

International Journal on Emerging Mathematics Education

Vol. 3, No. 1, March 2019, pp. 57-68

P-ISSN: 2549-4996, E-ISSN: 2548-5806, DOI: <http://dx.doi.org/10.12928/ijeme.v3i1.11314>

Prospective Elementary Teachers' Knowledge of Comparing Decimals**¹Zetra Hainul Putra, ²Carl Winsløw**¹University of Riau, Kampus Bina Widya, Km 12,5, Simpang Baru, Pekanbaru 28293, Indonesia²University of Copenhagen, Øster Voldgade 3, 1350 København, Denmarke-mail: zetra.hainul.putra@lecturer.unri.ac.id**Abstract**

The aim of this study is to investigate prospective elementary teachers' (PsETs) mathematical and didactical knowledge of comparing decimals. Thirty-two fourth-year PsETs from an elementary school teacher education study program in Indonesia participated in this study. Each PsET is asked to solve a mathematical task of comparing decimals presented in the hypothetical teacher task (HTT), and then the PsETs use their mathematical knowledge to build their didactical knowledge collectively (pairs). Their mathematical and didactic knowledge is analyzed based on the anthropological theory of the didactic, especially praxeology. The findings indicate that PsETs have various techniques to solve the comparing decimal task, but some of them find it difficult to explain those techniques.

Keywords: Anthropological Theory of the didactic, praxeologies, hypothetical teacher tasks, mathematical and didactical knowledge

How to Cite: Putra, Z.H., & Winsløw, C. (2019). Prospective Elementary Teachers' Knowledge of Comparing Decimals. *International Journal on Emerging Mathematics Education*, 3(1), 57-68. <http://dx.doi.org/10.12928/ijeme.v3i1.11314>

INTRODUCTION

The results from the Program for International Student Assessment (PISA) in 2015 ranked the performance of Indonesian pupils 62 out of 70 countries, and most of the pupils were only able to solve problems directly related to routine procedures (mostly at level 1 and 2 on the PISA framework)(OECD, 2015). These results reflect how pupils learn mathematics at school, and how teachers teach mathematics to their pupils. Among many factors affecting pupils' low mathematical achievement, teachers' mathematical knowledge becomes the main concern for some studies because teachers' knowledge support for the success of pupils' learning: teachers' pedagogical content knowledge (Kuntur, et al. 2013) and mathematical knowledge for teaching (Hill, Rowan, & Ball, 2005) significantly affects their pupils' achievement.

Actually, many studies have been conducted on teachers' knowledge concerning specific mathematical knowledge (e.g. Ma, 1999), including scale comparative studies of teachers' knowledge (Tatto, et al. 2008). Ma (1999) has studied teachers' performance about rational numbers especially on calculations and representations of a division of fractions. She evaluated teachers' knowledge through posing two tasks: to compute and to represent meaning for the resulting mathematical sentences. Meanwhile, Teacher Education and Development Study in Mathematics (TEDS-M) studied teachers' knowledge through questionnaires (Tatto, et al. 2008; Tatto, et al. 2018). They used three question formats: multiple-choice, complex multiple-choice, and open constructed-response. They argued that only the third one allows teachers to demonstrate the depth of their thinking on mathematics knowledge and mathematics

teaching knowledge. However, both studies investigate individual teachers' knowledge through written tests. This method is commonly used by other studies and sometimes followed by an individual interview of selecting teachers.

Teachers' knowledge is, thus, a complex phenomenon that can be studied through different approaches or methods, depending on the aims (eg. large scale quantitative comparison or capturing more of the complexity). Here, as we are interested in an in-depth analysis of prospective Indonesian teachers' capability to address challenging incidences in mathematics teaching, we choose to design open constructed tasks based on pupils' difficulties and misconceptions. The tasks might examine both teachers' individual and collective mathematical and didactical knowledge. This study focuses on designing a model for teachers' mathematical and didactical knowledge of rational numbers, specifically on comparing decimals. This topic provides some challenges for teachers to teach and for pupils to learn various aspects, such as place value, that can support pupils' knowledge on the numerical system, and also the density properties of decimal numbers. As been shown by Widjaja, Stacey, and Steinle (2008), the Indonesian prospective elementary teachers (PsETs) seem to have considerable difficulties to understand the density set of rational numbers. They tended to overgeneralize their knowledge about the integers to the case of the decimal numbers. In addition, pupils also find this topic so challenge because of their lack of place value of the numbers involved in the decimal comparison (Pramudiani, Zulkardi, Hartono, & van Amerom, 2011). Based on this situation, we formulate the research question of this study: How do PsETs deal with the teacher task of comparing decimals? What mathematical and didactical knowledge is shared by them when dealing with the teacher task of comparing decimals?

Teacher Knowledge with the Anthropological Theory of the Didactic

Many studies about teacher knowledge refer to the notions of content knowledge (CK) and pedagogical content knowledge (PCK) introduced by Shulman (1986). These notions also have influenced most later studies on mathematics teacher education (Ma, 1999; Hill et al., 2005; Winsløw & Durand-Guerrier, 2007). Winsløw and Durand-Guerrier (2007) categorised three components of teacher knowledge as content knowledge (pertaining to mathematical concepts: use of techniques, theories etc.), pedagogical knowledge (concerning education, learning and teaching in general), and didactical knowledge (regarding the conditions and mechanisms of mathematics teaching and learning, requiring an analysis specific to the target mathematical knowledge). Meanwhile, we focus this study to investigate teachers' mathematical and didactical knowledge specifically on comparing decimals. Mathematical knowledge means the knowledge used by teachers to solve a mathematical task. For instance, to compare two decimal numbers, a teacher may change both decimals into fractions with a common denominator and then compare the numerators. Didactical knowledge is related to the knowledge of teaching mathematical practice and theory to pupils. Some pedagogical aspects may influence the didactical knowledge (and practice), but they are then related to the mathematical knowledge to be taught. For instance, a teacher may suggest pupils work in small groups (a pedagogical aspect), but the teacher has also to describe what mathematical knowledge is discussed, and how it is discussed.

Teachers' mathematical and didactical knowledge is related to what teachers need to know and perform in situations of mathematics teaching. It is a didactic phenomenon because it involves the production, teaching, learning, and practice of

mathematical activities (Hardy, 2009). To study teacher knowledge, ATD provides an epistemological model to describe mathematical knowledge as a human activity among others (Chevallard, 2006). The model states that any knowledge, including teachers' knowledge, can be investigated in term of a *praxeology*. We use this notion as a framework to study teachers' mathematical and didactical knowledge of comparing decimals.

A praxeology consists of two main interrelated components: *praxis* (practical block) and *logos* (theoretical block) (Bosch & Gascón, 2006; Chevallard, 2006; Putra, 2018; Putra, 2019). Both practical and theoretical block of praxeology are divided into two elements. The practical block is made of a *type of tasks* (T) and corresponding *techniques* (τ) which apply to accomplish tasks of type T. An example of a type of mathematical tasks (T) is to compare two given decimal numbers. To solve this task, a technique (τ) is needed; for instance, one can use the lexicographic technique of comparing decimals from right to left until a difference is met. The theoretical block is made of *technologies* (θ) and *theories* (Θ). A technology (θ) is a discourse used to explain and justify the techniques (τ), while a theory (Θ) explains and justifies the technology (θ) with more formal elements such as definitions, general rules and so on. An example of technology is an explanation of available methods to decide which of two different given decimals is greater, when the methods work or are more efficient, etc. The order structure of rational numbers is a mathematical theory (Θ) which can be used to justify and explain the technology (θ).

A praxeology can be used not only to model mathematical knowledge but also to model didactical knowledge. The praxeology used to describe didactical knowledge is known as a *didactical praxeology*. Like a mathematical praxeology, the didactical praxeology includes a type of didactical tasks, didactical techniques, didactical technologies and theories (Rodríguez, Bosch & Gascón, 2008). The didactical praxeology is thus closely related to the mathematical praxeology because the didactical praxeology is based on the task of teaching a (specific) mathematical praxeology. An example of a type of didactical tasks is to teach pupils about comparing two decimals. A didactical technique is to directly present a mathematical technique for comparing two decimals and then ask pupils to apply this technique for other similar mathematical tasks. A technological discourse to justify this didactical technique could be developed from the assumption that pupils might learn better if they directly get an example from the teacher how to solve a mathematical task. Also, learning based on direct instruction and exercise can be considered a didactical theory to justify this didactical technology.

RESEARCH METHOD

This study takes an approach based on the *anthropological theory of the didactic* (ATD), specifically on the notion of *praxeology* (Chevallard, 2006). It is used because it provides a basic unit to model a human action, e.g., teachers' knowledge. Winsløw and Durand-Guerrier (2007) and Durand-Guerrier, Winsløw, and Yoshida (2010) have developed a tool based on this notion to model teachers' specific mathematical and didactical knowledge that is known as *hypothetical teacher task* (HTT). We apply this idea by designing HTT about comparing decimals that can be used to investigate not only PsETs' individual knowledge but also more shared or collective elements. The case study of comparing decimals is as a part of the first author's Ph.D. project about PsETs' knowledge of rational numbers (Putra, 2018).

The notion of HTT has been used by Durand-Guerrier et al. (2010) and Winsløw and Durand-Guerrier (2007) to investigate pre-service lower secondary teachers' knowledge. Any HTT consists of mathematical and didactical tasks for teachers. The mathematical task is one that is problematic to pupils in the hypothetical situation, often related to some common misconceptions. Teachers have to analyze this task and student answers, and provide some mathematical techniques. They work individually for this task and then share their ideas for the discussion on the didactical task. The didactical task asks, with variations depending on the situation described, what could be done to further pupils' overcoming of particular difficulties with the mathematical task. So the didactical task strongly relates to the mathematical task.

The HTT about comparing decimals was designed based on known misconceptions related to place value (Irwin, 2001). As an example, pupils may argue that 0.15 is greater than 0.2 because 0.15 is longer than 0.2 or 15 is greater than 2. Beginning with a situation where pupils hold such views, the HTT reads in Figure 1.

Fifth-grade pupils are asked to compare the size of 0.5 and 0.45. Some pupils answer that 0.45 is greater than 0.5, while others say that 0.5 is greater than 0.45.

- a. Analyze the pupils' answers. Explain your ideas to handle the situation in this class? (to be solved individually in 3 minutes)*
- b. How do you use this situation to further the pupils' learning? (to be discussed and solved in pairs within 5 minutes)*

Figure 1. HTT about comparing two decimals

The HTT was originally written by the first author in English, and then it was translated into Indonesian. Two Indonesian researchers checked the translations for consistency. The HTT was also piloted with a pair of recently graduated students from the Elementary School Teacher Education (ESTE) study program at University of Riau, Indonesia. We asked for the students' comments and used them to revise the HTT. The data consist of PsETs' written answers for the first question and video recording of the discussion for the second question. We transcribed the video recording for all groups using the NVivo computer program. Then, the written answers and video transcripts were analyzed based on the mathematical and didactical praxeologies, to identify the techniques produced, and also possible technologies and theories. The subjects for the implementation of HTT were 32 (16 pairs) fourth-year PsETs from the ESTE study program, and the data were collected in March 2016. All participants wrote their answers on the worksheets for the individual question a, and then they used their answers to support a common discussion for the question b. A more comprehensive analysis of these data was based on the techniques identified among individual pairs, and the technologies and possible theories. While to keep the reliability of the data analysis, some questionable points from the PsETs' written answers and their collaborative work are discussed with one mathematics researcher who has some experiences on the similar study and knows about the ATD framework.

In the first phase of the analysis, we focus on the practical blocks (i.e. types of tasks and techniques). The mathematical task (T_m) contained in the HTT (Figure 1) can be stated as follows:

T_m : given two different decimal numbers, $0 < a < 1$ and $0 < b < 1$, decide if $a > b$ or $a < b$.

There are many possible mathematical techniques to solve a mathematical task of type T_m which could be developed by the PsETs individually, or during their discussion. We describe some of them in Table 1.

Table 1. Mathematical Techniques for the Mathematical Task of Type T_m

Code of techniques	General description of techniques
τ_1	Change a and b into integers by multiplying both by an appropriate power of ten.
τ_2	Use lexicographical order to compare the decimals.
τ_3	Add 0 digits where required to get the same number of digits in both decimals.
τ_4	Change decimals into fractions with a common denominator and compare the numerators.
τ_5	Subtract b from a or divide a by b . When the result is less than 0 (for subtraction) or less than 1 (for division), $a < b$, otherwise $a > b$.

In addition, there are several possible mathematical techniques based on diagrammatical representations and number lines. For instance, one can represent both decimals by a rectangle or a circle diagram and then compare areas or sizes (τ_6), or locate both decimals on a number line and compare the positions (τ_7). Furthermore, to each correct mathematical technique, one might associate with one or more incorrect mathematical techniques. For example, when someone multiplies both decimals with different powers of ten, one may end up with an incorrect mathematical technique similar to τ_1 . This mathematical technique is denoted as τ_1^- , where the minus means “incorrect variation of τ_1 ”. Typically, there will be at least as many incorrect mathematical techniques as correct ones.

The question b, and also part of question a, contains a didactical task (T_d) as follows:

T_d : given that pupils’ answers as stated to a task of type T_m , determine what to do as a teacher to facilitate pupils’ learning.

Most didactical techniques to solve T_d relate to the mathematical techniques proposed to solve the task of type T_m . When PsETs recommend teaching pupils by simply explaining a mathematical technique, for instance τ_1 , this technique is coded as τ_1^* , and τ_{1x}^* is a code from applying the mathematical technique τ_1^- , so similar numbers of didactical techniques can be derived from the previous mathematical techniques. In addition, some didactical techniques can be variants of those didactical techniques. For instance, PsETs provide pupils with similar problems, such as comparing 0.5 and 0.25 (τ_{10}^*), they choose these decimals because pupils might simply recognize both decimals as a half and a quarter, and may then realize their original mistake. Many other possible didactical techniques might appear during the discussion. One common didactical technique is to build the mathematical task into a real word problem (τ_8^*). PsETs may even say that the mathematical task presented in the HTT is too abstract for pupils, so they need to present it within a more familiar situation (Table 2). Such a justification furnishes a technological discourse for the didactical technique, could conceivably even invoke a didactic theory.

Table 2. Variants of didactical techniques to the didactical task of type T_d

Code of techniques	General description of techniques
τ_{2a}^*	Teach pupils about place value or positioning numerical systems.
τ_{5a}^*	Present the task as addition or subtraction of two decimals
τ_{7a}^*	Propose a contextual/real-world problem related to measurement, e.g. measure or compare two bottles of water, one contains 0.45 liter and the other one contains 0.5 liters.
τ_8^*	Explain through a contextual/real-world problem related to dividing and comparing objects like cakes (τ_{8x}^* is an inappropriate contextual/real-world problem)
τ_9^*	Organize a class discussion about the two solutions, to have pupils realize what is the correct answer.
τ_{10}^*	Provide pupils with other problems related to comparing decimals, such as giving some common decimal numbers.
τ_{11}^*	Explain through rounding decimals to the nearest tenths, hundredths, thousandths, and so on. Let pupils aware that the decimals after rounding can be more or less depending on the type of round. Ex. 0.45 is rounded into 0.5, so $0.45 < 0.5$ because it is rounded the decimal up.

RESULTS AND DISCUSSION

The results of this study are presented into two sections. The first section presents the analysis of PsETs' mathematical praxeologies, especially from their written answers. The second section focuses on PsETs' didactical praxeologies from their collaborative work, and how they link their written answer to build their didactical knowledge.

PsETs' Mathematical Praxeologies

The analysis of answers to the task of type T_m was mainly based on the PsETs' written solutions, but we also looked at the video transcripts when there were some difficulties in categorizing the mathematical techniques from the written solutions. In general, almost all mathematical techniques described in the reference models appeared in PsETs' written answers, but some techniques were more common than others. The mathematical techniques presented by PsETs are summarised in Table 3.

Table 3 shows that the most common mathematical technique is to add 0s to equalize the number digits after the decimal point (τ_3). PsETs provide two technological discourses to support this technique. The first and most frequent one is to justify τ_3 based on the equivalent representation of decimals. This means that one can write a decimal number into several different representations, such as 0.5; 0.50; or 50%, but they still have the same value. It is illustrated by a PsET' s written answer from group 1:

I think the answers from some pupils are incorrect because 0.45 has a smaller value than 0.5, because 0.5 means 0.50. For 0.50 it is rarely written the zero after the digit behind the comma (S_{1a}).

Another PsET also supported the mathematical technique τ_3 based on to the order structures of decimal numbers.

We should first explain that the value of 0.5 is the same as 0.50, so if it is compared to 0.45, the answer is that 0.5 or 0.50 is greater than 0.45,

because from 0.45 to 0.5, there are still many (decimal) numbers, such as 0.45; 0.46, ..., 0.5 (S_{2b}).

Knowing the equivalent value of two different rational representations can help pupils to consider that there are also many numbers can be found in between two decimal numbers, and this situation could also let them recognize which decimal numbers are greater.

Table 3. A summary of PsETs' mathematical techniques for T_m

Mathematical techniques	Number of answers
τ_1	2
τ_1^-	1
τ_2	2
τ_3	10
τ_4	6
τ_4^-	5
τ_5	1
τ_6	1
τ_6^-	1
τ_7	3
τ_7^-	2
No answers	1
Total	35

Changing decimals into fractions is also the other common mathematical technique given by prospective teachers, but five of them could not change 0.45 into a fraction.

To have pupils better understand, a teacher can give an instruction to the pupils that $0.5 = \frac{1}{2}$ and $0.45 = \dots$, and then those can be represented by diagrams (S_{4b}).

S_{4b} has the idea of equivalent value between decimal and fraction representations, and 0.5 could be known as a half, but she does not know how to convert a decimal into a fraction. During the discussion, she also mentioned that

We can change decimals into fractions, but I do not know how to change 0.45 into a fraction (S_{4b}).

While her partner did not also know how to convert 0.45 into a fraction, and in fact, they did not have any appropriate mathematical technique to solve that mathematical task T_m . Among six PsETs who gave a correct mathematical technique of τ_4 , only two PsETs changed the fractions to have a common denominator and then compared numerators, whereas the others presented both decimals into simple fractions and compared intuitively. The technology underlying this technique is also the equivalent value between decimal and fraction representations.

Five PsETs also provided the mathematical technique of representing decimals on a number line, but two of them placed the numbers in incorrect positions on the number line. One of those PsETs stated on her worksheet:

I think the correct answer is that 0.5 is greater than 0.45 when compared, and one of the solutions is to present them on a number line

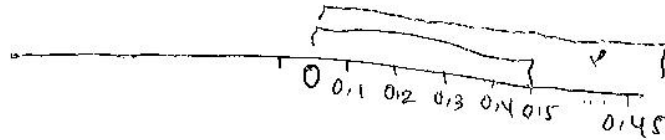


Figure 2. S_{4b}'s incorrect number line representation of decimals

S_{4b} agreed that 0.5 is greater than 0.45, and also represented the decimal numbers incorrectly on the number line (Figure 2). She did not realize that 0.45 should be put between 0.4 and 0.5. There was also a mistake for a PsET who applied the mathematical technique τ_6 even if she answered correctly that 0.5 was bigger than 0.45 (Figure 3). Overall, only 71% of the mathematical techniques presented by the PsETs are correct.

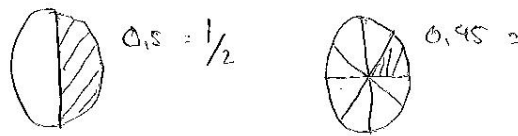


Figure 3. An incorrect diagram representation of decimals

PsETs' Didactical Praxeologies

The total number of didactical techniques proposed by PsETs is greater than the number of those mathematical techniques because some pairs presented more than one didactical technique during their discussion. The most common didactical technique was direct instruction of a specific mathematical technique, used by the PsETs themselves to solve the pupils' task of type T_m. Some PsETs who gave incorrect or incomplete mathematical techniques also tended to instruct pupils improperly. In general, the didactical techniques discussed by PsETs are presented in Table 4.

Table 4. A Summary of PsETs' Didactical Techniques for T_a

Didactical techniques	Number of answers	Didactical techniques	Number of answers
τ_1^*	2	τ_6^*	4
τ_{1x}^*	3	τ_{6x}^*	2
τ_2^*	2	τ_7^*	5
τ_{2a}^*	2	τ_{7a}^*	1
τ_3^*	8	τ_{7x}^*	2
τ_4^*	2	τ_8^*	2
τ_{4a}^*	4	τ_{8x}^*	1
τ_{4x}^*	2	τ_9^*	2
τ_5^*	1	τ_{10}^*	4
τ_{5a}^*	1	τ_{11}^*	1
Total		51	

Table 4 shows that the didactical technique of demonstrating how to add 0 after 0.5 (τ_{3}^*) and to convert decimals into fractions were the most common didactical techniques suggested by PsETs. Four pairs who discussed the converting technique were only able to compare the two fractions intuitively (τ_{4a}^*). This means that they knew that 0.5 equals to $\frac{1}{2}$, and 0.45 equals to $\frac{9}{20}$, but they did not explain why $\frac{1}{2}$ is greater than $\frac{9}{20}$. None of them suggests any technological discourse such as using a benchmark or equivalent fractions ($\frac{1}{2} = \frac{10}{20}$) to clarify that technique. And two pairs did not even know how to find a fraction representation for 0.45 (τ_{4x}^*).

The didactical technique related to number line representations was also discussed by eight pairs of PsETs, but two of them placed 0.45 incorrectly in relation to 0.5. For example, the following discussion shows how two PsETs shared their incorrect mathematical techniques τ_{4-} and τ_{7-} in order to produce possible didactical techniques.

- S_{4a}: Let's use a number line. Here is 0, and here is 0.1; 0.2. (She explained her drawing presented in Figure 1.)
- S_{4b}: And so on.
- S_{4a}: So, 0.5 is greater than 0.45.
- S_{4b}: How can we know that 0.5 is greater than 0.45? I thought, using your number line, that one is greater than the other.
- S_{4a}: How do you think?
- S_{4b}: I am confused. I change them into fractions. From fractions, they can be represented in rectangle diagrams, so we can see them. For instance, we know that 0.5 is equal to a half.
- S_{4a}: Hmm.
- S_{4b}: If this is 0.45, what fraction is it? Later, it is drawn. From the drawing, pupils can compare, to see which one is greater.

From the discussion, S_{4b} might realize that her partner placed the two decimals incorrectly on the number line, but she did not have any idea on how to fix it. Instead, she proposed to change decimals into fractions and then suggested to represent the fractions into rectangle diagrams. However, it turned out that they could not change 0.45 into a fraction or represent it by a correct rectangle diagram. They appeared to lack a general technique to convert decimals into fractions.

Five pairs suggested explaining to pupils how to change decimals into percentages or into natural numbers, but three of them were, in fact, unable to do so correctly (τ_{1x}^*). For example, one prospective elementary teacher presented to his partner the mathematical technique of changing decimals into fractions. He changed 0.5 into $\frac{5}{100}$ and assumed that was similar to 500%, but no one realized the mistake. Furthermore, some PsETs also considered presenting the mathematical task into a contextual or real-life problem, providing other decimal comparison problems, or giving some technological elements, such as writing 0s after the decimal point is rarely written but may be useful. In general, twelve pairs suggested reasonable didactical techniques, most of the techniques being classified as direct instruction of mathematical techniques. Two pairs suggested both reasonable and unreasonable didactical techniques, and the other two totally could not recommend any didactical technique.

As for the theoretical block of the didactical praxeologies produced, four pairs argued that one can always add 0 digits at the end of a decimal because they will not change the value of the number represented. This technology was used by PsETs to justify the didactical technique τ_3^* . Meanwhile, a technological discourse mentioned by some PsETs to justify the didactical techniques such as τ_6^* , τ_7^* , and τ_8^* is based on their belief that concrete models or examples will accelerate pupils' learning process, but some of them did not have relevant mathematical knowledge to support it.

CONCLUSION

An important point for this study is to explore PsETs' mathematical and didactical knowledge of comparing decimals. The design of tasks involves open constructed-responses and conversations among pairs of informants. This situation challenges PsETs to produce more than one single technique for each task. They shared their mathematical knowledge to provide didactical techniques to further pupil learning through a collaborative effort.

To deal with the mathematical task of comparing decimals, the PsETs proposed several mathematical techniques. The most common mathematical technique shared by PsETs was to put 0s after numbers behind the comma to equalize the number of digits for both decimals (τ_3). This mathematical technique can be simply applied by PsETs because it reduces the comparison to the more familiar task of comparing two integers. The technique is valid for comparing two decimal numbers in $[0,1]$, but it does not work as immediately in other cases; so it is a more limited technique than, for instance, τ_4 .

When PsETs discuss how they might handle the didactical task, they tend to just explain, based on their mathematical techniques, how to solve the mathematical task. In fact, when they have an inappropriate mathematical technique for the mathematical task, they then struggle to provide an appropriate didactical technique during the discussion. With subtle didactical techniques in mind, they could conceivably realize their mathematical mistake; unfortunately, this was not observed in any case. In addition, we may argue that PsETs' difficulties cover all area of decimals, from the density properties (Widjaya, et al. 2008) to the arithmetic operation of decimals (Putra, 2018; Putra & Winslow, 2018).

Finally, we conclude this study with two remarks. First, the mathematical task designed in the HTT did not involve a contextual or real-life situation. Such a situation could both facilitate and add to the difficulty of the HTT, and variations of this type would be interesting to investigate. The second one is related to the PsETs' collective discussion on didactical techniques. We expected that they could resolve their difficulties in constructing didactical techniques during their discussion in pairs, but some could not do that because none of them had an adequate mathematical technique for the first part. Therefore, we may recommend to a future study to apply a problematic HTT as an instructional tool for a classroom discussion in the teacher education program, in order to overcome both the PsETs' own mathematical misconceptions and to construct didactical techniques for their future tasks as teachers.

Acknowledgment

The authors thank the Ministry of Research, Technology, and Higher Education of the Republic of Indonesia for funding this project. The preliminary result of this study has

been presented in the 10th Congress of European Research in Mathematics Education, Dublin, February 1-5, 2017.

REFERENCES

- Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICME Bulletin*, 58, 51-65.
- Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In Bosch, M. (Ed.) *Proceedings of the IV Congress of the European Society for Research in Mathematics Education* (p. 21-30). Barcelona, Spain: FUNDEMI-IQS. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Steps_towards_a_New_Epistemology.pdf
- Durand-Guerrier, V., Winsløw, C., & Yoshida, H. (2010). A model of mathematics teacher knowledge and a comparative study in Denmark, France, and Japan. *Annales de didactique et des sciences cognitives*, 15, 147-172. https://mathinfo.unistra.fr/fileadmin/upload/IREM/Publications/Annales_didactique/vol_15/adsc15-2010_006.pdf
- Hardy, N. (2009). Students' perceptions of institutional practices: the case of limits of functions in college level Calculus courses. *Educational Studies in Mathematics*, 72(3), 341. <https://doi.org/10.1007/s10649-009-9199-8>
- Hill, H.C., Rowan, H.B., & Ball, D.L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371-406. <http://lmt.soe.umich.edu/files/hillrowanball.pdf>
- Irwin, K.C. (2001). Using everyday knowledge of decimals to enhance understanding. *Journal for research in mathematics education*, 32(4), 399-420. <http://jwilson.coe.uga.edu/EMAT7050/articles/Irwin.pdf>
- Kuntur, M., Baumert, J., Voss, T., Klusmann, U., Richter, D., & Hachfeld, A. (2013). Professional competence of teachers: effects on instructional quality and student development. *Journal of Educational Psychology*, 105(3), 805-820. <http://dx.doi.org/10.1037/a0032583>
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.
- OECD. (2015). *PISA 2015 Results in Focus*. Retrieved from <https://www.oecd.org/pisa/pisa-2015-results-in-focus.pdf>
- Pramudiani, P., Zulkardi, Hartono, Y., & van Amerom, B. (2011). A concrete situation for learning decimals. *Journal on Mathematics Education*, 2(2), 215-230. <https://doi.org/10.22342/jme.2.2.750.215-230>
- Putra, Z.H. (2018). A comparative study of Danish and Indonesian pre-service teachers' knowledge of rational numbers (Doctoral dissertation), Copenhagen: University of Copenhagen. Retrieved from https://www.ind.ku.dk/begivenheder/2018/putra/Thesis_Zetra_Putra.pdf
- Putra, Z.H. (2019). Elementary teachers' knowledge on fraction multiplication: An anthropological theory of the didactic approach. *Journal of Teaching and Learning in Elementary Education*, 2(1), 47-52. <https://dx.doi.org/10.33578/jtlee.v2i1.6964>
- Putra, Z.H., & Winsløw, C. (2018). Teachers' collective knowledge: the case of equivalent fractions. *Journal of Physics: Conference Series*, 1088(1), 012003. <https://doi.org/10.1088/1742-6596/1088/1/012003>

- Rodríguez, E., Bosch, M., & Gascón, J. (2008). A networking method to compare theories: metacognition in problem solving reformulated within the Anthropological Theory of the Didactic. *ZDM*, 40(2), 287-301. <https://doi.org/10.1007/s11858-008-0094-z>
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14. <https://doi.org/10.3102/0013189X015002004>
- Tatto, M.T., Rodriguez, M.C., Smith, W.M., Reckase, M.D., & Bankov, K. (2018). *Exploring the Mathematical Education of Teachers Using TEDS-M Data*. Switzerland: Springer International Publishing. <https://doi.org/10.1007/978-3-319-92144-0>
- Tatto, M.T., Schwille, J., Senk, S., Ingvarson, L., Peck, R., & Rowley, G. (2008). *Teacher Education and Development Study in Mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics. Conceptual framework*. East Lansing, MI: Teacher Education and Development International Study Center, College of Education, Michigan State University. https://msu.edu/~mttatto/documents/TEDS_FrameworkFinal.pdf
- Widjaja, W., Stacey, K., & Steinle, V. (2008). Misconceptions about density of decimals: Insights from Indonesian pre-service teachers' work. *Journal for Science and Mathematics Education in Southeast Asia*, 31(2), 117-131.
- Winsløw, C., & Durand-Guerrier, V. (2007). Education of lower secondary mathematics teachers in Denmark and France. *Nordic Studies in Mathematics Education*, 12(2), 5-32. http://ncm.gu.se/pdf/nomadopen/12_2_005032_winslow.pdf