lemma is an easy consequence of Lemma 6, by setting $q = 1$, $U = V(G)$, $\mathcal{Q} = \{\{w\} \mid w \in V(C)\}$, and $\mathscr{F} = \mathscr{C}$. □

Now we are able to present a characterization for (p, q)-clique-Helly graphs. The cases $p = 1$ and $p > 1$ will be dealt with separately, as in Section 2.

Theorem 24 *Let G be a graph, and let W be the union of the q^+-cliques of G. Then G is a $(1, q)$-clique-Helly graph if and only if $G[W]$ contains q universal vertices.*

Proof.

($\Rightarrow$) Assume that G is a $(1, q)$-clique-Helly graph. Consider the subfamily $\mathscr{F}$ of the cliques of G formed by the q^+-cliques only.

If $w \in W$, then w clearly belongs to a q^+-clique of G. This implies that $w \in V(G[\mathscr{F}]_c)$. On the other hand, if $w' \in V(G[\mathscr{F}]_c)$, then w' belongs to a

q^+-clique of G, and therefore $w' \in W$. This shows that $G[\mathscr{F}]_c$ is a spanning subgraph of $G[W]$.

Since $\mathscr{F}$ is $(1,q)$-intersecting by hypothesis, it has a q^+-core. This means that $G[\mathscr{F}]_c$ contains (at least) q universal vertices. Hence, $G[W]$ contains q universal vertices.

($\Leftarrow$) Assume that $G[W]$ contains q universal vertices forming a q-complete Q. Let $\mathscr{F} = \{C_1, \ldots, C_k\}$ be a $(1,q)$-intersecting subfamily of cliques of G. Then $|C_i| \geq q$, that is, every $w \in C_i$ is contained in a q-complete of G, for $i = 1, \ldots, k$. This implies that every C_i is an induced subgraph of $G[W]$. Therefore, every $u \in Q$ is adjacent to all the vertices of $C_i \backslash \{u\}$. By the maximality of C_i, it contains all the vertices $u \in Q$, for $i = 1, \ldots, k$. Hence, $\mathscr{F}$ has a q^+-core, as required. $\square$

Theorem 25 *Let $p > 1$ be an integer. A graph G is a (p,q)-clique-Helly graph if and only if every $(p+1)$-expansion of $\Phi_q(G)$ contains a universal vertex.*

Proof.

($\Rightarrow$) Suppose that G is a (p,q)-clique-Helly graph and there exists a $(p+1)$-expansion T, relative to a $(p+1)$-complete C of $\Phi_q(G)$, such that T contains no universal vertex.

Let $\mathscr{C}$ be the subfamily of cliques of $H = \Phi_q(G)$ that contain at least p vertices of C. Let $\mathscr{F} = \Phi_q^{-1}(\mathscr{C})$. Consider a p^--subfamily $\mathscr{F}' \subseteq \mathscr{F}$. Let $\mathscr{C}' = \Phi_q(\mathscr{F}')$. By Lemma 23, $\mathscr{C}'$ has a 1^+-core. That is, $H[\mathscr{C}']_c$ contains a universal vertex. This implies, by Lemma 22, that $G[\mathscr{F}']_c$ contains q universal vertices. Thus, $\mathscr{F}'$ has a q^+-core, that is, $\mathscr{F}$ is (p,q)-intersecting. Since G is (p,q)-clique-Helly, we conclude that $\mathscr{F}$ has a q^+-core and $G[\mathscr{F}]_c$ contains q universal vertices. By using Lemma 22 again, $H[\mathscr{C}]_c$ contains a universal vertex. Moreover, by Lemma 18, $H[\mathscr{C}]_c$ is a spanning subgraph of T. However, T contains no universal vertex. This is a contradiction. Therefore, every $(p+1)$-expansion of $H = \Phi_q(G)$ contains a universal vertex.

($\Leftarrow$) Assume by contradiction that G is not (p,q)-clique-Helly. Let $\mathscr{F} = \{C_1,\dots,C_k\}$ be a minimal (p,q)-intersecting subfamily of cliques of G which does not have a q-core. Clearly, $k > p$.

By the minimality of $\mathscr{F}$, the subfamily $\mathscr{F} \backslash C_i$ has a q^+-core Q_i, for $i = 1,\dots,k$. It is clear that $Q_i \not\subseteq C_i$. Moreover, every two distinct Q_i, Q_j are contained in a same clique, since $k \geq 3$. Hence the sets $Q_1, Q_2, \dots, Q_{p+1}$ correspond to a $(p+1)$-complete C in $\Phi_q(G)$.

Let $\mathscr{C}$ be the subfamily of cliques of $H = \Phi_q(G)$ that contain at least p vertices of C. Let $\mathscr{C}' = \Phi_q(\mathscr{F}) = \{\Phi_q(C_1),\dots,\Phi_q(C_k)\}$. Since every $C_i \in \mathscr{F}$ contains at least p sets from $Q_1, Q_2, \dots, Q_{p+1}$, it is clear that the clique $\Phi_q(C_i)$ of H contains at least p vertices of C. Therefore, $\Phi_q(C_i) \in \mathscr{C}$, for $i = 1,\dots,k$.

Let T be the $(p+1)$-expansion of H relative to C. By Lemma 18, $H[\mathscr{C}]_c$ is a spanning subgraph of T. Therefore, $V(Q) \subseteq V(T)$, for every $Q \in \mathscr{C}$. In particular, $V(\Phi_q(C_i)) \subseteq V(T)$, for $i = 1,\dots,k$. By hypothesis, T contains a universal vertex x. Then x is adjacent to all the vertices of $\Phi_q(C_i) \backslash \{x\}$, for $i = 1,\dots,k$. This implies that $\Phi_q(C_i)$ contains x, otherwise $\Phi_q(C_i)$ would not be maximal. Thus, $\mathscr{C}'$ has a 1^+-core and $H[\mathscr{C}']_c$ contains a universal vertex. By Lemma 22, $G[\mathscr{F}]_c$ contains q universal vertices, that is, $\mathscr{F}$ has a q^+-core. This contradicts the assumption for $\mathscr{F}$. Hence, G is a (p,q)-clique-Helly graph. □

4 Complexity Aspects

Let p and q be fixed positive integers. If $p = 1$, testing whether the union of the q^+-cliques of G contains q universal vertices (Theorem 24) can be easily done in polynomial time. If $p > 1$, testing the existence of a universal vertex in every $(p+1)$-expansion of $\Phi_q(G)$ (Theorem 25) can also be done in polynomial time, since the number of such $(p+1)$-expansions is $O(|V(G)|^{q(p+1)})$. Thus:

Corollary 26 *For fixed positive integers p, q, there exists a polynomial time algorithm for recognizing (p, q)-clique-Helly graphs.* □

Now we will show that when p or q is not fixed, the problem of deciding whether a given graph G is (p, q)-clique-Helly is NP-hard. We first recall the following NP-complete problems [6]:

SATISFIABILITY: Given a boolean expression $\mathscr{E}$ in the conjunctive normal form, is there a truth assignment for $\mathscr{E}$?

CLIQUE: Given a graph G and a positive integer k, is there a k^+-clique in G?

The NP-hardness of CLIQUE can be proved by a transformation from SATISFIABILITY (see [6]): given a boolean expression $\mathscr{E}$ with m clauses in the conjunctive normal form, construct the graph $\mathscr{G}(\mathscr{E})$ by defining a vertex for each occurrence of a literal in $\mathscr{E}$, and by creating an edge between two vertices if and only if the corresponding literals lie in distinct clauses and one is not the negation of the other. Moreover, set $k = m$. The following fact is easy to prove:

Fact 27 *The boolean expression $\mathscr{E}$ with m clauses in the conjunctive normal form is satisfiable if and only the graph $\mathscr{G}(\mathscr{E})$ contains an m-clique.*

Let us first show the NP-hardness proof when p is fixed and q is variable:

Theorem 28 *Let p be a fixed positive integer. Given a graph G and a positive integer q, the problem of deciding whether G is (p, q)-clique-Helly is NP-hard.*

Proof. Transformation from CLIQUE. Given a graph G and a positive integer k, construct the graph G' by adding $2p + 2$ new vertices forming a

$(p+1)$-complete $Z = \{z_1, z_2, \ldots, z_{p+1}\}$ and a $(p+1)$-independent set $W = \{w_1, w_2, \ldots, w_{p+1}\}$. Add the edges (z_i, w_j), for $i \neq j$, and the edges (v, u), for $v \in V(G)$ and $u \in Z \cup W$. The construction of G' is finished. Figure 2 shows the construction, where non-edges between Z and W are represented by dashed lines linking z_i to w_i.

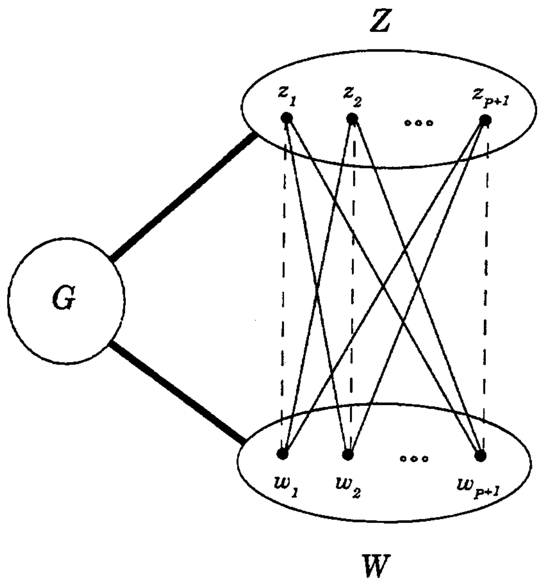

Figure 2: The graph G' for Theorem 28.

Define $q = k + 1$. We will show that G contains a $(q-1)$-clique if and only if G' is not (p, q)-clique-Helly. Assume first that G contains a $(q-1)$-clique C. Consider the following $p+1$ cliques of G':

$$C \cup \{w_j\} \cup (Z \backslash \{z_j\}), \text{ for } 1 \leq j \leq p+1.$$

These cliques are (p, q)-intersecting, but do not have a q^+-core. Therefore, G' is not (p, q)-clique-Helly.

Conversely, assume that the cliques of G have size at most $q - 2$. Since $G'[Z \cup W]$ is $K_{(p+2)}$-free, its cliques have size at most $(q-2) + (p+1) =$

$q + p - 1$, that is, G' is $K_{(p+q)}$-free. By Lemma 15, G' is (p, q)-clique-Helly, as desired. □

Now we prove the NP-hardness in the case where q is fixed and p is variable:

Theorem 29 *Let q be a fixed positive integer. Given a graph G and a positive integer p, the problem of deciding whether G is (p, q)-clique-Helly is NP-hard.*

Proof. Transformation from SATISFIABILITY. Given a boolean expression $\mathscr{E} = (\mathscr{E}_1, \ldots, \mathscr{E}_m)$ in the conjunctive normal form, let us construct a graph G'.

First, construct the graph $\mathscr{G}(\mathscr{E})$ described above in the transformation from SATISFIABILITY to CLIQUE. Define $\mathscr{V}_i$ as the subset of vertices of $V(\mathscr{G}(\mathscr{E}))$ corresponding to ocurrences of literals in clause $\mathscr{E}_i$, $1 \leq i \leq m$.

Next, add m new vertices, one for each $\mathscr{E}_i$, forming an m-independent set $W = \{w_1, w_2, ..., w_m\}$. For $i = 1, \ldots, m$, add the edges (w_i, v) where $v \in V(\mathscr{G}(\mathscr{E}))$ and $v \notin \mathscr{V}_i$.

Finally, add $q - 1$ new vertices forming a $(q-1)$-complete $Z = \{z_1, ..., z_{q-1}\}$, and add the edges (z, v), for $z \in Z$ and $v \in W \cup \mathscr{G}(\mathscr{E})$. The construction of G' is finished. Clearly, every vertex of Z is universal in G', and every clique of G' contains these $q-1$ vertices. Figure 3 shows a scheme of the construction, where the dashed lines mean that w_i is not adjacent to the vertices of $\mathscr{V}_i$, for $1 \leq i \leq m$.

Set $p = m-1$. We will show that $\mathscr{E}$ is satisfiable if and only if G' is not (p, q)-clique-Helly. Assume first that $\mathscr{E}$ is satisfiable. By Fact 27, $\mathscr{G}(\mathscr{E})$ contains a $(p+1)$-clique $C = \{v_1, v_2, ..., v_{p+1}\}$, where $v_j \in \mathscr{V}_j$. By the construction of G', it contains the $(p+q)$-cliques

$$C_i = (C \backslash \{v_j\}) \cup \{z_j\} \cup Z, \text{ for } 1 \leq j \leq p+1.$$

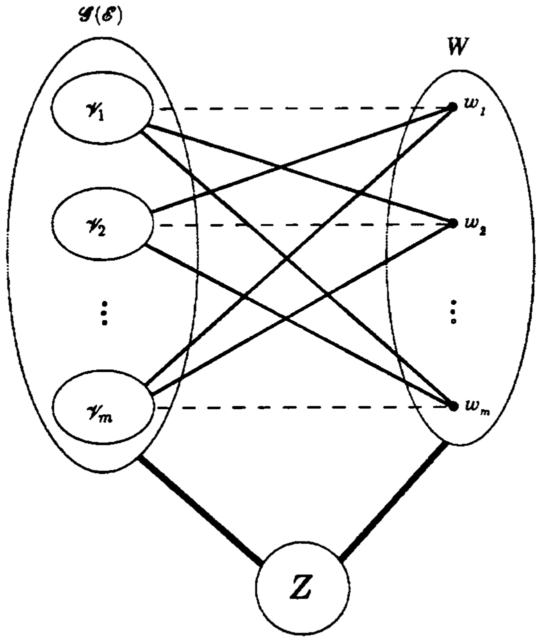

Figure 3: The graph G' for Theorem 29.

These $p+1$ cliques are (p,q)-intersecting, but do not have a q^+-core. Thus, G' is not (p,q)-clique-Helly.

Conversely, assume that $\mathscr{E}$ is not satisfiable. In this case, by Fact 27, $\mathscr{G}(\mathscr{E})$ is $K_{(p+1)}$-free. Thus, every clique of G' contains exactly a vertex of W, since for any p^--subset $S \subseteq V(\mathscr{G}(\mathscr{E}))$, there exists at least one vertex of W adjacent to all the vertices of S.

Let $\mathscr{Q}$ be a $(p+1)$-family of q-completes contained in a same clique of G', and let $\mathscr{F}$ be the the subfamily of cliques of G' that contain at least p members of $\mathscr{Q}$. By Observation 13, we need to prove that $\mathscr{F}$ has a q^+-core. (Recall that $\mathscr{F}$ has the $(q-1)$-core Z.)

If Univ($\mathscr{Q}$) is contained in a $(p+q-1)^-$-complete of G', Lemma 14 guarantees that $\mathscr{F}$ has a q^+-core, and nothing remains to prove. Hence, let us assume that Univ($\mathscr{Q}$) is a $(p+q)^+$-complete of G'.

Since $\mathscr{G}(\mathscr{E})$ is $K_{(p+1)}$-free, a maximum clique C of G' is of size at most $(q-1)+1+p = p+q$. Therefore, $\mathrm{Univ}(\mathscr{Q})$ is in fact a $(p+q)$-clique of G'.

Write $C = \mathrm{Univ}(\mathscr{Q})$. Then C is of the form $C = Z \cup \{w_k\} \cup P$, where $k \in \{1,\ldots,p+1\}$ and P is a p-complete contained in $V(\mathscr{G}(\mathscr{E}))$. It is clear that the ocurrences of literals corresponding to the vertices of P lie in distinct clauses of $\mathscr{E}$. This means that there is exactly one vertex $v \in P \cap \mathscr{V}_j$, for every $j \in \{1,\ldots,p+1\} \backslash \{k\}$. Thus, write $P = \{v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_{p+1}\}$, where $v_j \in \mathscr{V}_j$ for $j \in \{1,\ldots,p+1\} \backslash \{k\}$.

Let $v \in \{w_k\} \cup P$. If v belongs simultaneously to two members of $\mathscr{Q}$, then v belongs to all the members of $\mathscr{F}$. In other words, $Z \cup \{v\}$ is a q-core of $\mathscr{F}$, as desired. Therefore, it only remains to analyze the case in which

$$\mathscr{Q} = \{\ Z \cup \{v_j\} \mid 1 \le j \le p+1, j \ne k\ \} \cup \{\ Z \cup \{w_k\}\ \}.$$

In this case, let us show that w_k belongs to every member of $\mathscr{F}$. Suppose that some $C' \in \mathscr{F}$ does not contain w_k. Recall that C' contains a vertex $w_j, j \ne k$. Moreover, v_j is not adjacent to w_j. This implies that C' cannot contain the member of $\mathscr{Q}$ which v_j belongs to. Since C' does not contain w_k, C' can neither contain the member of $\mathscr{Q}$ which w_k belongs to. A contradiction arises, since C' should contain p members of $\mathscr{Q}$. Thus, w_k indeed belongs to every member of $\mathscr{F}$, and $Z \cup \{w_k\}$ is a q-core of $\mathscr{F}$, as desired. $\square$

From Theorems 28 and 29, we conclude:

Corollary 30 *The recognition of (p,q)-clique-Helly graphs, for p or q variable, is NP-hard.* $\square$

5 Some Questions

It remains open the question of deciding whether there exists a recognition algorithm for (p, q, r)-families of sets which is polynomial on the size of the input family, for fixed integers p, q and r.

Define a graph to be (p, q, r)-clique-Helly if its family of cliques is (p, q, r)-Helly. Another interesting question is to obtain a characterization for (p, q, r)-clique-Helly graphs that might possibly lead to a polynomial time recognition algorithm on the size of the input graph, for fixed p, q and r.

References

[1] C. Berge. *Graphs and Hypergraphs.* North-Holland, Amsterdam, 1973.

[2] C. Berge. *Hypergraphs.* Elsevier Science Publishers B. V., Amsterdam, 1989.

[3] C. Berge and P. Duchet.. A generalization of Gilmore's theorem. *Recent Advances in Graph Theory* (M. Fiedler, ed.), Acad. Praha, Prague 1975, 49–55.

[4] P. L. Butzer, R. J. Nessel, and E. L. Stark. Eduard Helly (1884-1943): In memoriam. *Resultate der Mathematik* 7 (1984).

[5] G. Chartrand, S. F. Kapoor, T. A. McKee, and F. Saba. Edge-clique graphs. *Graphs and Combinatorics* 7 (1991) 253–264.

[6] S. A. Cook. The complexity of theorem-proving procedures. Proc. 3rd Ann. ACM Symp. on Theory of Computing (1971), New York, 151-158.

[7] L. Danzer, B. Grünbaum, and V. L. Klee. Helly's theorem and its relatives. *Proc. Symp. on Pure Math AMS* Vol.7 (1963) 101–180.

[8] F. F. Dragan. *Centers of Graphs and the Helly Property* (in Russian). Doctoral Thesis, Moldava State University, Chisinău, 1989.

[9] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Academic Press, New York, 1980.

[10] M. C. Golumbic and R. E. Jamison. The edge intersection graphs of paths in a tree. *J. Comb. Theory. Series B* 38 (1985) 8–22.

[11] E. Helly. Ueber Mengen konvexer Koerper mit gemeinschaftlichen Punkter, Jahresber. *Math.-Verein.* 32 (1923) 175–176.

[12] R. E. Jamison and R. Nowakowski. A Helly theorem for convexity in graphs. *Discrete Mathematics* 51 (1984) 35–39.

[13] J. W. Moon and L. Moser. On cliques in graphs. *Israel J. Math.* 3 (1965) 23–28.

[14] E. Prisner. *Graph Dynamics.* Pitman Research Notes in Mathematics 338, Longman, London (1995).

[15] J. L. Szwarcfiter. Recognizing clique-Helly graphs. *Ars Combinatoria* 45 (1997) 29–32.