RELATÓRIO TÉCNICO

ON EDGE TRANSITIVITY OF DIRECTED GRAPHS

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ABSTRACT

We examine edge transitivity of directed graphs. The class of local comparability graphs is defined as the underlying graphs of locally edge transitive digraphs. The latter generalize edge transitive orientations, while local comparability graphs include comparability, anti-comparability and circle graphs. Recognizing local comparability graphs is NP-complete, however they are differences of comparability graphs. We define dimension so as to generalize that of an edge transitive digraph. Connected proper interval graphs are characterized as exactly the class of local comparability graphs of dimension one. Finally, a characterization of circle graphs is given also in terms of edge transitivity.

RESUMO

Examinamos transitividade em arestas de grafos direcionados. A classe dos grafos de comparabilidade local e' definida como os grafos subjacentes dos digrafos localmente transitivos em arestas. Estes ultimos generalizam orientacoes transitivas em arestas, enquanto que grafos de comparabilidade local incluem os de comparabilidade, anti-comparabilidade e circulares. Reconhecer grafos de comparabilidade local e' NP-completo, contudo eles constituem diferencas de grafos de comparabilidade. Definimos dimensao de modo a generalizar a de um digrafo transitivo em arestas. Os grafos conexos de intervalo proprio sao caracterizados exatamente como a classe dos de comparabilidade local de dimensao um. Finalmente, uma caracterizacao dos grafos circulares e' apresentada em termos de transitividade em arestas.

1. Introduction

We examine a special orientation of an undirected graph, called locally edge transitive, a generalization of edge transitive orientation. For simplicity, we write instead, locally transitive and transitive orientation, respectively. The graphs admiting such an orientation form the class of local comparability graphs. They include comparability, anti-comparability and circle graphs. As a motivation, we mention that there are special methods for finding the maximum clique and counting the number of maximal and maximum cliques of a local comparability graph, given a locally transitive orientation of it [14]. Both these problems can be solved by algorithms requiring a number of steps of the order of the product of the sizes of the vertex and edge sets of the graph. Clearly, in general finding a maximum clique is a classical NP-hard problem [3], while the enumeration problem is #P-complete [16]. It is worth noting that the number of maximum cliques of a local comparability graph can grow exponentially with the number of vertices. In fact, the examples of Moon and Moser [9] of graphs having a maximum (exponential) number of maximum cliques are circle graphs.

The recognition problem for local comparability graphs is NP-complete. It might then be difficult to find a good characterization for the class. However, every graph of it can be obtained as the difference of two suitable comparability graphs. This is the subject of Section 3. Further, we consider the dimension of a local comparability graph, a generalization of partially ordered set dimension. In Section 4, we describe a characterization of proper interval graphs in terms of dimensions. That is, proper interval graphs are exactly the local comparability graphs of dimension one. Other characterizations of proper interval graphs have been described by Duchet [4], Fishburn [5] and Roberts [13]. Next, we consider circle graphs. Read, Rotem and Urrutia [12] proved that the complement of a circle graph can be split into two subgraphs satisfying certain conditions of transitivity. In Section 5, we show that adding a few restrictions to those of [12] leads to a set of necessary and sufficient conditions for circle graphs. Other characterizations of circle graphs are those of Bouchet [2], Fournier [6], Gabor, Supowit and Hsu [7] and Naji [10]. That of [7] corresponds to the recognition algorithm with the least complexity. Section 2 contains basic definitions, while the formulation of related problems form the last section.

2. Preliminaries

G denotes a simple undirected graph with vertices $V(G) = \{v_1,\dots,v_n\}$ and edge set $E(G)$. Let G_1 and G_2 be two graphs sharing a common vertex set. The *difference* $G_1 - G_2$ is the graph having the same vertices as G_1 and those edges belonging to $E(G_1) - E(G_2)$. Let $\vec{G}$ be an acyclic orientation of G, and v_i, v_j $\in V(G)$. If $\vec{G}$ contains a v_i-v_j path then v_i is an *ancestor* of v_j, and v_j a *descendant* of v_i. Denote by $\langle v_i,v_j\rangle$ the subset of vertices simultaneously descendants of v_i and ancestors of v_j in $\vec{G}$. When necessary we write $\langle v_i,v_j\rangle_{\vec{G}}$ instead. $\vec{G}$ is *transitive* whenever $(v_i,v_j) \in E(\vec{G})$ and $(v_j,v_k) \in E(\vec{G}) \rightarrow (v_i,v_k) \in E(\vec{G})$, and is *anti-transitive* when $(v_i,v_j) \in E(\vec{G})$ and $(v_j,v_k) \in E(\vec{G}) \rightarrow (v_i,v_k) \notin E(\vec{G})$. G is a *comparability* or *anti-comparability* graph when it admits a transitive or anti-transitive orientation, respectively. The *transitive closure* of $\vec{G}$ is its minimum transitive supergraph, while the *Hasse diagram* is the anti-transitive subgraph of $\vec{G}$, which preserves the transitive closure.

Let $\vec{G}$ be a transitive digraph. A *linear extension* of $\vec{G}$ is a maximum spanning transitive supergraph of it. Let $\vec{L}_1,\dots,\vec{L}_k$ be linear extensions of $\vec{G}$. The *dimension* (*dim*) of $\vec{G}$ is the least integer k such that $E(\vec{L}_1) \cap \dots \cap E(\vec{L}_k) = E(\vec{G})$. The *dimension* of its subjacent comparability graph G is defined as equal the dimension of $\vec{G}$. This is not ambiguous, since whenever $\vec{G}_1$, $\vec{G}_2$ are transitive orientations of G, then dim G_1 equals dim $\vec{G}_2$

(Trotter, Moore and Summer [15], Gysin [8]).

Now, let S be a family of non-empty subsets S_i of some set. The *intersection graph* $\alpha(S)$ is the one having vertex set S and edges (S_i,S_j) iff $S_i \cap S_j \neq \phi$ for $i \neq j$. The *overlap graph* of S, denoted $\beta(S)$, has again vertex set S and edges (S_i,S_j) whenever $S_i \cap S_j$ is different from both ϕ and $S_i \cup S_j$. Suppose S is a set of intervals of a real line L. Then $\alpha(S)$ is called an *interval graph*, while $\beta(S)$ is a *circle graph*. Observe that the overlap graphs of the intervals of a line coincide with the intersection graphs of the chords of a circle. If no interval of S properly contains another one then $\alpha(S) = \beta(S)$ and the graph is called *proper interval* (*unit interval or indifference*). In any case, we refer to S as an *interval representation* for $\alpha(S)$ or $\beta(S)$. Each interval $S_i \in S$ corresponds to a vertex v_i of $\alpha(S)$ or $\beta(S)$ and is represented as $[v'_i,v''_i]$, where v'_i and v''_i are the *extreme points* which determine S_i in L. We denote by $s(v'_i)$ and $s(v''_i)$ the distances from v'_i and v''_i, respectively to some fixed origin, located in L at the left of the first interval. Then $s(v'_i) < s(v''_i)$, for all v_i. Without loss of generality, we assume that no two extreme points coincide in L.

Let G be a graph, H a subgraph of it and R a linear ordering $v_1,\dots,v_n$ of the vertices of G. An orientation $\vec{H}$ of H is *induced* by R when $(v_i,v_j) \in E(\vec{H}) \rightarrow i < j$.

Finally, let S be an interval representation for some interval or circle graph G, and H a subgraph of G. Let R be a linear ordering $v_1,\dots,v_n$ of the vertices of G, such that $i < j$ if and only if $s(v'_i) < s(v''_j)$. Then R is called a *canonical ordering* and the orientation $\vec{H}$ induced by it is a *canonical orientation* for S.

3. Locally Transitive Orientations

Let $\vec{G}$ be an acyclic orientation of an undirected graph G. $\vec{G}$ is called *locally transitive* when the subgraph induced in it by the subset of vertices $\langle v,w\rangle$ is transitive, for all edges $(v,w) \in E(\vec{G})$. In this case, G is a *local comparability* graph. For example, figures 1(b) and 1(c) are orientations of the graph of 1(a). That of 1(c) is locally transitive, while 1(b) is not. Hence 1(a) is a local comparability graph.

In particular, if $\vec{G}$ is transitive so is every of its induced subgraphs. Therefore every comparability graph is local comparability. On the other hand, if $\vec{G}$ is an anti-transitive orientation of G then $\langle v,w\rangle = \{v,w\}$, for every edge $(v,w) \in E(\vec{G})$. It follows that $\vec{G}$ is locally transitive. In addition, we know that G does not contain triangles, otherwise $\vec{G}$ would have an edge implied by transitivity. Conversely, if G is a triangleless local comparability graph then any locally transitive orientation of it is anti-transitive. Otherwise, $\vec{G}$ would contain an edge (v,w) implied by transitivity, that is $|\langle v,w\rangle| > 2$ and any v-w path of length greater than 2 would correspond to a triangle in G, a contradiction. Therefore, the classes of anti-comparability graphs and triangleless local comparability graphs coincide. Recognizing anti-comparability graphs is NP-complete (Nesetril and Rodl [11]). Consequently, the same is true for local comparability ones.

From the above observation it follows that it might be difficult to formulate good characterizations for local comparability graphs. A necessary condition is given below.

Theorem 1: Let G be a local comparability graph. Then G is the difference of two comparability graphs.

Proof: Since G is a local comparability graph it admits a locally transitive orientation $\vec{G}$. Let $\vec{C}_1$ be the transitive closure of $\vec{G}$ and $\vec{C}_2 = \vec{C}_1 - \vec{G}$. Clearly, C_1 is a comparability graph. We show that the same is true for C_2. Let $(v,w),(w,z) \in E(\vec{C}_2)$. Then $(v,w),(w,z) \in E(\vec{C}_1)$. That is, $\vec{G}$ contains both v-w and w-z paths, i.e. $w \in \langle v,z\rangle_{\vec{G}}$. Because $\vec{C}_1$ is transitive, it follows $(v,z) \in E(\vec{C}_1)$. If $(v,z) \in E(\vec{G})$ the fact $(v,w),(w,z) \notin E(\vec{G})$ implies that the subgraph induced in $\vec{G}$ by $\langle v,z\rangle$ is not transitive, which contradicts $\vec{G}$ being locally transitive. Therefore $(v,z) \in E(\vec{C}_2)$, that is $\vec{C}_2$ is transitive. Since $G = C_1 - C_2$, the theorem follows. □

4. Dimension.

In this section we discuss the dimension of a local comparability graph, a generalization of that of a comparability graph.

Let G be a local comparability graph, $\vec{G}$ a locally transitive orientation of it and $\vec{C}$ the transitive closure of $\vec{G}$. Define *dim* $\vec{G}$ = dim $\vec{C}$, and *dim* G to be the minimum dimension of any locally transitive orientation of G.

Observe that unlike the case of comparability graphs, the dimensions of different locally transitive orientations of a same local comparability graph are not necessarily equal.

It is well known that the comparability graphs of dimension one are precisely the complete graphs, those of dimension two are the permutation graphs, while it is NP-complete to recognize comparability graphs of dimension three (Yannakakis [17). The theorem below characterizes local comparability graphs of dimension one.

Theorem 2: G is a local comparability graph of dimension one if and only if G is a connected proper interval graph.

Proof: (⇒) Let G be a dim 1 local comparability graph. Then it admits a dim 1 locally transitive orientation $\vec{G}$. Let $\vec{C}$ be the transitive closure of $\vec{G}$. Then $\vec{C}$ has a Hamilton path $v_1, \ldots, v_n$. That is, the Hasse diagram of $\vec{C}$ is $v_1, \ldots, v_n$ and such a path is also present in $\vec{G}$. Then G is connected. Now, we construct an interval representation for G, as follows. Each $v_i \in V(G)$ corresponds to an interval $[v'_i, v''_i]$ such that

$$s(v'_n) = n,\ s(v''_n) = n + 1, \text{ and for each } j,\ 1 \le j < n,$$
$$s(v'_j) = j \text{ and } s(v''_j) = k + j/n,$$

where k is chosen so as to satisfy $(v_j, v_k) \in E(\vec{G})$ and $(v_j, v_{k+1}) \notin E(\vec{G})$. Let us consider the relative position of two intervals $[v'_i, v''_i]$ and $[v'_j, v''_j]$, $i < j$. Then $s(v'_i) < s(v'_j)$. If $s(v''_i) \geq s(v''_j)$, there must be an edge $(v_i, v_k) \in E(\vec{G})$ such that $(v_j, v_k) \notin E(\vec{G})$, $j < k$. However, $v_j \in \langle v_i, v_k \rangle$ and therefore the subgraph induced in $\vec{G}$ by $\langle v_i, v_k \rangle$ is transitive, which implies $(v_j, v_k) \in E(\vec{G})$, a contradiction. Hence no interval properly contains any other. It remains to show that $[v'_i, v''_i]$, $[v'_j, v''_j]$ intersect iff $(v_i, v_j) \in E(G)$. When the intervals intersect then $s(v'_i) < s(v'_j) < s(v''_i)$. That is, following the construction of the intervals, there is an edge $(v_i, v_k) \in E(\vec{G})$ satisfying $i < j < k$. Again, because $v_j \in \langle v_i, v_k \rangle$ and $\vec{G}$ is locally transitive it follows $(v_i, v_j) \in E(\vec{G})$. Conversely, if $(v_i, v_j) \in E(\vec{G})$ then $i < j$ and $s(v''_i) \geq j + i/n$. Because $s(v'_i) = i$ and $s(v'_j) = j$ it follows that $[v'_i, v''_i]$, $[v'_j, v''_j]$ intersect. Then G is a proper interval graph.

($\Leftarrow$) Let G be a proper connected interval graph and S an interval representation for it. Let $v_1, \ldots, v_n$ be a canonical ordering and $\vec{G}$ a canonical orientation for S. Since G is connected, $(v_i, v_j) \in E(\vec{G})$, for some $j \geq i + 1$ and $i < n$. That is, $s(v'_i) < s(v'_j) < s(v''_i) < s(v''_j)$. Since $s(v'_i) < s(v'_{i+1}) \leq s(v'_j)$ and $s(v''_i) < s(v''_{i+1}) \leq s(v''_j)$ we conclude that $[v'_i, v''_i]$, $[v'_{i+1}, v''_{i+1}]$ intersect and therefore $v_1, \ldots, v_n$ is a Hamilton path of $\vec{G}$. We need to show that $\vec{G}$ is locally transitive. Let $(v_i, v_j) \in E(\vec{G})$ and $i < k \leq j$. Then $s(v'_i) < s(v'_k) \leq s(v'_j)$. Since no interval properly contains any other, $s(v'_j) < s(v''_i) < s(v''_k) \leq s(v''_j)$. Hence $(v_i, v_k) \in E(\vec{G})$ and for $j \neq k$ we have $(v_j, v_k) \in E(\vec{G})$. Because $v_k \in \langle v_i, v_j \rangle$ the subgraph induced in $\vec{G}$ by $\langle v_i, v_j \rangle$ is transitive. Therefore $\vec{G}$ is locally transitive. Because it contains a Hamilton path its dimension is 1. This completes the proof. □

5. Circle Graphs.

In this section we discuss aspects of circle graphs, related to edge transitivity.

The following is a necessary condition for circle graphs, in terms of locally transitive orientations.

Theorem 3 ([14]): Let G be a circle graph, S an interval representation of it and $\vec{G}$ the orientation of G canonical for S. Then $\vec{G}$ is locally transitive.

Hence circle graphs are local comparability.

The next theorem is another necessary condition.

Theorem 4 (Read, Rotem and Urrutia [12]): Let G be a circle graph. Then there exists an edge disjoint partition $\bar{G} = Q \cup I$ of its complement $\bar{G}$ and a linear ordering R of its vertices such that the orientations $\vec{Q}$, $\vec{I}$ induced by R satisfy:

for $i < j < k$,

(i) $(v_i, v_j) \in E(\vec{I}) \rightarrow (v_i, v_k) \in E(\vec{I})$

(ii) $(v_i, v_j) \in E(\vec{Q})$ and $(v_j, v_k) \in E(\vec{Q}) \rightarrow (v_i, v_k) \in E(\vec{Q})$

It follows that the complement of a circle graph is an edge disjoint union of a permutation and the complement of an interval graph.

Next, we show that adding restrictions to theorem 4 gives necessary and sufficient conditions for circle graphs.

Theorem 5: G is a circle graph if and only if there exists an edge disjoint partition $\bar{G} = Q \cup I$ of its complement $\bar{G}$ and a linear ordering R of its vertices such that the orientations $\vec{G}$, $\vec{Q}$ and $\vec{I}$ induced by R satisfy:

for $i < j < k$,

(i)	$(v_i, v_j) \in E(\vec{I})$	$\rightarrow$	$(v_i, v_k) \in E(\vec{I})$
(ii)	$(v_i, v_j) \in E(\vec{Q})$ and $(v_j, v_k) \in E(\vec{Q})$	$\rightarrow$	$(v_i, v_k) \in E(\vec{Q})$
(iii)	$(v_i, v_j) \in E(\vec{G})$ and $(v_j, v_k) \in E(\vec{G})$	$\rightarrow$	$(v_i, v_k) \notin E(\vec{Q})$
(iv)	$(v_i, v_j) \in E(\vec{G})$ and $(v_j, v_k) \in E(\vec{I})$	$\rightarrow$	$(v_i, v_k) \in E(\vec{I})$
(v)	$(v_i, v_j) \in E(\vec{Q})$ and $(v_j, v_k) \in E(\vec{G})$	$\rightarrow$	$(v_i, v_k) \notin E(\vec{I})$

Proof: ($\rightarrow$) Let G be a circle graph, S an interval representation of it and $v_1, \dots, v_n$ a canonical ordering of its vertices. Split the edges $(v_i, v_j) \in E(\bar{G})$ into subgraphs Q and I, as follows:

$$(v_i, v_j) \in E(Q) \quad \text{iff} \quad s(v_j'') < s(v_i'')$$
$$(v_i, v_j) \in E(I) \quad \text{iff} \quad s(v_i'') < s(v_j')$$

Let $\vec{G}$, $\vec{Q}$ and $\vec{I}$ be the corresponding orientations of G, Q and I which are induced by S. Then (i)-(ii) are true because they are theorem 4. To show (iii), note that $(v_i, v_j) \in E(\vec{G})$ and $(v_j, v_k) \in E(\vec{G})$ implies that $s(v_j'') > s(v_i'')$ and $s(v_k'') > s(v_j'')$. That is, $s(v_k'') > s(v_i'')$, i. e. $(v_i, v_k) \notin E(\vec{Q})$. Next, we prove (iv). If $(v_i, v_j) \in E(\vec{G})$ and $(v_j, v_k) \in E(\vec{I})$ then $s(v_i'') < s(v_j'')$ and $s(v_j'') < s(v_k')$. That is, $s(v_i'') < s(v_k')$, i. e. $(v_i, v_k) \in E(\vec{I})$. Finally, consider implication (v). Let $(v_i, v_j) \in E(\vec{Q})$ and $(v_j, v_k) \in E(\vec{G})$. Then $s(v_j'') < s(v_i'')$ and $s(v_k') < s(v_j'')$. Hence $s(v_k') < s(v_i'')$, meaning that $(v_i, v_k) \notin E(\vec{I})$.

($\leftarrow$) Let G be a graph, $\bar{G} = Q \cup I$ and such that there exists an ordering $v_1, \dots, v_n$ of its vertices for which the induced orientations $\vec{G}$, $\vec{Q}$ and $\vec{I}$ satisfy (i)-(v). We prove that G must be a circle graph. The argument is by induction on n. If $n = 1$ there is nothing to prove. The induction hypothesis is that we can construct intervals $[v_i', v_i'']$, such that

for $i < j < n$

(1)	$(v_i, v_j) \in E(\vec{G})$	iff	$s(v_j') < s(v_i'') < s(v_j'')$
(2)	$(v_i, v_j) \in E(\vec{I})$	iff	$s(v_i'') < s(v_j')$.

As a consequence of (1)-(2) it follows:

(3) $(v_i, v_j) \in E(\vec{Q})$ iff $s(v''_j) < s(v''_i)$.

Implications (i)-(v) all remain valid when we restrict to induced subgraphs of $G \cup \bar{G}$. Therefore we can apply the induction hypothesis to $G-v_n$ and construct a set of intervals $S-v_n$, with intervals corresponding to the vertices $v_1, \ldots, v_{n-1}$ and such that (1)-(2) are satisfied. Next, we obtain a new set of intervals S by properly inserting in $S-v_n$ a new interval $[v'_n, v''_n]$ corresponding to vertex v_n, without changing the relative positions of the other intervals, as follows.

(a): *Position of* v'_n: Insert v'_n as to be the leftmost possible extreme point at the right of v'_{n-1}, such that no point v''_i, $(v_i, v_n) \in E(\vec{I})$, lies at the right of v'_n.

(b): *Position of* v''_n: Insert v''_n as to be the leftmost possible extreme point at the right of v'_n, such that no point v''_i, $(v_i, v_n) \in E(\vec{G})$, lies at the right of v''_n.

We prove that the induction hypothesis is valid for graphs with n vertices. That is, we show that S satisfies:

(4) $(v_i, v_n) \in E(\vec{G})$ iff $s(v'_n) < s(v''_i) < s(v''_n)$

(5) $(v_i, v_n) \in E(\vec{I})$ iff $s(v''_i) < s(v'_n)$.

Proof for (4): ($\rightarrow$) Assume $(v_i, v_n) \in E(\vec{G})$. If $s(v'_n) < s(v''_i) < s(v''_n)$ is false, since (b) asures $s(v'_n) < s(v''_n)$ it follows $s(v''_i) < s(v'_n)$ or $s(v''_n) < s(v''_i)$. The latter does not occur by (b) and we show next that neither the former does. There are two possibilities:

Case 1: $s(v''_i) < s(v'_{n-1})$

Then by (2) of the induction hypothesis, $(v_i, v_{n-1}) \in E(\vec{I})$. Using (i), we know that $(v_i, v_{n-1}) \in E(\vec{I})$ implies $(v_i, v_n) \in E(\vec{I})$, a contradiction. Therefore this case can not occur.

Case 2: $s(v''_i) > s(v'_{n-1})$

Then by (a) there exists v''_j such that $(v_j, v_n) \in E(\vec{I})$ and $s(v''_i) < s(v''_j) < s(v'_n)$.

Case 2.1: $i < j$

If $s(v'_j) < s(v''_i)$ then by (1), $(v_i, v_j) \in E(\vec{G})$. In addition, because of (iv) we conclude that $(v_i, v_n) \in E(\vec{I})$, a contradiction. Otherwise, $s(v'_j) > s(v''_i)$ and by (2) it follows $(v_i, v_j) \in E(\vec{I})$. Because of (i) the latter implies $(v_i, v_n) \in E(\vec{I})$, a contradiction.

Case 2.2: $i > j$

By (3) it follows $(v_j, v_i) \in E(\vec{Q})$ and by (v) we conclude that $(v_j, v_n) \notin E(\vec{I})$, a contradiction.

Therefore case 2 can also not occur and the only possibility is $s(v'_n) < s(v''_i) < s(v''_n)$.

($\Leftarrow$) Conversely, assume that $s(v'_n) < s(v''_i) < s(v'_n)$ and let us show that $(v_i, v_n) \in E(\vec{G})$. Suppose the latter is not true. Then by (b) we know that S contains an extreme point v''_j such that $(v_j, v_n) \in E(\vec{G})$ and $s(v'_n) < s(v''_i) < s(v''_j) < s(v''_n)$. There are two possibilities, below discussed. We show that neither of them can occur. This implies that $(v_i, v_n) \in E(\vec{G})$.

Case 1: $i < j$

Then by (1), $(v_i, v_j) \in E(\vec{G})$. Using (iii), we conclude that either $(v_i, v_n) \in E(\vec{G})$ or $(v_i, v_n) \in E(\vec{I})$. Neither of these alternatives can occur, the former because of the initial assumption and the latter by (a).

Case 2: $i > j$

Then by (3), $(v_j, v_i) \in E(\vec{Q})$. Since $(v_j, v_n) \in E(\vec{G})$ and using (ii) it follows $(v_i, v_n) \notin E(\vec{Q})$. However, by the construction (a) we know that $(v_i, v_n) \notin E(\vec{I})$. Therefore, the only possibility is $(v_i, v_n) \in E(\vec{G})$, contrary to the assumption.

This completes the proof of (4).

Proof for (5):

($\Rightarrow$) Assume $(v_i, v_n) \in E(\vec{I})$. Then by (a), $s(v''_i) < s(v'_n)$.

($\Leftarrow$) Suppose $s(v''_i) < s(v'_n)$ and let us show that $(v_i, v_n) \in E(\vec{I})$. Consider the two cases below.

Case 1: $s(v''_i) < s(v'_{n-1})$

Then $i \neq n - 1$ and by (2), $(v_i, v_{n-1}) \in E(\vec{I})$. Using (i), it follows $(v_i, v_n) \in E(\vec{I})$.

Case 2: $s(v''_i) > s(v_{n-1})$

Assume $(v_i, v_n) \notin E(\vec{I})$. Then by (a) there exists an extreme point v''_j such that $(v_j, v_n) \in E(\vec{I})$ and $s(v''_i) < s(v''_j) < s(v'_n)$. We discuss the following alternatives.

Case 2.1: $i < j$

If $s(v'_j) < s(v''_i)$ then $(v_i, v_j) \in E(\vec{G})$ by (1), and using (iv) we conclude that $(v_i, v_n) \in E(\vec{I})$, a contradiction. Therefore $s(v'_j) > s(v''_i)$. In the latter possibility $(v_i, v_j) \in E(\vec{I})$, by (2). Because of (i) it follows $(v_i, v_n) \in E(\vec{I})$, a contradiction.

Case 2.2: $i > j$

Then by (3), $(v_j, v_i) \in E(\vec{Q})$. If $(v_i, v_n) \in E(\vec{Q})$ then because of (ii) it follows $(v_j, v_n) \in E(\vec{Q})$, a contradiction. Therefore $(v_i, v_n) \in E(\vec{G})$. In the latter possibility, (v) implies that $(v_j, v_n) \notin E(\vec{I})$, also a contradiction.

Therefore case 2 does not occur.

The only possibility is $(v_i, v_n) \in E(\vec{I})$, which completes the proof of (5).

Hence the induction hypothesis remains true for graphs with n vertices. In particular, (1) implies that G is a circle graph. We have then proved theorem 5. □

6. Conclusions

We have considered the class of local comparability graphs, a generalization of comparability, anti-comparability and circle graphs. The following questions are related.

(i) The intersection and overlap graphs of the intervals of a line are the interval and circle graphs, respectively. The intersection graphs of the intervals taken on a tree are path graphs. Which are the overlap graphs of the intervals of a tree ?

(ii) The graphs which admit a transitive orientation having a tree as its Hasse diagram are those not containing an induced P_4 nor C_4 [1]. Which are the graphs having a loccally transitive orientation whose Hasse diagram is a tree ?

(iii) Local comparability graphs of dimension 1 are the proper interval graphs, while it is NP-complete to recognize local comparability graphs of dimension 3. Which are the local comparability graphs of dimension 2 ?

(iv) Given a locally transitive orientation $\vec{G}$, is there an efficient method for finding the maximum independent set of G, and also the number of maximum and maximal of these sets ?

(v) Which are the graphs having an acyclic orientation such that the distance from each vertex to any of its descendants is at most 2 ? This class contains complements of circle graphs [12], among others.

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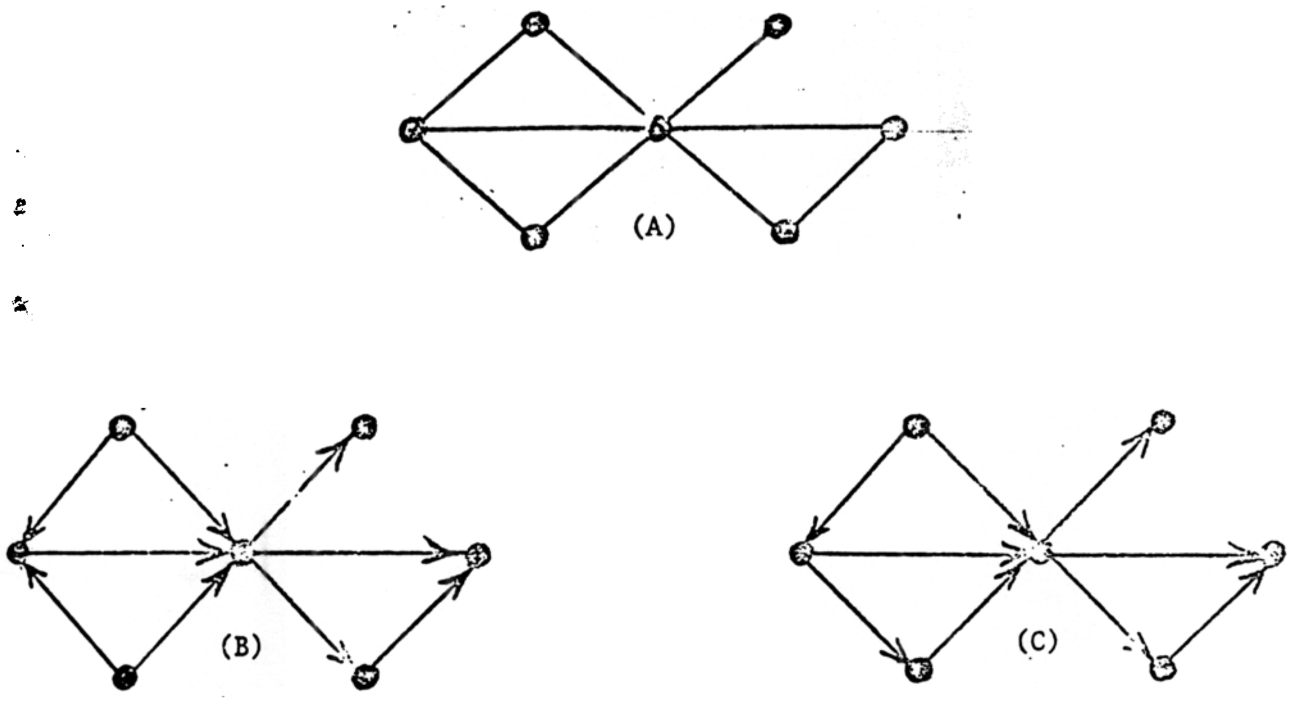

FIGURE ONE