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CLIQUE GRAPHS OF CHORDAL AND PATH GRAPHS

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Resumo

Nós caracterizamos os grafos clique de grafos cordais e de caminhos. É apresentada ainda uma classe de grafos chamada **árvores expandidas**. Elas formam uma subclasse dos grafos disk-Helly. Mostra-se que o grafo clique de todo grafo cordal (portanto dos de caminho) é uma árvore expandida. Mais ainda, que toda árvore expandida é o grafo clique de algum grafo de caminho (portanto cordal). Diferentes caracterizações de árvores expandidas são descritas, conduzindo a um algoritmo de tempo polinomial para o reconhecimento de grafos nesta classe.

Abstract

We characterize clique graphs of chordal and path graphs. A special class of graphs called **expanded trees** is introduced. They form a subclass of disk-Helly graphs. It is shown that the clique graph of every chordal (hence path) graph is an expanded tree. In addition, every expanded tree is the clique graph of some path (hence chordal) graph. Different characterizations of expanded trees are described, leading to a polynomial time algorithm for recognizing clique graphs of chordal and path graphs.

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Abstract

We characterize clique graphs of chordal and path graphs. A special class of graphs called **expanded trees** is introduced. They form a subclass of disk-Helly graphs. It is shown that the clique graph of every chordal (hence path) graph is an expanded tree. In addition, every expanded tree is the clique graph of some path (hence chordal) graph. Different characterizations of expanded trees are described, leading to a polynomial time algorithm for recognizing clique graphs of chordal and path graphs.

1 Introduction

We examine clique graphs of chordal graphs. Bandelt and Prisner [2] proved that they are disk-Helly. Chen and Lih [4] and independently Bandelt and Prisner [1] showed that the second iterated clique graph of a chordal graph is again chordal. Here it is shown that clique graphs of chordal graphs correspond to a class named expanded trees, in the sence that the clique graph of a chordal graph is always an expanded tree and every expanded tree is the clique graph of some chordal graph. We characterize the class of expanded trees and describe a polynomial time algorithm for recognizing them. Next it is shown that expanded trees form a proper subclass of disk-Helly graphs. Finally, path graphs are considered. We prove that the class of clique graphs of path graphs is no more restricted than that of chordal graphs. Every expanded tree is also the clique graph of some path graph.

Expanded trees are closely related to dismantlable graphs. Those graphs were examined by Bandelt and Prisner [2], Prisner [9] and Nowakovski and Winkler [8]. Disk-Helly graphs are a proper subclass of dismantlable graphs, and they can be recognized in polynomial time, according to an algorithm described by Bandelt and Pesch [1]. See also Nowakovski and Rival [7] and Quilliot [10].

G denotes a simple undirected graph, V(G) and E(G) are its vertex and edge sets respectively, $n = |V(G)|$ and $m = |E(G)|$. A(v) is the set of vertices adjacent to $v \in V(G)$, while N(v) is defined as $A(v) \cup \{v\}$. The vertex $v \in V(G)$ is **dominated** by $w \in V(G)$ in G, when v,w are distinct and $N(v) \subset N(w)$. A **clique** is a complete subgraph of G. The **clique graph** of G, denoted by K(G), is the intersection graph of the maximal cliques of G. A **chordal graph** is the intersection graph of subtrees of a tree T. The subtree of T corresponding to a vertex $v \in V(G)$ is called **representative subtree** of v and is denoted by T(v). The tree T together with the representative subtrees form a **tree representation** for G. A **minimal representation** is a tree representation such that $|V(T)|$ is the least possible. Gavril [5] and Buneman [3] showed that a minimal representation is precisely one in which each vertex of T corresponds to a maximal clique of G. In addition, for each $v \in V(G)$, the subtree T(v) is formed exactly by the vertices of T corresponding to those maximal cliques of G which contain v. A **path graph** is the intersection graph of paths of a tree. Monma and Wei [6] characterized path graphs and variations of this class in terms of their minimal representations.

G is a **dismantlable** graph if there exists a sequence $v_1, v_2, \ldots, v_n$ of its vertices such that, for $i < n$, v_i is dominated in $G - \{v_1, \ldots, v_{i-1}\}$. If additionally the maximal cliques of G satisfy the Helly property then G is **disk-Helly**.

2 Expanded trees

G is an **expanded tree** when it admits a spanning tree T(G), such that for each edge $(v,w) \in E(G)$ the vertices of the v–w path in T induce a clique in G. In this case, T(G) is a **canonical tree** of G.

Lemma 1: Let G be a connected chordal graph and T a minimal tree representation of it. Then K(G) is the graph obtained from T by adding exactly the edges that transforms each representative subtree T(v) into a |T(v)| clique.

Proof: T and K(G) have the same vertex set. Two vertices of K(G) are adjacent precisely when their corresponding maximal cliques in G intersect. That is, all vertices of T contained in each of the representative subtrees T(v) must be adjacent in K(G), because T(v) corresponds to the maximal cliques of G intersecting at v. On the other hand, suppose that there exist two vertices of T such that there is no representative subtree containing simultaneously both of them. Then the corresponding maximal cliques of G have no common vertex. Therefore these two vertices are not adjacent in K(G).

Lemma 2: Let G be an expanded tree and T(G) a canonical tree of it. Then the vertices of each maximal clique of G induce a (connected) subtree in T(G).

Proof: Suppose the contrary. Let C be a maximal clique of G such that $C \cap T(G)$ has at least two distinct connected components A and B. Let P be the path of T between A and B, that is, the vertices of P are $x_1, \ldots, x_t$, with $t > 2$, $x_1 \in V(A)$, $x_t \in V(B)$ and $x_i \notin V(A \cup B)$, $1 < i < t$. In particular, there must be some vertex $x_j \in V(P)$ such that $x_j \notin V(C)$, otherwise $A \cup B \cup P$ becomes a connected component of $C \cap T$, a contradiction. Let y be an arbitrary vertex of C and denote by P' the path in T between y and x_j. If $x_1 \in V(P')$ then x_j belongs to the y–x_t path in T. G is an expanded tree and

$(y,x_t) \in E(G)$ thus following $(y,x_j) \in E(G)$. Similarly, if $x_t \in V(P')$ then x_j belongs to the $y-x_1$ path in T, and $(y,x_j) \in E(G)$. In the remaining alternative, $x_1, x_t \notin V(P')$, we again conclude $(y,x_j) \in E(G)$, since x_j necessarily belongs to at least one of the paths, $y-x_1$ or $y-x_t$. This implies $x_j \in V(C)$, a contradiction. Hence $C \cap T(G)$ is a subtree of $T(G)$.

The theorem below characterizes the class of clique graphs of chordal graphs.

Theorem 1: The following are equivalent:

(i) G is the clique graph of some connected chordal graph H.

(ii) G admits a spaning tree T, such that for each $v \in V(G)$, $N_G(v) \cap T$ is a (connected) subtree of T.

(iii) G is an expanded tree.

Proof:

(i) → (ii) Since H is chordal, we know by Lemma 1 that G can be obtained from the minimal tree representation T of H, by adding the edges that transforms each representative subtree into a clique. It follows that each vertex $v \in V(G)$ is adjacent to the vertices belonging to the representative subtrees of T, which contain v. $N_G(v)$ is the union of the vertex sets of the representative subtrees of T which contain v, and hence $N_G(v) \cap T$ is connected.

(ii) → (iii) - It sufices to show that T is a canonical tree of G. Let $(v,w) \in E(G)$ and $v = v_0, \ldots, v_r = w$ be the v–w path in T. Suppose by induction that $\{v_0, \ldots, v_{r-1}\}$ induces a clique in G, $r > 1$. Since v and w are adjacent and $N_G(v) \cap T$ is connected it follows that $v_1, \ldots, v_r$ are all adjacent to v. Similarly, $v_0, \ldots, v_{r-1}$ are all adjacent to w. Consequently, $\{v_0, \ldots, v_r\}$ induces a clique in G, that is, G is an expanded tree.

(iii) → (i) - Given an expanded tree G, we construct a connected chordal graph H such that $G = K(H)$. Let $X = \{V(C_i) \ / \ C_i$ is a maximal clique of $G\}$, $Y = \{\{v_i\} \ / \ v_i \in V(G)\}$. Let H be the intersection graph of the elements of $X \cup Y$. Denote by T a canonical tree of G. By Lemma 2, $C_i \cap T$ is a subtree of T. Hence H is the intersection graph of subtrees of a tree T, that is, H is chordal. In addition, each vertex v of T corresponds to a maximal clique of H, namely to that formed by the maximal cliques of G which contain v and by v itself. Hence T together with X and Y define a minimal representation of G. Using Lemma 1 we conclude that K(H) = G.

3 Recognition of Expanded Trees

A sequence S_k of vertices $v_1,...,v_k$, $k \leq n$, of a graph G is **canonical** when for each $1 \leq i \leq k$, either $i = n$ or v_i is dominated by some vertex v_j, $i < j$, in the graph $G(S_i)$, defined as

$$V(G(S_i)) = V(G)$$

$$E(G(S_i)) = E(G) - \{(x,y) \in E(G) \;/\; x \in \{v_1,...,v_{i-1}\},\ y \in \{v_i,...,v_n\} \text{ and } |N_{G_i}(x) \cap \{v_i,...,v_n\}| = 1\}$$

We call the vertex v_i **canonical** in $G(S_i)$. The value k is the **length** of S_k. If $k = n$ then S_k is **complete**. S_k is **maximal** when it is complete or $G(S_{k+1})$ has no canonical vertex. Clearly, if S_k is canonical then any of the subsequences $v_1,...,v_i$ is so, $1 \leq i \leq k$.

Expanded trees can be also characterized as follows.

Theorem 2: G is an expanded tree if and only if it admits a complete canonical sequence.

Proof: ($\rightarrow$) Let T be a canonical tree of G. Let S_n be a sequence $v_1,...,v_n$ of the vertices of G, such that v_i is a leaf of T_i, $1 \leq i \leq n$, where $T_1 = T$ and for $i > 1$ $T_i = T_{i-1} - v_{i-1}$. We show by induction that S_i is canonical. Assume it is true for all subsequences of length $< i$. If $i = n$ there is nothing to prove. Otherwise let v_j, $j > i$, be the vertex adjacent to v_i in T_i. We claim that v_j dominates v_i in $G(S_i)$. This is clear for $i = 1$. When $i > 1$, suppose the claim is false. In this case, there is a vertex v_p, such that $(v_p,v_i) \in E(G(S_i))$ and $(v_p,v_j) \notin E(G(S_i))$. T is canonical; thus the following two conditions must hold:

$p < i$ and $(v_p, v_q) \notin E(G)$ for all $q > i$, otherwise (v_p, v_j) would belong to $E((G(S_i))$, a contradiction.

In this case, $|N_{G_i}(v_p) \cap \{v_i, \ldots, v_n\}| = 1$, and $(v_p, v_i) \notin E(G(S_i))$, again a contradiction. Therefore v_j dominates v_i in $G(S_i)$, that is, S_i is canonical.

($\leftarrow$) Let $v_1, \ldots, v_n$ be a canonical sequence of G. Each v_i, $1 \leq i < n$, has a dominator v_j in $(G(S_i))$, $i < j$, and we will write $v_j = dom(V_i)$. Let T be the graph defined as follows:

$$V(T) = V(G)$$

$$E(T) = \{(v_i, dom(v_i)) \in E(G) \;/\; 1 \leq i < n\}.$$

When $n > 1$, every vertex of G is incident to some edge of T; and since $|E(T)| = n - 1$, T is a spanning tree of G. We show that T is canonical. Suppose it is not canonical. In this case, there is an edge $(v_a, v_b) \in E(G)$, such that the $v_a - v_b$ path in T is not a clique of G. By choice of (v_a, v_b), we can have $(v_b, v_c) \notin E(G)$, where v_c is the vertex following v_a in the $v_a - v_b$ path in T. Let $\vec{T}$ be an orientation of T obtained by directing each edge $(v_i, v_j) \in E(T)$ from v_i to v_j, when $i > j$. $\vec{T}$ is a directed rooted tree.

Examine the edge $(v_a, v_b) \in E(G)$ and suppose $a < b$. Then $a < c$ and $(v_c, v_a) \in E(\vec{T})$ implies that v_c dominates v_a in $G(S_a)$. Since $a < b$, we must have $(v_a, v_b) \in E(G(S_a))$. Hence $(v_b, v_c) \in E(G(S_a))$ and therefore $(v_b, v_c) \in E(G)$, a contradiction. The case $a > b$ is similar.

Lemma 3: All maximal canonical sequences of G have the same length.

Proof: Let us assume the lemma is false. Then there are two distinct maximal sequences $v_1, \ldots, v_k$ and $v'_1, \ldots, v'_l$ denoted respectively by S_k and S'_l, $l > k$. Then S'_l contains a vertex $v'_j \notin S_k$.In this case, $j < n$. Otherwise if $j = n$ and v'_j is the only vertex which belongs to S'_l and not to S_k; then $k = n - 1$ and S_k is not maximal, a contradiction. Let w be the dominator of v'_j in $G(S_j)$. The following cases can occur:

Case 1: $w \notin S_k$

If $A_G(v_j) = \{w\}$ then w dominates v_j in $G(S_{k+1})$. Otherwise, examine each vertex $z\ (\neq w) \in A_G(v_j)$. It follows that z is adjacent to at least two vertices of $V(G) - S_k$, namely v_j and w. In additition, $(v_j,w) \in E(G(S_{k+1}))$. Therefore $A_G(v_j) = A_{G(S_{k+1})}(v_j)$ and $N_G(v_j) \subset N_{G(S_{k+1})}(w)$, that is, w again dominates v_j in $G(S_{k+1})$. Hence S_k is not maximal, a contradiction.

Case 2: $w \in S_k$

Let $v_i = w$, $i \leq k$, and denote by w' the dominator of v_i in $G(S_i)$. Then w' also dominates v_j in $G(S_i)$. If $w' \in S_k$ then apply case 2 again, with w replaced by its dominator. Iteratively repeat the argument until a dominator w' is found satisfying $w' \notin S_k$. Since $k < l$, and at each iteration the index i increases, $w' \notin S_k$ will eventually be reached. Then case 1 applies.

Theorem 2 and Lemma 3 lead to a greedy algorithm for recognizing expanded trees. Construct a maximal canonical sequence S_k of vertices $v_1,...,v_k$ of the graph G. Clearly, G is an expanded tree if and only if $k = n$. For $i < n$ each v_i can be arbitrarily chosen among the dominated vertices in $G(S_i)$, if existing. The algorithm terminates within $O(n^2m)$ steps. A canonical tree T can be obtained as a by-product: For $i < n$, include in E(T) the edge (v_i,w) where w is the dominator of v_i in $G(S_i)$.

Similar results as those above presented can be formulated for the class of dismantlable graphs, with corresponding formulations of Theorem 2, Lemma 3 and the recognition algorithm.

4 Disk-Helly Graphs

In this section, we compare the classes of disk-Helly graphs and expanded trees.

Theorem 3: The class of disk-Helly graphs properly contains that of expanded trees.

Proof: Let G be an expanded tree. By theorem 2, G admits a sequence S_n of its vertices $v_1, ..., v_n$, such that each v_i is dominated in $G(S_i)$, $i < n$. Clearly, $G - \{v_1, ..., v_{i-1}\}$ is an induced subgraph of $G(S_i)$. In addition, v_i dominated in $G(S_i)$ implies that v_i has a dominator belonging to $\{v_{i+1}, ..., v_n\}$. Therefore v_i is also dominated in $G - \{v_1, ..., v_{i-1}\}$, that is, G is dismantlable. Let T be a canonical tree of G. By lemma 2, the maximal cliques of G correspond to subtrees of T. Hence they satisfy the Helly property, and G is disk-Helly. It remains to show that the containment is proper. The graph H of Figure 1 is known to be disk-Helly [1]. Each of the seven degree 5 vertices $v_1, ..., v_7$ of it is canonical in H. The sequence $v_1, ..., v_7$ is also canonical. Denote it by S_7. There is no canonical vertex in $G(S_8)$. Hence S_7 is maximal and H is not an expanded tree.

5 Path Graphs

Below is a characterization of clique graphs of path graphs.

Theorem 4: H is an expanded tree if and only if it is the clique graph of some connected path graph G.

Proof:

(←) Apply theorem 1, since path graphs are chordal.

(→) Let H be an expanded tree. We construct a path graph G such that H = K(G). By theorem 1, there is a connected chordal graph F such that H = K(F). Let T be a minimal tree representation of F. If all representative subtrees are paths of T then define G = F and the theorem is proved. Otherwise T contains a representative subtree T(s), of some $s \in V(F)$, such that T(s) has more than two leaves. Let $a,b \in V(T)$ be two distinct leaves of T(s). Let F' be the intersection graph of the same subtrees of T that represents the vertices of F, except T(s), and adding the following new three subtrees:

(i) The a–b path in T(s);

(ii) The subtree $T(s) - a$;

(iii) The subtree $T(s) - b$.

Then F' is also chordal. Next we prove that the tree representation T of F' is again minimal. For $v \in V(T)$, denote by $K_G(v)$ the maximal clique of G, formed by the vertices of G whose representative subtrees of T contain v. It follows that T is minimal if and only if for each pair of distinct vertices $v,w \in V(T)$, we have $K_G(v) \neq K_G(w)$.

Compare the relative position of v and w in T, for both graphs F and F'. If $v,w \notin T(s)$ then $K_F(v) = K_{F'}(v)$, and $K_F(w) = K_{F'}(w)$; since T is minimal for F it follows that $K_{F'}(v) \neq K_{F'}(w)$.

If $v,w \in T(s)$ then $K_F(v) \neq K_F(w)$ implies that there exists a representative subtree $T(z)$, $z \neq s$, which contains just one of v,w, and thus $K_{F'}(v) \neq K_{F'}(w)$. The remaining possibility is $v \in T(s)$ and $w \notin T(s)$. In this case, v must belong to at least two among the three new subtrees (i) – (iii), and w cannot be contained in any of them. Consequently, $K_{F'}(v) \neq K_{F'}(w)$ in all cases, that is, T is also minimal for F'.

Applying lemma 1 for F and F' leads to $K(F) = K(F')$. On the other hand, the subtree (i) is already a path while each of the subtrees (ii) and (iii) contains one leaf less than $T(s)$. This completes the proof, since eventually a path graph will be constructed.

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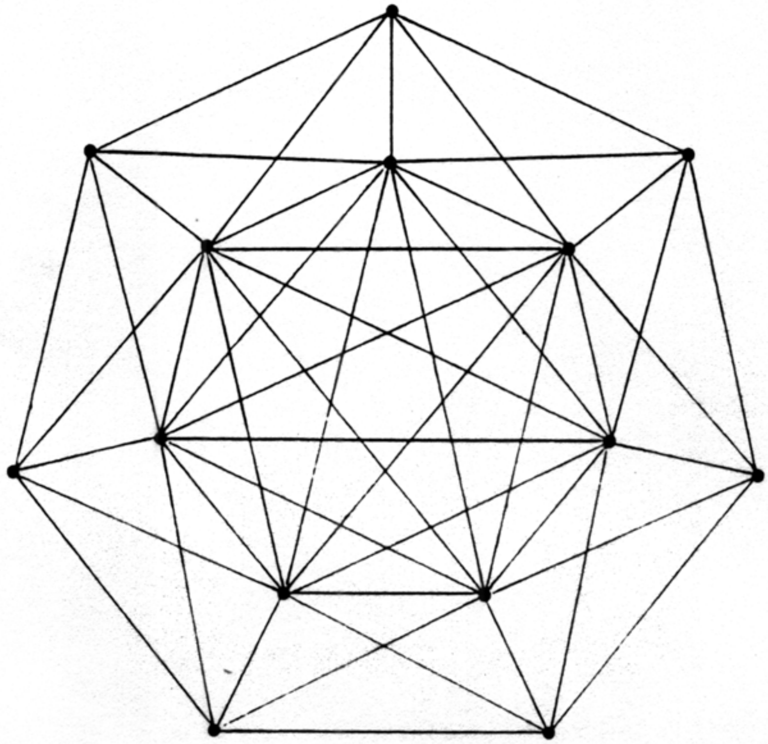

Figure 1